Turk J Math
26 (2002) , 229 – 235.

Remarks on Bounded Operators in Köthe Spaces

P. Djakov, T. Terzioğlu, M. Yurdakul, V. Zahariuta

Abstract

We prove that if $\lambda(A), \lambda(B)$ and $\lambda(C)$ are Köthe spaces such that $L(\lambda(A), \lambda(B))$ and $L(\lambda(C), \lambda(A))$ consist of bounded operators then each operator acting on $\lambda(A)$ that factors over $\lambda(B)\widehat{\otimes}_\pi \lambda(C)$ is bounded.

If X and Y are topological vector spaces then a linear operator $T : X \to Y$ is *bounded* if there exists a neighborhood of zero U in X such that $T(U)$ is a bounded set in Y. We write $(X,Y) \in \mathcal{B}$ if each continuous linear operator from X into Y is bounded. As in [2] we say that a pair (X,Y) has the *bounded factorization* property and write $(X,Y) \in \mathcal{BF}$ if each linear continuous operator $T : X \to X$ that factors over Y (i.e. $T = S_1S_2$, where $S_2 : X \to Y$ and $S_1 : Y \to X$ are linear continuous operators) is bounded. There is still no general characterization of pairs of Fréchet spaces with $\mathcal{BF}$ property (see Question 1 in [2]).

First, let us note some obvious properties of the relation $\mathcal{BF}$:

(i) if $E \subset X$ and $F \subset Y$ are, respectively, complemented subspaces of X and Y, then $(X,Y) \in \mathcal{BF}$ implies $(E,F) \in \mathcal{BF}$;

(ii) if $(X,Y) \in \mathcal{B}$ and $(Z,X) \in \mathcal{B}$ then $(X, Y \times Z) \in \mathcal{BF}$.

In [2] we used the second property in order to construct non-trivial examples of pairs of Fréchet spaces with $\mathcal{BF}$ property.

Our aim here is to prove that if X, Y, Z are Köthe spaces such that $(X,Y) \in \mathcal{B}$ and $(Z,X) \in \mathcal{B}$ then $(X, Y\widehat{\otimes}_\pi Z) \in \mathcal{BF}$ (Theorem 3). Our approach is based on a

AMS Subject Classification [1991]: Primary 46A45.

characterization of pairs of Köthe spaces with property $\mathcal{B}$ ([4], Theorem 3.3; see also [1] and Proposition 1 below), which is a modification of Vogt's result [9], Satz 1.5, where ℓ_∞-Köthe space $\lambda^\infty(B)$ is considered instead of $\lambda(B)$. Here we provide a direct proof of Proposition 1 in the spirit of [9] as suggested in [4]. The proof in [1] was based on a characterization of $\mathcal{B}$ in terms of quasi-diagonal operators obtained in ([3, 8, 1]). We observe (using an argument from [1]) that the quasi-diagonal characterization of $\mathcal{B}$ itself can be obtained as a consequence of Proposition 1.

Let $\mathbf{I}$ be a countable set, and let $A = (a_{ip})_{i\in\mathbf{I},p\in\mathbb{N}}$ be a matrix of real numbers such that $0 \le a_{ip} \le a_{i,p+1}$. A Köthe space $\lambda(A)$ is the space of all sequences $x = (x_i)$ of real (or complex) numbers such that $\|x\|_p = \sum_{i\in\mathbf{I}} |x_i| a_{ip} < \infty \quad \forall p \in \mathbb{N}$; regarded with the system of seminorms $\|x\|_p$, $p \in \mathbb{N}$ it is a Fréchet space. As usual, we denote the canonical basis of $\lambda(A)$ by $(e_i)_{i\in\mathbf{I}}$.

An operator $T : \lambda(A) \to \lambda(B)$ is called *quasi-diagonal* if there exist a map $m : i \to m(i)$ and a sequence of numbers (t_i) such that $T(e_i) = t_i e_{m(i)}$, $\forall i \in \mathbf{I}$.

If X and Y are locally convex spaces we denote by $X\widehat{\otimes}_\pi Y$ the completion of their projective tensor product. In case $X = \lambda(A), A = (a_{ip})_{i,p\in\mathbb{N}}$ and $Y = \lambda(B), B = (b_{jp})_{i,p\in\mathbb{N}}$ the space $X\widehat{\otimes}_\pi Y$ is naturally isomorphic to the space $\lambda(C)$, $C = (c_{\nu p}), c_{\nu p} = a_{ip} b_{jp}, \nu = (i,j) \in \mathbf{I} = \mathbb{N} \times \mathbb{N}$ (e.g. [10]).

We use the following notation for the operator seminorms of a linear operator T between Fréchet spaces X and Y (which may take as a value ∞):

$$|T|_{p,q} := \sup\{\|Tx\|_p : \|x\|_q \le 1\}.$$

Recall that the operator T is continuous if and only if there is a map $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$|T|_{p,\pi(p)} < \infty, \quad p \in \mathbb{N},$$

while T is bounded if and only if there is r such that

$$|T|_{q,r} < \infty, \quad q \in \mathbb{N}.$$

We refer to [5, 6, 7] for terminology used, but not defined here.

Proposition 1 *([9, 4]) If $\lambda(A)$, $A = (a_{ip})$ and $\lambda(B)$, $B = (b_{\nu q})$ are Köthe spaces then $(\lambda(A), \lambda(B)) \in \mathcal{B}$ if and only if for each $\pi : \mathbb{N} \to \mathbb{N}$ there is $k \in \mathbb{N}$ such that for each*

$r \in \mathbb{N}$ there are $q_0 \in \mathbb{N}$ and $M > 0$ such that

$$\frac{b_{\nu r}}{a_{ik}} \leq M \max_{1 \leq q \leq q_0} \left\{ \frac{b_{\nu q}}{a_{i\pi(q)}} \right\} \tag{1}$$

holds for all i, ν.

Proof. The necessity part follows from [9], Satz 1.1, if applied to the set of all one-dimensional operators $T := e_\nu \otimes e'_i$, since $|e_\nu \otimes e'_i|_{q,\pi(q)} = \dfrac{b_{\nu q}}{a_{i\pi(q)}}$.

To prove that (1) is sufficient we fix an operator $T \in L(\lambda(A), \lambda(B))$, find $\pi(q)$ from the continuity of T and choose k as in (1). If $Te_i = \sum_{\nu=1}^{\infty} u_{\nu,i} e_\nu$ then $|T|_{q,\pi(q)} = \sup_{i \in \mathbb{N}} \left\{ \sum_{\nu=1}^{\infty} |u_{\nu,i}| \dfrac{b_{\nu q}}{a_{i\pi(q)}} \right\} < \infty$. Now,

$$\begin{aligned}
|T|_{r,k} &= \sup_i \left\{ \sum_{\nu=1}^{\infty} |u_{\nu,i}| \frac{b_{\nu r}}{a_{ik}} \right\} \leq M \sup_i \sum_{\nu=1}^{\infty} |u_{\nu,i}| \max_{1 \leq q \leq q_0} \left\{ \frac{b_{\nu q}}{a_{i\pi(q)}} \right\} \\
&\leq M \sup_i \left\{ \sum_{\nu=1}^{\infty} |u_{\nu,i}| \sum_{q=1}^{q_0} \frac{b_{\nu q}}{a_{i\pi(q)}} \right\} \leq M \sum_{q=1}^{q_0} \sup_i \left\{ \sum_{\nu=1}^{\infty} |u_{\nu,i}| \frac{b_{\nu q}}{a_{i\pi(q)}} \right\} \\
&= \sum_{q=1}^{q_0} M |T|_{q,\pi(q)} < \infty.
\end{aligned}$$

This means T is bounded.

The following result generalizes the corresponding fact for nuclear spaces ([3, 8]); it is obtained in [1], and used there to prove Proposition 1. Now we observe (using an argument from ([1])) that it can be obtained as a corollary of Proposition 1.

Corollary 2 *A pair of Köthe spaces $\lambda(A)$ and $\lambda(B)$ has the property $\mathcal{B}$ if and only if each continuous quasi-diagonal operator from $\lambda(A)$ into $\lambda(B)$ is bounded.*

Proof. Obviously it is enough to prove that if (1) fails then there exists a continuous unbounded quasi-diagonal operator from $\lambda(A)$ into $\lambda(B)$.

Suppose (1) fails; then there exists a map $q \to \pi(q)$ such that

$$\forall k \, \exists r_k \, \forall n \in \mathbb{N} \, \exists i_n, \nu_n \quad : \quad \frac{b_{\nu_n r_k}}{a_{i_n k}} \geq n \max_{1 \leq q \leq n} \frac{b_{\nu_n q}}{a_{i_n \pi(q)}} \, ,$$

where the sequences $(i_n) = (i_n(k))$ and $(\nu_n) = (\nu_n(k))$ depend on k.

There exist new sequences (for convenience we use the same notations (i_n) and (ν_n)) such that the sequence (i_n) is strictly increasing and for each k there exists a subsequence (n_j) with $i_{n_j} = i_{n_j}(k)$, $\nu_{n_j} = \nu_{n_j}(k)$, $\forall j$. Indeed, let $\mathbb{N} = \cup_s N_s$ be a representation of $\mathbb{N}$ as a sum of disjoint infinite subsets. Choose one after another elements $i_n = i_{k_n}(s)$ and $\nu_n = \nu_{k_n}(s)$ for $n \in N_s$ so that $i_n > i_{n-1}$.

Consider a quasi-diagonal operator $T : K(a) \to K(b)$ defined by

$$Te_i = 0 \quad \text{for} \quad i \neq i_n, \qquad Te_{i_n} = t_n \tilde{e}_{\nu_n},$$

where

$$t_n^{-1} := \max_{1 \le q \le n} \frac{b_{\nu_n q}}{a_{i_n \pi(q)}} .$$

By the choice of constants t_n the operator T is continuous. On the other hand for each k there exists r_k such that for some subsequence (n_j) we have

$$t_{n_j} b_{\nu_{n_j} r_k} / a_{i_{n_j} k} \ge n_j,$$

hence the operator T is unbounded.

Theorem 3 *Suppose $A = (a_{np}), B = (b_{iq})$ and $C = (c_{jq})$ are Köthe matrices and $\lambda(A), \lambda(B)$ and $\lambda(C)$ are the corresponding Köthe spaces. If $(\lambda(A), \lambda(B)) \in \mathcal{B}$ and $(\lambda(C), \lambda(A)) \in \mathcal{B}$ then $(\lambda(A), \lambda(B) \hat{\otimes}_\pi \lambda(C)) \in \mathcal{BF}$.*

Proof. The tensor product $\lambda(B) \hat{\otimes}_\pi \lambda(C)$ is isomorphic to the Köthe space generated by the matrix $D = (b_{iq} c_{jq})$; we denote the elements of the canonical basis of $\lambda(D)$ by e_{ij}, then $|e_{ij}|_q = b_{iq} c_{jq}$.

Let $T : \lambda(A) \to \lambda(D)$ and $S : \lambda(D) \to \lambda(A)$ be arbitrary continuous operators, and let (T_n^{ij}), (S_{ij}^m) be their matrix representations, that is $Te_n = \sum_{ij} T_n^{ij} e_{ij}$, $Se_{ij} = \sum_m S_{ij}^m e_m$. In order to prove the theorem we show that the composition $ST : \lambda(A) \to \lambda(A)$ is a bounded operator.

Since the operator S is continuous there is a map $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_m |S_{ij}^m| a_{mp} \le C_p \, b_{i\pi(p)} c_{j\pi(p)}, \quad (i, j) \in \mathbb{N}^2 \tag{2}$$

holds with some constant C_p, $p \in \mathbb{N}$.

Since $(\lambda(C), \lambda(A)) \in \mathcal{B}$, by Proposition 1 there is l such that for every $r \in \mathbb{N}$ there exist $p_0 \in \mathbb{N}$ and $M > 0$ such that

$$\frac{a_{mr}}{c_{jl}} \leq M \max_{1 \leq p \leq p_0} \left\{ \frac{a_{mp}}{c_{j\pi(p)}} \right\} \tag{3}$$

holds for all m, j.

Since the operator T is continuous there exists a map $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{ij} |T_n^{ij}| b_{iq}\, c_{jq} \leq \tilde{C}_q a_{n\sigma(q)}, \quad n \in \mathbb{N} \tag{4}$$

holds for any $q \in \mathbb{N}$ with some constant $\tilde{C}_q > 0$. Without loss of generality we can assume that the above inequality remains true if c_{jq} is replaced by c_{jl} since the map σ and the constant $\tilde{C}$ may be chosen so that $\sigma(q) = \sigma(l)$ and $\tilde{C}_q = \tilde{C}_l$ for $q \leq l$. Now, again by Proposition 1, the relation $(\lambda(A), \lambda(B)) \in \mathcal{B}$ implies that for the map σ there is $k \in \mathbb{N}$ such that for every $s \in \mathbb{N}$ there are $q_0 \in \mathbb{N}$ and $M_1 > 0$ such that

$$\frac{b_{is}}{a_{nk}} \leq M_1 \max_{1 \leq q \leq q_0} \left\{ \frac{b_{iq}}{a_{n\sigma(q)}} \right\}, \quad i, n \in \mathbb{N}. \tag{5}$$

Let $R = ST$, then $Re_n = \sum_m \left(\sum_{ij} T_n^{ij} S_{ij}^m \right) e_m$. By (3), (2), we have

$$\begin{aligned} \|Re_n\|_r &\leq \sum_m \sum_{ij} \left| T_n^{ij} S_{ij}^m \right| a_{mr} \leq \sum_m \sum_{ij} \left| T_n^{ij} \right| \left| S_{ij}^m \right| M \max_{1 \leq p \leq p_0} \left\{ \frac{a_{mp}}{c_{j\pi(p)}} \right\} c_{jl} \\ &\leq M \sum_{p=1}^{p_0} \sum_{ij} \left| T_n^{ij} \right| C_p\, b_{i\pi(p)} c_{jl}. \end{aligned}$$

Now, by (5) with $s = \pi(p)$, we get

$$\begin{aligned} \|Re_n\|_r &\leq M \sum_{p=1}^{p_0} C_p \sum_{ij} \left| T_n^{ij} \right| M_1(\pi(p)) \max_{1 \leq q \leq q_0(\pi(p))} \left\{ \frac{b_{iq}}{a_{n\sigma(q)}} \right\} a_{nk} c_{jl} \\ &\leq M \sum_{p=1}^{p_0} C_p M_1(\pi(p)) \sum_{q=1}^{q_0(\pi(p))} \sum_{ij} \left| T_n^{ij} \right| \frac{b_{iq}}{a_{n\sigma(q)}} a_{nk} c_{jl}. \end{aligned}$$

Finally, applying (4) with c_{jl} instead of c_{jq}, we have

$$\|Re_n\|_r \leq \left[M \sum_{p=1}^{p_0} C_p M_1(\pi(p)) \sum_{q=1}^{q_0(\pi(p))} \tilde{C}_q \right] a_{nk} = D\, a_{nk},$$

i.e. $|R|_{k,r} \leq D$, which means that the operator R is bounded. The theorem is proved.

References

[1] P.B.Djakov, M.S.Ramanujan, *Bounded and unbounded operators between Köthe spaces*, to appear in Studia Math.

[2] P. B. Djakov, T. Terzioğlu, M. Yurdakul, V. P. Zahariuta, *Bounded operators and isomorphisms of Cartesian products of Fréchet spaces*, Mich. Math. J., **45**, 599-609, 1998.

[3] M.M.Dragilev, *Riesz classes and multi-regular bases,* (Russian), Theory of functions, functional analysis and their applications, Kharkov, vol. **15**, 512-525, 1972.

[4] J. Krone, D. Vogt, *The splitting relation for Köthe spaces,* Math. Z., **190,** 387-400, 1985.

[5] G. Köthe, *Topological vector spaces I,* Berlin-Heidelberg-New York, 1969.

[6] G. Köthe, *Topological vector spaces II,* Berlin-Heidelberg-New York, 1979.

[7] R. Meise, D. Vogt, *Introduction to Functional Analysis,* Clarendon Press, Oxford, 1997.

[8] Z.Nurlu, T.Terzioğlu, *Consequences of the existence of a non-compact operator,* Manuscripta Math. **47**, 1-12, 1984.

[9] D. Vogt, *Frécheträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J.Reine. Angew. Math., **345**, 182-200, 1983.

[10] V.P.Zahariuta, *Linear Topologic Invariants and their applications to Isomorphic Classification of Generalized Power Spaces*, Turkish Journal of Mathematics, **20**, 237-289, 1996.

Received 29.11.2001

P. B. DJAKOV
Department of Mathematics
Sofia University
1164 Sofia-BULGARIA
Ohio State University
Columbus, OH 43210, USA
e-mail: djakov@fmi.uni-sofia.bg
e-mail: djakov@math.ohio-state.edu
T. TERZİOĞLU
Sabancı University
İstanbul-TURKEY
e-mail:tosun@sabanciuniv.edu.tr
M. YURDAKUL
Department of Mathematics
Middle East Technical University
06531 Ankara, Turkey
e-mail: myur@metu.edu.tr
V. P. ZAHARIUTA
Department of Mathematics
Rostov State University
Rostov-on-Don-RUSSIA
Department of Mathematics
Middle East Technical University
06531 Ankara-TURKEY
e-mail: zaha@math.metu.edu.tr