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Deformation inequivalent complex conjugated complex structures and applications

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Abstract

We start from a short summary of our principal result from [KK]: an example of a complex algebraic surface which is not deformation equivalent to its complex conjugate and which, moreover, has no homeomorphisms reversing the canonical class. Then, we generalize this result to higher dimensions and construct several series of higher dimensional compact complex manifolds having no homeomorphisms reversing the canonical class. After that, we resume and broaden the applications given in [KK] and [KK2], in particular, as a new application, we propose examples of (deformation) non equivalent symplectic structures with opposite canonical classes.

1. Introduction

Many of achievements in real algebraic geometry appeared as applications of complex geometry. By contrary, the results of the present paper, which has grown from a solution of some questions from the real algebraic geometry, can be considered as applications in the opposite direction. They may be of an interest for symplectic geometry too.

To state the principal questions we need to fix some definitions. We choose the language of complex analytic geometry since in this setting the results sometimes look stronger than if we restricted ourselves to algebraic or Kähler varieties, though the Kähler hypothesis could simplify several proofs and could allow one to extend the set of examples.

Thus, we define a *real structure* on a complex manifold X as an antiholomorphic involution $c : X \to X$. By a *deformation equivalence* we mean the equivalence generated by local deformations of complex manifolds. Since we are interested only in compact manifolds, this, commonly used, local deformation equivalence relation can be defined in the following way: given a proper holomorphic submersion $f : W \to B_1$, $B_1 = \{|z| < 1\} \subset \mathbb{C}$, the complex varieties **isomorphic** to $X_t = f^{-1}(t), t \in B_1$, are said to be *deformation equivalent*.

Note that anti-holomorphic automorphisms (*anti-automorphisms* for short) of a complex manifold $X = (M, J)$, where M is the underlying smooth manifold and J is a complex

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structure on it, can be regarded as isomorphisms between X and its complex conjugated twin, $\bar{X} = (M, -J)$. All the anti-automorphisms together with the automorphisms form a group which we denote by Kl and call *Klein group*. Clearly, the Klein group is either a $\mathbb{Z}/2$-extension of the automorphism group Aut, $1 \to \mathrm{Aut} \to \mathrm{Kl} \xrightarrow{\mathrm{kl}} \mathbb{Z}/2 \to 1$, or coincides with it, $\mathrm{Kl} = \mathrm{Aut}$.

Note also that the underlying smooth manifolds of deformation equivalent varieties are necessary diffeomorphic. Moreover, the diffeomorphisms coming from a deformation preserve the complex orientation and the canonical class. Certainly, in the same time, on a given differentiable manifold there may exist isomorphic complex structures with different canonical classes or even with different complex orientations.

Below are two closely related questions from real algebraic geometry, which initiated our study. *Can any complex compact manifold be deformed to a manifold which can be equipped with a real structure or at least to a manifold with an anti-automorphism* (not necessarily of order 2)?

It is clear that the response is in the affirmative as long as the (complex) dimension of the manifold is $\leqslant 1$, i.e., for points and Riemann surfaces: obviously, any Riemann surface can be deformed to a real one. To our knowledge, starting from $\dim_{\mathbb{C}} = 2$ the both questions remained open. We have proved the following theorem.

Theorem 1.1. *In any dimension $\geqslant 2$ there exists a compact complex manifold X which can not be deformed to $\bar{X}$; in particular, X cannot be deformed to a real manifold or to a manifold with an anti-automorphism.*

Explicit examples are given below in Sections 2 and 3. Many of them have a stronger property (Corollary 3.13):

In any dimension $\geqslant 2$ there exists a projective complex manifold X of general type which has no homeomorphisms $h : X \to X$ such that $h^* c_1(X) = -c_1(X)$.

It is worth noticing that in dimension 2 all our examples are rigid, i.e., they have no local nontrivial deformations at all. Moreover, they are strongly rigid, i.e., up to isomorphism and conjugation, they have only one complex structure. Their strong rigidity is not used in the proof of Theorem 1.1 in its dimension 2 part, but we use it in higher dimensions.

The examples mentioned above provide the promised applications, which are: new counter-examples to the Dif = Def problem for complex surfaces (and higher dimensional varieties); first (to our knowledge) counter-examples to the ambient Dif $\Rightarrow$ Def problem for plane cuspidal curves; and first (to our knowledge) examples of symplectic structures ω not equivalent, up to diffeomorphisms and deformations, to their reverse $-\omega$.

Deformation equivalent complex manifolds are orientedly ($\mathcal{C}^\infty$-) diffeomorphic and it is the converse which is called the Dif = Def *problem* (the, probably, first mentioning of this problem is found in [FM]). Obviously, the response is in the affirmative in $\dim \leqslant 1$. For complex surfaces the question remained open for quit a while. The first counter-examples were found by Manetti only a couple of years ago, see [Ma]. The surfaces X and $\bar{X}$ from our Theorem 1.1 are different and, by our opinion, more simple counter-examples. At least, they have the following, useful for our further applications, properties: our

inequivalent complex structures are opposite to each other ($J_0 = -J_1$; in other words, the complex structures are complex conjugated), so that their canonical classes are opposite ($c_1(J_0) = -c_1(J_1)$), and, as in Corollary 3.13, there is no homeomorphism transforming $c_1(J_0)$ in $c_1(J_1)$. Moreover, in our examples the structures J_0 and J_1 are not equivalent even in the class of almost-complex structures, see the concluding Remark in Section 4.

The counter-examples to the Dif = Def problem in dim $\geqslant 3$ were known before. In dimension 3, at least in the category of Kähler manifolds, one can take products of the Riemann sphere with Dolgachev surfaces, see [R]; as it follows from the classification of 6-dimensional manifolds [Z], they are diffeomorphic, and their stability under deformations in the category of Kähler manifolds follows from [F] (note that, as is proven in [R], these counter-examples work as well in the symplectic category). In dimension 4 and in higher even dimensions one can take products of Dolgachev surfaces and argue on divisors of the canonical class which are invariant under deformation, see [FM]; such products are diffeomorphic due to h-cobordance of Dolgachev surfaces, which follows from their homotopy equivalence, see [W]. Note that in our dim $\geqslant 3$ examples the non equivalence of the varieties is not related to the divisibility of the canonical class and, in each dimension, among the examples there are varieties of general type.

It is worth noticing also that the convention to fix the orientation in the statement of the Dif = Def problem is not necessary in the case of complex dimension 2, at least in the category of Kähler surfaces. As it follows from Kotschick's results, see [Ko], in this dimension in the Kähler case the existence of two complex structures with opposite orientations (which is an extremely rare phenomenon in this dimension) implies the existence of an orientation-reversing diffeomorphism, and, thus, the set of equivalence classes does not change if the orientation convention is removed.

Now, let us discuss the second application, which concerns the *ambient* Dif $\Rightarrow$ Def *problem for plane curves*. As in the absolute case, (equisingular) deformation equivalent pairs (P^2, C), where $C \subset P^2$ stands for a plane curve, are diffeomorphic whenever C is a cuspidal curve, and the problem consists in the converse statement (for non cuspidal curves the diffeotopy should be replaced by an isotopy; in fact, in all cases one can speak already on Dif $\Rightarrow$ Iso problem; note that the isotopy coming from an equisingular deformation certainly always can be made smooth outside the singular points, and for some types of singular points, as in the case of ordinary nodes and cusps, the isotopy can be made smooth everywhere). If the curve is nonsingular or if it has only ordinary nodes, the response to the Dif $\Rightarrow$ Def problem is in the affirmative (for the nonsingular case it is trivial, for curves with ordinary nodes it follows from the Severi assertion on the irreducibility of the corresponding spaces). We have produced, using our surfaces from theorem 1.1, an infinite sequence of counter-examples in which the curves are cuspidal. Our curves are irreducible and in each example the diffeomorphism $P^2 \to P^2$ transforming one curve into another is, in fact, the standard complex conjugation. They are obtained as the visible contours of the surfaces from theorem 1.1 (see more details in Section 4).

To conclude the introduction, let us mention also an application to a Dif = Def *problem in symplectic geometry*, which may also be of a certain interest. Consider, on a given

oriented smooth 4-manifold, all the possible symplectic structures compatible with the orientation. Call two structures *equivalent* if one can be obtained from the other by a deformation followed, if necessary, by a diffeomorphism. Our surfaces from theorem 1.1 provide examples where the number of equivalence classes is at least 2; namely, in these examples any Kähler symplectic form ω and its reverse, $-\omega$, are not equivalent to each other; it is due to the fact that there is no diffeomorphism (or even homeomorphism) reversing the canonical class of ω, see section 4 for details. To our knowledge, in the previously known examples of nonequivalent symplectic structures ω_1, ω_2 (see [MT], [LeB], and [Sm]) the assertion states only that $\omega_1 \neq \pm\omega_2$.

It is probably worth to note that in our examples the nonequivalence can not be detected by the divisibility properties of the canonical class or by the Seiberg-Witten invariant: we deal with surfaces of general type, our complex and symplectic structures are opposite, as well as their canonical classes, and thus in our examples the Seiberg-Witten functions coincide.

Remarks. In dimension two, in our counter-examples to the Dif = Def problem the number of equivalence classes is equal to 2, and, moreover, the moduli space is merely a two point set. Additional dimension two examples of deformation inequivalent complex conjugated complex structures are worked out by F.Catanese in [Ca].

2. Principal examples.

Our first, main, example is a suitably chosen ramified Galois $(\mathbb{Z}/5 \times \mathbb{Z}/5)$-covering of $\mathbb{C}\mathrm{p}^2$ (its construction is inspired by some Hirzebruch's one, cf., [H]). The ramification locus is the configuration of 9 lines dual to the inflection points of a nonsingular cubic in the dual plane, $(\mathbb{C}\mathrm{p}^2)^*$. What follows does not depend on a particular choice of the cubic, moreover, even the configuration of the 9 lines does not depend, up to projective transformation, from this choice. In proper chosen homogeneous coordinates it is defined by equation $(x_1^3 - x_2^3)(x_2^3 - x_3^3)(x_3^3 - x_1^3) = 0$ (as it follows, indeed, from the famous elementary geometry Céva theorem; the equation given corresponds to the Ferma cubic $x_1^3 + x_2^3 + x_3^3 = 0$). In these configuration there are 12 triple points and they are the only multiple points.

Birationally, our covering is given by equations

$$z^5 = l_1 l_2 l_3 l_4^3 l_5^3 l_9, \quad w^5 = l_1 l_3 l_4^3 l_6 l_7 l_8^2 l_9,$$

where the lines $l_1 = 0, \dots, l_9 = 0$ are numbered like in the table below following the identification of inflection points with the points of order 3 in the Jacobian of the cubic. In the homogeneous coordinates like above one can take

$$\begin{array}{ccc} l_1 = x_1 - x_3, & l_2 = x_1 - \mu^2 x_3, & l_3 = x_1 + \mu x_3, \\ l_4 = x_2 - \mu^2 x_3, & l_5 = x_2 - x_3, & l_6 = x_2 + \mu x_3 = 0, \\ l_7 = x_1 + \mu x_2, & l_8 = x_1 - \mu^2 x_2, & l_9 = x_1 - x_2. \end{array}$$

where $\mu = e^{\pi i/3}$.

Correspondence between the lines and the points of order 3 in the Jacobian.

l_1	l_4	l_7
l_2	l_5	l_8
l_3	l_6	l_9

Table 1

As any Galois covering, our one is determined by the corresponding homomorphism $\phi = (\phi_1, \phi_2) : H_1(\mathbb{C}\mathrm{p}^2 \setminus \cup l_r) \to \mathbb{Z}/5 \times \mathbb{Z}/5$ (here and further, we do not make difference between the notation of the lines and their linear form expressions). The exponents in the equations of the covering encode the value of ϕ on the standard, dual to the lines, generators $\lambda_r \in H_1(\mathbb{C}\mathrm{p}^2 \setminus \cup l_r)$:

$$\begin{array}{lll} \phi(\lambda_1) = (1,1), & \phi(\lambda_2) = (1,0), & \phi(\lambda_3) = (1,1), \\ \phi(\lambda_4) = (3,3), & \phi(\lambda_5) = (3,0), & \phi(\lambda_6) = (0,1), \\ \phi(\lambda_7) = (0,1), & \phi(\lambda_8) = (0,2), & \phi(\lambda_9) = (1,1), \end{array}$$

Thus defined $(\mathbb{Z}/5 \times \mathbb{Z}/5)$-covering of $\mathbb{C}\mathrm{p}^2$, $X \to \mathbb{C}\mathrm{p}^2$, has isolated singularities, which arise from the 12 triple points of the ramification locus. To get its (minimal) resolution it is sufficient to blow up $\mathbb{C}\mathrm{p}^2$ in the triple points and to take the induced $(\mathbb{Z}/5 \times \mathbb{Z}/5)$-covering of the blown up plane, $\tilde{X} \to \mathbb{C}\mathrm{p}^2(12)$.

Let denote by $D_{ijk} \subset \tilde{X}$ the full transform of the blown up curves $E_{ijk} \subset \mathbb{C}\mathrm{p}^2(12)$ and by $C_r \subset \tilde{X}$ the strict transform of the lines $l_r \subset \mathbb{C}\mathrm{p}^2$. All of them belong to the ramification locus of $\tilde{X} \to \mathbb{C}\mathrm{p}^2(12)$ and they all have the ramification index equal to 5. For the intermediate strict transforms $L_r \subset \mathbb{C}\mathrm{p}^2(12)$ of l_r one has $L_r^2 = -3$. So, elementary pull back calculation shows that

$$C_r^2 = -3, \quad D_{ijk}^2 = -1, \quad \text{and} \quad 3K = 7\sum C_r + 12\sum D_{ijk}. \tag{1}$$

In addition,

$$(C_r, K) = 9 \quad \text{and} \quad (D_{ijk}, K) = 3. \tag{2}$$

Proposition 2.1. *The surface $\tilde{X}$ is a strongly rigid surface of general type with ample canonical divisor. The group $\mathrm{Kl}(\tilde{X})$ coincides with the covering transformations group $G = \mathbb{Z}/5 \times \mathbb{Z}/5$. In particular, there does not exist neither a real structure nor even an anti-holomorphic diffeomorphism on $\tilde{X}$.*

Here and everywhere further, by the *strong* or *Mostow-Siu rigidity* of a compact complex manifold M we mean the following property: whatever are a compact complex Kähler variety Y and a continious map $p : Y \to M$ with nonzero $p_* : H_{2m}(Y;\mathbb{Z}) \to H_{2m}(M;\mathbb{Z})$, $m = \dim_{\mathbb{C}} M$, then p is homotopic to a holomorphic or anti-holomorphic map $Y \to M$. According to Siu's results [S], any non singular compact quotient of an irreducible bounded Hermitian symmetric domain of dimension ≥ 2 has such a property. In addition, as is

known, for any such quotient M one has $H^1(M;\Theta)=0$ (Θ states for the sheaf of holomorphic tangent fields), which implies, in particular, that M is *locally rigid*, i.e., its complex structure has no non trivial local deformations.

Sketch of the proof of Proposition 2.1 (see [KK] for calculations and combinatorial details). A direct calculation shows that $K^2_{\tilde X}=333$ and $e(\tilde X)=111$ (e states for the Euler characteristic), so $\tilde X$ is a so-called Miyaoka-Yau surface; their universal cover is a complex ball, see, for example, [BPV], and, thus, their strong rigidity follows from [S] (all the complex structures on the underlying smooth manifold are Kähler, as it follows, for example, from $b_1=0$; the strong rigidity applied to the identity map implies also the absence of complex structures with opposite orientation). Due to (1), (2) and the Nakai-Moishezon criterion, K is ample.

Since the ramification locus contains the strict transform of at least two different pairs of lines, to prove $\mathrm{Kl}=G$ it is sufficient to show, first, that both the union of strict transforms C_r of the lines and the union of the exceptional divisors D_{ijk} are invariant under Klein transformations of $\tilde X$, and, second, that there is no Klein transformation of $\mathbb{C}\mathrm{p}^2$, different from id, which can be lifted from $\mathbb{C}\mathrm{p}^2$ to $\tilde X$ (*respecting the covering* for short).

Suppose that $\cup C_r$ is not preserved under some $h\in\mathrm{Kl}$, i.e., there is i such that $h(C_i)\not\subset C=\cup C_r$. Then, $(h(C_i),\sum C_r)=a\geqslant 0$, $(h(C_i),\sum D_{ijk})=b\geqslant 0$, and $3K=7\sum C_r+12\sum D_{ijk}$ implies

$$7a+12b=3(h(C_i),K)=27.$$

To get a contradiction it remains to note that the latter equation has no solutions in $a,b\in\mathbb{Z}_+$ (here, counting the intersection numbers we equip all the curves, including $h(C_i)$, with their complex orientation induced from $\tilde X$). The proof of $h(D)\subset D$, $D=\cup D_{ijk}$, is completely similar.

Now, it follows that every $h\in\mathrm{Kl}(\tilde X)$ is lifted from $\mathbb{C}\mathrm{p}^2$ and thus it remains to prove that the only $g\in\mathrm{Kl}(\mathbb{C}\mathrm{p}^2)$ respecting the covering is $g=\mathrm{id}$. Since g respects the covering, it acts on the set of intermediate $\mathbb{Z}/5$-coverings $Y\to\mathbb{C}\mathrm{p}^2$ and their deck transformations. Namely, such a covering with a marked deck transformation is given by an epimorphism $\psi: H_1(\mathbb{C}\mathrm{p}^2\setminus\cup l_r)\to\mathbb{Z}/5$ and g transforms it into $(-1)^{\mathrm{kl}\,g}\psi\circ g_*$ (recall that $\mathrm{kl}\,g$ is 0 if g is holomoprhic and 1 otherwise). The following is the table of rows representing all the epimorphisms $\psi=x\phi_1+y\phi_2: H_1(\mathbb{C}\mathrm{p}^2\setminus\cup l_r)\to\mathbb{Z}/5$ in the base λ_i:

$(1,1,1,3,3,0,0,0,1)$	$(2,2,2,1,1,0,0,0,2)$	$(3,3,3,4,4,0,0,0,3)$	$(4,4,4,2,2,0,0,0,4)$
$(1,0,1,3,0,1,1,2,1)$	$(2,0,2,1,0,2,2,4,2)$	$(3,0,3,4,0,3,3,1,3)$	$(4,0,4,2,0,4,4,3,4)$
$(2,1,2,1,3,1,1,2,2)$	$(4,2,4,2,1,2,2,4,4)$	$(1,3,1,3,4,3,3,1,1)$	$(3,4,3,4,2,4,4,3,3)$
$(3,1,3,4,3,2,2,4,3)$	$(1,2,1,3,1,4,4,3,1)$	$(4,3,4,2,4,1,1,2,4)$	$(2,4,2,1,2,3,3,1,2)$
$(4,1,4,2,3,3,3,1,4)$	$(3,2,3,4,1,1,1,2,3)$	$(2,3,2,1,4,4,4,3,2)$	$(1,4,1,3,2,2,2,4,1)$
$(0,1,0,0,3,4,4,3,0)$	$(0,2,0,0,1,3,3,1,0)$	$(0,3,0,0,4,2,2,4,0)$	$(0,4,0,0,2,1,1,2,0)$.

The elements in a row represent, in fact, the weights of the deck transformation on the components of the ramifications locus. Thus, they should be preserved up to permutation.

In particular, the functions $a \mapsto r_i(a)$ counting the number of coordinates of a row a equal i should be invariant under the action of g.

An easy running over, taking into account, in addition, the incidence relations between the lines l_i, shows that the only possibility for g is to keep invariant each line. Hence, $g = \mathrm{id}$. $\square$

Theorem 1.1 in dimension two is a straightforward consequence of the second statement of Proposition 2.1. In fact, our proof of this part of Proposition 2.1 uses only the local rigidity of $\tilde{X}$, which is an easier result than the strong rigidity. As to the latter, it implies the following property which is crucial for the other applications and for the proof of Theorem 1.1 in higher dimensions.

Lemma 2.2. *The homeotopy group of X (i.e., the group of homotopy classes of homeomorphisms $X \to X$) coincides with the covering transformations group $G = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. In particular, for any homeomorphism $f : \tilde{X} \to \tilde{X}$ one has $f^*[K] = [K], [K] \in H^2(X; \mathbb{Z})$.*

Proof. By strong rigidity, any homeomorphism $\tilde{X} \to \tilde{X}$ is homotopic to an automorphism or to an anti-automorphism. Due to Proposition 2.1, there are no anti-automorphisms and the group of automorphisms coincides with the covering transformations group G. It remains to note that the canonical class is preserved by any automorphism. $\square$

Another important property of $\tilde{X}$ is that its irregularity $q = q(\tilde{X})$ is zero, see Appendix.

There are other surfaces which have similar properties and for which Proposition 2.1 and Lemma 2.2 hold as well. As is proved in [KK], $\tilde{X}$ can be replaced, in particular, by a fake projective plane. Recall that, by definition, a *fake projective plane* is a surface of general type with $p_g = q = 0$ and $K^2 = 9$. Such surfaces do really exist, see [Mu].

3. Higher-dimensional examples

In this section we prove that part of Theorem 1.1 which concerns the dimensions $\geqslant 3$, the case of surfaces is covered by Section 2. For this purpose, we develop several series of examples: the examples from Proposition 3.7 cover even dimensions $\geqslant 4$ and those from Proposition 3.8 odd dimensions $\geqslant 3$. When applying Propositions 3.7 and 3.8 to prove this part of Theorem 1.1, it is sufficient to take as M_1 the surface constructed in Section 2 and as M_2 a fake projective plane.

Another series of examples sufficient for the proof of Theorem 1.1 is given by Propositions 3.9. and 3.11, see Corollary 3.13.

We start from some auxiliary results which, with one exception, we did not find in the literature; they may be of an independent interest. We use them in the proof of Propositions 3.7, 3.8, 3.9 and 3.11.

Lemma 3.1. [LS] *Let X and X_0 be compact complex manifolds contained in a deformation family over irreducible base space. Suppose that the canonical class of X_0 is ample. Then, X is a Moishezon variety, i.e., the transcendence degree of meromorphic function field on X coincides with* $\dim X$. $\square$

Lemma 3.2. *Let V_1 and V_2 be minimal nonsingular surfaces of general type. If there exists a regular map $f : V_1 \to V_2$ of degree $d \geq 1$, then $K^2_{V_1} \geq K^2_{V_2}$. In addition, $K^2_{V_1} = K^2_{V_2}$ if and only if f is an isomorphism (i.e., $d = 1$).*

Proof. According to the pull-back formula, $K_{V_1} = f^*(K_{V_2}) + R$, where $R = \sum r_i R_i$ is the ramification divisor of f, R_i are irreducible components of R, $r_i \geq 0$ (by ramification divisor we mean here the locus of points where the jacobian matrix is not of maximal rank). On the other hand, $K^2_{V_1} > 0$, $K^2_{V_2} > 0$, and $(f^*(K_{V_2}), R) \geq 0$, since, by Bombieri theorem (see, f.e., [BPV]), the linear system $\mid 5K_{V_2} \mid$ has no fixed components. Besides, $(K_{V_1}, R_i) = (f^*(K_{V_2}) + R, R_i) \geq 0$ for each irreducible curve R_i lying on a minimal model. Therefore,

$$\begin{aligned} K^2_{V_1} &= dK^2_{V_2} + 2(f^*(K_{V_2}), R) + R^2 \\ &= dK^2_{V_2} + (f^*(K_{V_2}), R) + (f^*(K_{V_2}) + R, R) \\ &= dK^2_{V_2} + (f^*(K_{V_2}), R) + \sum r_i (f^*(K_{V_2}) + R, R_i) \geq K^2_{V_2} \end{aligned}$$

if $d \geq 1$ and we have the equality iff $d = 1$ and $R = 0$. Indeed, any degree one regular map f is an isomorphism, since V_1 and V_2 are minimal surfaces. □

Lemma 3.3. *Let U be a Moishezon variety and V a projective variety having no rational curves. Then, any meromorphic map $f : U \to V$ is holomorphic.*

Proof. By Hironaka theorem, there is a sequence $\sigma_i : U_i \to U_{i-1}$, $U_0 = U$, $1 \leq i \leq k$, of monoidal transformations with non-singular centers such that $f_k = f \circ \sigma : U_k \to V$ is holomorphic, where $\sigma = \sigma_1 \circ \cdots \circ \sigma_k$. Note that for any $p \in U$ the preimage $\sigma^{-1}(p)$ is rationally connected, i.e., any two distinct points from $\sigma^{-1}(p)$ belong to a connected union of a finite number of rational curves. Therefore, $f_k(\sigma^{-1}(p))$ is a point, since there are no rational curves lying on V. Thus, f_k factors through the holomorphic map f. □

Lemma 3.4. *Let Z be a compact Moishezon manifold and X be a projective one. Then, every holomorphic map f $Z \to X$ inducing an isomorphism in homology, f_* $H_*(Z; \mathbb{Z}) \to H_*(X; \mathbb{Z})$, should be biholomorphic.*

Proof. To show that f is an isomorphism, it is sufficient to check that there is no complex subvariety Y of Z such that $\dim f(Y) < \dim Y$.

First, let us eliminate the case $\dim Y = \dim Z - 1$. If $\dim f(Y) < \dim Y$ then Y is homological to 0, since f_* is an isomorphism. But, since X is a projective variety, one can find a curve $B \subset X$ meeting $f(Y)$ and not lying in $f(Y)$. Therefore, the strict preimage $f^{-1}(B)$ and Y (we assume here that $Y = f^{-1}(f(Y))$) have positive intersection number, which contradicts to $Y \simeq 0$.

Now, consider any Y such that $\dim f(Y) < \dim Y$. By above, we have $\dim Y < \dim Z - 1 = \dim X - 1$. One can find a meromorphic volume form ω on X such that $f(Y)$ does not belong to the support of the divisor (ω). But it is impossible, since in this case in a neighborhood of generic point of Y the form $f^*(\omega)$ would have zero along a submanifold of X of codimension at least two. □

Lemma 3.5. *Let M_1 and M_2 be nonsingular regular surfaces of general type. If M_1, $\bar{M}_1$, M_2, $\bar{M}_2$ are pairwise non isomorphic and contain no rational curves, then:*

(i) *the products $M_1^a \times \bar{M}_1^b \times M_2^c \times \bar{M}_2^d$ with $(a,b,c,d) \in \mathbb{N}^4$ are pairwise non isomorphic; in particular, such a product admits an anti-automorphism only if $a = b$ and $c = d$;*

(ii) *in a deformation $\pi\, X \to B_1$ each fiber manifold $X_z = \pi^{-1}(z), z \in B_1$, is isomorphic to a product $M_1^j \times \bar{M}_1^m \times M_2^k \times \bar{M}_2^n$ with $j = j(z), m = m(z), k = k(z), n = n(z)$, only if the deformation is trivial.*

Proof. Let us first prove (i). Assume that there is an isomorphism

$$h : M_1^{j_1} \times \bar{M}_1^{m_1} \times M_2^{k_1} \times \bar{M}_2^{n_1} \to M_1^{j_2} \times \bar{M}_1^{m_2} \times M_2^{k_2} \times \bar{M}_2^{n_2}.$$

Without loss of generality, we can assume also that $K^2_{M_1} \geq K^2_{M_2}$ and $j_1 \leqslant j_2$.

Denote by $S_l \subset M_1^{j_1} \times \bar{M}_1^{m_1} \times M_2^{k_1} \times \bar{M}_2^{n_1}$, $l \geqslant 1$, a fiber of the projection of the whole product to the partial product taken over all the factors except the one with index l. So, S_l is canonically isomorphic to M_1 for $1 \leq l \leq j_1$, to $\bar{M}_1$ for $j_1 < l \leqslant j_1 + m_1$, to M_2 for $j_1 + m_1 < l \leqslant j_1 + m_1 + k_1$, and to $\bar{M}_2$ for $j_1 + m_1 + k_1 < l$. Denote also by Σ_s the similar fibers of $M_1^{j_2} \times \bar{M}_1^{m_2} \times M_2^{k_2} \times \bar{M}_2^{n_2}$.

Consider, first, the restrictions $p_{s|_l}$ to $h_1(S_l)$ of the projections $p_s : M_1^{j_2} \times \bar{M}_1^{m_2} \times M_2^{k_2} \times \bar{M}_2^{n_2} \to \Sigma_s$. Let us show that for each $l \geq j_1 + 1$ the image $p_{s|_l}(h_1(S_l))$ is a point as soon as $s \leq j_2$. In fact, $p_{s|_l}(h_1(S_l))$ can not be a curve, since there are no rational curves in $M_1, \bar{M}_1, M_2$ or in $\bar{M}_2$, and if $p_{s|_l}(h_1(S_l))$ was a curve of positive genus g, then the irregularity of S_l would be at least g. And since M_1 is not isomorphic to any of $\bar{M}_1, M_2$, and $\bar{M}_2$, Lemma 3.1 can be applied and it implies that $p_{s|_l}(h_1(S_l))$ can not be a surface neither.

Therefore,

$$h_* : H_2(M_1^{j_1} \times \bar{M}_1^{m_1} \times M_2^{k_1} \times \bar{M}_2^{n_1}) \to H_2(M_1^{j_2} \times \bar{M}_1^{m_2} \times M_2^{k_2} \times \bar{M}_2^{n_2})$$

maps the direct Künneth summand $H_2(\bar{M}_1^{m_1}) \oplus H_2(M_2^{k_1}) \oplus H_2(\bar{M}_2^{n_1})$ into the direct Künneth summand $H_2(\bar{M}_1^{m_2}) \oplus H_2(M_2^{k_2}) \oplus H_2(\bar{M}_2^{n_2})$. So, h_* can be an isomorphism only if $j_1 = j_2$. Similarly, $m_1 = m_2$ (one can simply exchange M_1 and $\bar{M}_1$ in the above arguments).

Now, consider a multi-fiber isomorphic to $M_2^{k_1} \times \bar{M}_2^{n_1}$. From what is proved above it follows that such a multi-fiber is mapped by h to a multi-fiber of $M_1^{j_2} \times \bar{M}_1^{m_2} \times M_2^{k_2} \times \bar{M}_2^{n_2}$ isomorphic to $M_2^{k_2} \times \bar{M}_2^{n_2}$. Applying Lemma 3.2 and the arguments as above to such a map $M_2^{k_1} \times \bar{M}_2^{n_1} \to M_2^{k_2} \times \bar{M}_2^{n_2}$, we get, first, that it is an isomorphism and then that $k_1 = k_2$ and $n_1 = n_2$.

To prove (ii), let us consider a deformation $\pi\, X \to B_1$ such that each fiber manifold $X_z = \pi^{-1}(z), z \in B_1$, is isomorphic to a product $M_1^j \times \bar{M}_1^m \times M_2^k \times \bar{M}_2^n$ with $j = j(z), m = m(z), k = k(z), n = n(z)$. Since neither of the factors contains a rational curve, all the manifolds $X_z, z \in B_1$, have ample canonical bundles. Hence, by the Kodaira vanishing theorem, $\dim H^i(X_z, mK_z) = 0$ for any $m > 1$, $i \geqslant 1$, and the Riemann-Roch theorem imples that $\dim H^0(X_z, mK_z)$ is constant with respect to z. By Grauert's continuity

theorem (see [G], Section 7, Theorem 4), it implies that the canonical map $B_1 \to \mathcal{M}$, where $\mathcal{M}$ is the moduli space of projective complex structures on the smooth oriented manifold underlying $X_z, z \in B_1$, is defined and holomorphic. Since, under our hypothesis, the image of this map is countable, it is constant. □

Note, that when it is only question of applying Lemma 3.5 (i) to products of Miayoka-Yau surfaces, this Lemma can be replaced by more traditional arguments based on the De Rham theorem of uniqueness of the decomposition in irreducible factors (see, f.e., [KN]) applied to the universal covering equipped with a Bergmann metric.

Lemma 3.6. *Let M_1 and M_2 be nonsingular regular surfaces of general type and C_1 and C_2 be non-singular curves of genus $g > 0$. If $M_1, \bar{M}_1, M_2, \bar{M}_2$ are pairwise non isomorphic and contain no rational curves, then the products $M_{a_1,b_1,c_1,d_1,C_1} = M_1^{a_1} \times \bar{M}_1^{b_1} \times M_2^{c_1} \times \bar{M}_2^{d_1} \times C_1$ and $M_{a_2,b_2,c_2,d_2,C_2} = M_1^{a_2} \times \bar{M}_1^{b_2} \times M_2^{c_2} \times \bar{M}_2^{d_2} \times C_2$, $(a_i, b_i, c_i, d_i) \in \mathbb{N}^4$, are isomorphic if and only if $(a_1, b_1, c_1, d_1) = (a_2, b_2, c_2, d_2)$ and C_1 is isomorphic to C_2. In particular, such a product admits an anti-automorphism only if $a_1 = b_1$ and $c_1 = d_1$.*

Proof. Since M_1 and M_2 are regular surfaces, the image $\alpha_i(M_{a_i,b_i,c_i,d_i,C_i})$ of the Albanese map $\alpha_i : M_{a_i,b_i,c_i,d_i,C_i} \to \mathrm{Alb}(M_{a_i,b_i,c_i,d_i,C_i})$, $i = 1, 2$, coincides with C_i. To conclude it suffices to apply the universal property of the Albanese map and Lemma 3.2. □

Proposition 3.7. *Let $M_1, \bar{M}_1, M_2$ and $\bar{M}_2$ be pairwise non-isomorphic compact regular nonsingular surfaces of general type satisfying the Mostow-Siu rigidity and having no rational curves. Let m and n be two non-negative integers and $X_{0,0}$ be a product*

$$S_1 \times \cdots \times S_m \times S_{m+1} \times \cdots \times S_{m+n},$$

where each S_i, $1 \le i \le m$, is isomorphic to M_1 and each S_i, $m + 1 \le i \le m + n$, is isomorphic to M_2. Then:

(i) *there are at least $(m + 1)(n + 1)$ distinct deformation classes of complex structures on the underlying oriented smooth manifold $X_{0,0}$; these classes are represented by*

$$X_{m-j,n-k} = M_1^j \times \bar{M}_1^{m-j} \times M_2^k \times \bar{M}_2^{n-k},$$

with $j = 0, \ldots, m$ and $k = 0, \ldots, n$ (here, M_l^j states for the product of j copies of M_l and $\bar{M}_l^s$ for the product of s copies of $\bar{M}_l$);

(ii) *there are no anti-automorphisms on $X_{p,q}$ except the case $m = 2p$, $n = 2q$; for other values of p, q the deformation class of $X_{p,q}$ is not invariant under reversing of complex structure, $\bar{X}_{p,q} = X_{m-p,m-q}$.*

(iii) *a complex manifold is deformation equivalent to $X_{p,q}$ if and only if it is isomorphic to $X_{p,q}$.*

Note that, as it follows from the proof below, the complex structures represented by $X_{p,q}$ are the only Moishezon complex structures on the smooth oriented manifold underlying $X_{0,0}$. Up to our knowledge, the existence of non Moishezon complex structures on this manifold is an open question. If all the complex structures on $X_{0,0}$ are Moishezon, the number of their deformation classes is exactly $(m + 1)(n + 1)$.

Proof. Clearly, (i) and (ii) are straightforward consequences of (iii). So, let us prove the part (iii) of the statement.

As it follows from Lemma 3.1 and Lemma 3.5 (ii), it is sufficient to show that any Moishezon complex structure on the underlying $X_{0,0}$ smooth manifold provides a complex manifold isomorphic to one of $X_{p,q}$.

So, consider a Moishezon complex compact variety X having the same underlying smooth oriented manifold as $X_{0,0}$. Pick a diffeomorphism between X and $X_{0,0}$ and consider the corresponding to it projections $p_i : X \to S_i$, $i = 1, \dots, m+n$, (we use here notations from Lemma 3.5). By Moishezon Theorem, there is a sequence $\sigma_i : Z_i \to Z_{i-1}$, $Z_0 = X$, $1 \le i \le r$, of monoidal transformations with non-singular centers such that $Z = Z_r$ is a projective variety. Denote by σ their composition. Every projection and every monoidal transformation induce an epimorphism in homology. So, according to the Mostow-Siu rigidity hypothesis, each $p_i \circ \sigma$, $i = 1, \dots, m+n$, is homotopic to a map $\widetilde{p}_i : Z \to S_i$ which is either holomorphic or anti-holomorphic. Let j be the number of holomorphic maps $\widetilde{p}_i$ for $i \le m$ and k be the number of holomorphic maps $\widetilde{p}_i$ for $i > m$. Thus, after a suitable renumbering of $\widetilde{p}_i$, we get a meromorphic map $f = \widetilde{p}_1 \circ \sigma^{-1} \times \dots \times \widetilde{p}_{m+n} \circ \sigma^{-1}\ X \to M_1^j \times \bar{M}_1^{m-j} \times M_2^k \times \bar{M}_2^{n-k}$. By Lemma 3.3 it is holomorphic, and, since σ induces an epimorphism in homology, from the construction of f it follows that f induces an isomorphism in homology. Now, Lemma 3.4 applies and it shows that X is isomorphic to $X_{j,k}$. □

Proposition 3.8. *Let M_1 and M_2 be as in Proposition 3.7, C_0 be a curve of genus $g > 1$, and $Y_{0,0}$ be a product $M_1^m \times M_2^n \times C_0$. Then:*

(i) *there are at least $(m+1)(n+1)$ distinct deformation classes of complex structures on the underlying oriented smooth manifold $Y_{0,0}$; these deformation classes are represented by*

$$Y_{m-j,n-k} = M_1^j \times \bar{M}_1^{m-j} \times M_2^k \times \bar{M}_2^{n-k} \times C_0,$$

$j = 0, \dots, m$ and $k = 0, \dots, n$;

(ii) *there are no anti-automorphisms on $Y_{p,q}$ except the cases when $m = 2p$, $n = 2q$ and C_0 admits an anti-automorphism; for all the other values of p, q the deformation class of $Y_{p,q}$ is not invariant under reversing of complex structure, $\bar{Y}_{p,q} = Y_{m-p,m-q}$.*

Here, the dimension of complex manifolds $Y_{p,q}$ is odd, and, thus, reversing the complex structure in 3.8(ii) we change the complex orientation. However, since $Y_{p,q}$ admit orientation reversing diffeomorphisms, there are as much deformation classes of complex structures on the underlying smooth manifold as with fixed or with any orientation.

Similarly to the situation of Proposition 3.7, if all the complex structures on $Y_{0,0}$ are Moishezon, the number of their deformation classes is exactly $(m+1)(n+1)$.

Proof. Let start of the first part of Proposition. As in the proof of Proposition 3.7, to prove the first part we should check that the complex structure which can be deformed to a Moishezon complex structure is isomorphic to some $M_1^j \times \bar{M}_1^{m-j} \times M_2^k \times \bar{M}_2^{n-k} \times C$,

where C is a curve of genus g, and then we should prove that a complex structure which is deformation equivalent to one of $Y_{p,q}$ is isomorphic to the complex structure on $M_1^p \times \bar{M}_1^{m-p} \times M_2^q \times \bar{M}_2^{n-q} \times C$ for some curve C.

First, consider a Moishezon variety Y having the same underlying smooth oriented manifold as $Y_{0,0}$. Consider the Albanese map $\alpha : Y \to \mathrm{Alb}(Y)$. The Albanese map $\alpha_{0,0} : Y_{0,0} \to \alpha_{0,0}(Y_{0,0}) \subset \mathrm{Alb}(Y_{0,0})$ coincides with the last factor projection $p_{m+n+1} : Y_{0,0} = M_1^m \times M_2^n \times C_0 \to C_0$. Since the irregularity $q(M_1^m \times M_2^n)$ is zero, the subring of $H^*(Y_{0,0};\mathbb{Z})$ generated by $H^1(Y_{0,0};\mathbb{Z})$ coincides with $\alpha_{0,0}^*(H^*(C_0;\mathbb{Z}))$. In particular, $\bigwedge^2 H^1(Y_{0,0},\mathbb{Z}) \subset H^2(Y_{0,0},\mathbb{Z})$ is a one-dimensional subspace and $\bigwedge^i H^1(Y_{0,0},\mathbb{Z}) = \{0\}$ for $i > 2$. Therefore, the image $\alpha(Y)$ is a curve C of genus g and $\alpha^*(H^*(\mathrm{Alb}(Y),\mathbb{Z}))$ also coincides with the subring generated by $H^1(Y,\mathbb{Z}) = H^1(Y_{0,0},\mathbb{Z})$.

Pick a diffeomorphism between Y and $Y_{0,0}$ and consider the corresponding to it projections $p_i : Y \to S_i$, $i = 1,\dots,m+n$, (we use here notations from Proposition 3.5), which are C^∞-submersions, and $p_{m+n+1} : Y \to C$. By Moisheson Theorem, there is a sequence $\sigma_i : Z_i \to Z_{i-1}$, $Z_0 = Y$, $1 \le i \le k$, of monoidal transformations with non-singular centers such that $Z = Z_k$ is a projective variety. Denote by σ their composition. Every projection and every monoidal transformation induce an epimorphism in homology. So, according to the Mostow-Siu rigidity hypothesis, each $p_i \circ \sigma$, $i = 1,\dots,m+n$, is homotopic to a map $\widetilde{p}_i : Z \to S_i$ which is either holomorphic or anti-holomorphic. Let j be the number of holomorphic maps $\widetilde{p}_i$ for $i \le m$ and k be the number of holomorphic maps $\widetilde{p}_i$ for $i > m$. Thus, we get a meromorphic map $f = \widetilde{p}_1 \circ \sigma^{-1} \times \dots \times \widetilde{p}_{m+n} \circ \sigma^{-1} \times \alpha : Y \to M_1^j \times \bar{M}_1^{m-j} \times M_2^k \times \bar{M}_2^{n-k} \times \alpha(Y)$. By Lemma 3.3 it is holomorphic, and, since σ induces an epimorphism in homology, from the construction of f it follows that f induces an isomorphism in homology. Now, Lemma 3.4 applies and it shows that Y is isomorphic to $Y_{j,k}$.

To finish the proof of the first part of Proposition, let us consider a deformation $X \to B_1$ of $Y_{m-j,n-k}$. As it is already proved, for any $z \in B_1$ the manifold X_z is isomorphic to $M_1^j \times \bar{M}_1^{m-j} \times M_2^k \times \bar{M}_2^{n-k} \times C_z$ with some $j = j(z), k = k(z)$. Hence, as it follows, for example, from Grauert's continuity theorem (see [G]) the associated Albanese varieties $\mathrm{Alb}X_z$ of $X_z, z \in B_1$, form a deformation $W \to B_1$. The Albanese map $X \to W$ is of constant rank equal to 2 (one unit comes from the images $C_z \subset \mathrm{Alb}X_z$ of the Albanese map and another one from B_1, since as any deformation the composite map $X \to W \to B_1$ is a submersion). It is proper and defines a deformation of $M_1^j \times \bar{M}_1^{m-j} \times M_2^k \times \bar{M}_2^{n-k}$. By Proposition 3.7 (iii), all elements of such a deformation are isomorphic to each other.

The second part of Proposition is a straightforward consequence of the first part and Lemma 3.6. □

Remark. If one restricts himself to Kähler (or Moishezon) manifolds and there deformations constituted only of Kähler (or Moishezon) manifolds, then using the same arguments as in the proof of Propositions 3.7 and 3.8 one can prove that $X_{p,q} \times T_1$ and $X_{k,l} \times T_2$, where T_1, T_2 are tori of equal dimensions and $X_{p,q}, X_{k,l}$ are as in Proposition 3.7, are equivalent if and only if $p = q$ and $k = l$. In particular, $X_{p,q} \times T_1$ and $X_{p,q} \times T_2$

are not equivalent except the case $2p = m, 2q = n$ (for the definition of m and n see Proposition 3.7).

Proposition 3.9. *Let M be a fake projective plane or the surface constructed in section 2, and let X be a Kähler manifold of Kodaira dimension ≤ 1 with $c_1(X) \in H^2(X, \mathbb{Z})$ divisible by some integer $p > 3, p \neq 6$ (in particular, $c_1(X)$ can be zero). Then, there is no homeomorphism $h : Y \to Y$, $Y = M \times X$, such that $h^*(c_1(Y)) = -c_1(Y)$.*

Proof. Assume that there is such a homeomorphism h. Since M is a regular surface, the Künneth decomposition of $H^2(Y; \mathbb{Z})$ takes the form $H^2(Y, \mathbb{Z}) = H^2(M, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$. Let $h^*(c_1(M)) = \omega_1 + \nu_1$ and $h^*(c_1(X)) = \omega_2 + \nu_2$, where $\omega_i \in H^2(M, \mathbb{Z})$ and $\nu_i \in H^2(X, \mathbb{Z})$. Then, according to our assumption, $\omega_1 + \omega_2 = -c_1(M)$ and $\nu_1 + \nu_2 = -c_1(X)$.

Denote by $p_1 : Y \to M$ the projection to the first factor, by $p_2 : Y \to X$ the projection to the second one, by $i_2 : X \to Y$ the canonical isomorphism with one of the fibers of p_1, and by $i_1 : M \to Y$ the canonical isomorphism with one of the fibers of p_2. Consider the composition $p_1 \circ h$. By Mostow-Siu rigidity, $p_1 \circ h$ is homotopic to $\widetilde{p}_1 : Y \to M$ which is either holomorphic or anti-holomorphic dominant map.

Since M is regular and has no rational curves, its image $(\tilde{p}_1 \circ i_1)M \subset M$ can not be a curve, and it can not be a point, since in this case the restriction of $\widetilde{p}_1$ to $i_2(X)$ should be a dominant map, but the latter is impossible because of Kod dim$X <$ Kod dim$M$. Therefore, Lemma 3.2 applies and it shows that $\widetilde{p}_1 \circ i_1$ is a holomorphic or anti-holomorphic isomorphism. Since M has no anti-automorphisms, $\widetilde{p}_1 \circ i_1$ is a holomorphic map. Thus, $\omega_1 = (p_1 \circ h \circ i_1)^* c_1(M) = (\tilde{p}_1 \circ i_1)^* c_1(M) = c_1(M)$. It implies $(p_1 \circ h \circ i_2)^* c_1(X) = \omega_2 = -2c_1(M)$, and to complete the proof it remains to note that $2c_1(M)$ is not divisible by $p > 3, p \neq 6$, which follows from $c_1^2(M) = 9$ in the case of fake projective planes and from $c_1^2(M) = 333$ in the case of the surface constructed in Section 2. □

Corollary 3.10. *Let Y be a variety as in Proposition 3.9. Then, it is not deformation equivalent to its conjugate and, in particular, can not be deformed to a variety with anti-automorphisms.* □

Remark. In the proof of Proposition 3.9 the hypothesis on Kodaira dimension of X is used only to exclude existence of dominant maps from X to M.

One more series of manifolds whose canonical class and its inverse are topologically distinct is given by the following proposition.

Proposition 3.11. *Let M_1 be the surface constructed in section 2, M_2 a fake projective plane, X as in Proposition 3.9, and $Y = M_1 \times M_2^a \times X$. Then there is no homeomorphism $h : Y \to Y$ such that $h^* c_1(Y) = -c_1(Y)$.*

The proof of Proposition 3.11 is essentially the same as the proof of Proposition 3.9. The principal new element with respect to that proof (except the analysis of the first component of the homeomorphism h, where we use arguments like in the proof of Proposition 3.7 in order to prove that the restriction to M_1 of the corresponding $\widetilde{p}_1$ is an isomorphism) is the following lemma applied to $Z = M_2$.

Lemma 3.12. *Let Z be a surface of general type such that for any holomorphic map $f : Z \to Z$ either* $\operatorname{codim} f(Z) \geq 2$ *or f is a biholomorphic. Then, the same is true for holomorphic maps from Z^a to Z^a for any $a \geqslant 1$.*

Proof. Let $f : Z^a \to Z^a$ be a holomorphic map. Denote by $p : Z^a \to Z^{a-1}$ one of the canonical projections and by $i : Z \to Z^a$ one of its fibers. If the projections of $(f \circ i)Z$ to each of the factors are all not dominant, the image of this fiber is a point. Then, the image of all the parallel fibers is a point, which implies $\operatorname{codim} f(Z^a) \geq 2$.

Otherwise, one of these projections provides an isomorphism $Z \to Z$. Since the automorphism group of Z is discrete, the map f has, up to permutation of the factors in the source and the target, the following triangle form $f(x, p) = (gx, h(x, p))$, where g is an automorphism of Z and h is a holomorphic map $Z^a \to Z^{a-1}$. By induction, we may assume that for some, and hence for any, $x \in Z$ the map $Z^{a-1} \to Z^{a-1}$ given by $p \mapsto h(x, p)$ either is biholomorphic or has a codimension ≥ 2 image. The above triangle form of f implies that it has the same property. □

Corollary 3.13. *For any $n \geq 2$ there is a projective n-dimensional manifold Y of general type which has no homeomorphisms h with $h^*c_1(Y) = -c_1(Y)$.*

Proof. Apply Proposition 3.11 to a curve X of genus $g > 4$ in the odd-dimensional case and to a point X in the even-dimensional case. □

In fact, the same arguments which were used in the proof of Proposition 3.7 (or more traditional arguments base on the De Rham theorem of uniqueness of the decomposition in irreducible factors) allow to deduce from Lemma 2.3 a complete description of the homeotopy group $\mathcal{H}(Y)$ in the case $Y = M^n$, where M is the surface constructed in Section 2.

Proposition 3.14. *Let M be the surface constructed in Section 2. Then, the group $\mathcal{H}(M^n) = (\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})^n \rtimes S_n$ is the semi-direct product, with standart action of the symmetric group S_n on the factors $(\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})$ of the direct product $(\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})^n$.* □

4. Applications

A. Non connected moduli spaces. Here, we discuss how Theorem 1.1 and our solution of the Dif = Def problem can be translated in the language of moduli spaces.

In most of our examples the varieties are of general type and, moreover, all varieties deformation equivalent to them are projective varieties of general type isomorphic to products of projective surfaces of general type and, possiblly, some curve of genus > 1 (see Propositions 2.1, 3.7, and 3.8). So, here we may restrict ourselves to the case when the *moduli space $\mathcal{M}_{\mathrm{Dif}}$ of complex structures* on a given smooth orientable manifold M is well defined as a complex space.

In fact, we fix the orientation of M only if the complex dimension of M is even. Then, obviously, in even complex dimensions the Dif = Def problem becomes the question of connectedness of $\mathcal{M}_{\mathrm{Dif}}$. In addition, if the complex dimension is even, the involution

conj $:X = (M, J) \mapsto \bar{X} = (M, -J)$ acts on $\mathcal{M}_{\mathrm{Dif}}$ and defines a canonical real structure on it.

Since in all our odd-dimensional examples M has orientation reversing diffeomorphisms, the Dif = Def problem as it is stated in Introduction remains the question of connectedness of $\mathcal{M}_{\mathrm{Dif}}$ even without introducing orientations in the definition of the moduli space. On the other hand, it allows still to consider the complex conjugation acting on M_{Dif} by conj $:X = (M, J) \mapsto \bar{X} = (M, -J)$.

To reinterpret Theorem 1.1 in its full content is more complicated, in general. Certainly, if X can be equipped with a real structure then it defines a real point of $\mathcal{M}_{\mathrm{Dif}}$, i.e., a fixed point of conj. But the reverse is not true, in general, since real points of $\mathcal{M}_{\mathrm{Dif}}$ are given also by complex manifolds having antiholomorphic automorphisms of any (even) order. The simplest examples are given by the Shimura curves [Shi] which are double coverings of $\mathbb{C}\mathrm{p}^1$ ramified in $4m + 2 \geqslant 6$ generic points invariant under the real structure of $\mathbb{C}\mathrm{p}^1$ acting without fixed points. The real structure is lifted to an anti-automorphism conj of order 4, and for generic points the curve has no other anti-automorphisms than conj and the other lift, which is its composition with a deck transformation and which is also of order 4. In fact, in these examples the automorphism group Aut is $\mathbb{Z}/2$ and the Klein group Kl, which includes all the anti-automorphisms and all the automorphisms, is $\mathbb{Z}/4$.

By contrary, the absence of deformation equivalence between X and $\bar{X}$ has a simple meaning in terms of $\mathcal{M}_{\mathrm{Dif}}$: it means that $\mathcal{M}_{\mathrm{Dif}}$ contains two connected components interchanged by conj. Thus, the following statement is a straightforward corollary of Theorem 1.1 and Propositions 3.7, 3.8. (Let us recall our convention: in this theorem, as well as above, we fix the orientation only if the complex dimension of the manifold is even.)

Theorem 4.1. *In any even real dimension $2n \geqslant 4$ there exist an oriented smooth compact manifold for which the moduli space of complex structures is disconnected and has at least $([\frac{n}{4}] + 1)([\frac{n}{2}] - [\frac{n}{4}] + 1) = N$ connected components. At most one of the components is invariant under complex conjugation.*

This statement can be refined a bit. For example, in the case of surfaces, the moduli space consists of two connected components and the complex structures from different components have opposite canonical classes.

B. Isotopy classes of cuspidal curves.

The theorem below shows that diffeomorphic plane cuspidal curves can be not equivalent under equisingular deformations. In fact, in our examples the curves are not even isotopic. Thus, the following question remains open: does existence of an isotopy imply existence of an equisingular deformation.

Theorem 4.2. *There are two infinite sequences, $\{C_{m,1}\}$ and $\{C_{m,2}\}$, of plane irreducible cuspidal curves of degree $\deg(C_{m,1}) = \deg(C_{m,2}) \to \infty$, such that the pairs $(\mathbb{C}\mathrm{p}^2, C_{m,1})$ and $(\mathbb{C}\mathrm{p}^2, C_{m,2})$ are diffeomorphic, but $C_{m,1}$ and $C_{m,2}$ are not isotopic and, in particular, they can not be connected by an equisingular deformation.*

To get Theorem 4.2 it is sufficient to apply Proposition 4.3 stated below to any of the surfaces given in Section 2.

Proposition 4.3. *Let X be a surface of general type with ample canonical class K. Suppose that there is no homeomorphism h of X such that $h^*[K] = -[K]$, $[K] \in H^2(X;\mathbb{Q})$. Then the moduli space of X consists of at least 2 connected components corresponding to X and $\overline{X}$ (the bar states for reversing of complex structure, $J \mapsto -J$), and for any $m \geq 5$ these two connected components are distinguished by the isotopy types of the branch curves of generic coverings $f_m : X \to \mathbb{C}\mathrm{p}^2$ and $\overline{f}_m : \overline{X} \to \mathbb{C}\mathrm{p}^2$ given by mK and $m\overline{K}$, respectively. In particular, the branch curves can not be connected by an equisingular deformation.*

When Proposition 4.3 is applied to the surface $X = \tilde{X}$ constructed in Section 2 by means of the Céva configuration, one finds (see general formulae in [K] , page 1155)

(1) $\deg f_m = 333\, m^2$;
(2) $C_{m,1}$ is a plane cuspidal curve of $\deg C_{m,1} = 333\, m(3m+1)$;
(3) the geometric genus g_m of $C_{m,1}$ equals $g_m = 333\,(3m+2)(3m+1)/2 + 1$;
(4) $C_{m,1}$ has $c = 111\,(36m^2 + 27m + 5)$ ordinary cusps.

If it is a fake projective plane which is taken as X, then

$$\deg f_m = 9\, m^2, \quad \deg C_{m,1} = 9\, m(3m+1),$$

$$g_m = \frac{9}{2}\,(3m+2)(3m+1) + 1, \quad c = 3\,(36m^2 + 27m + 5).$$

Sketch of the proof of Proposition 4.3 (see [KK2] for more details). Since K is ample, from Bombieri theorem it follows that the map $X \to \mathbb{C}p^{r_m}$, $r_m = \dim H^0(X, mK) - 1$, given by mK is an imbedding if $m \geqslant 5$. Let $m \geqslant 5$ and denote by X_m the image of X under the imbedding in $\mathbb{C}p^{r_m}$ given by mK, by $pr_m : \mathbb{C}p^{r_m} \to \mathbb{C}\mathrm{p}^2$ a linear projection generic with respect to X_m, by $f_m = pr_m|_{X_m}$ the restriction of pr_m to X_m, and by $C_{m,1} \subset \mathbb{C}\mathrm{p}^2$ the branch curve of f_m. As soon as we identify X_m and $\overline{X}_m$ as sets, the composition $\overline{f}_m = c \circ f_m : \overline{X}_m \to \mathbb{C}\mathrm{p}^2$ of f_m with the standard complex conjugation $c : \mathbb{C}\mathrm{p}^2 \to \mathbb{C}\mathrm{p}^2$ is a holomorphic generic covering with branch curve $C_{m,2} = c(C_{m,1})$. By construction, we have

$$\overline{f}_m^*(\Lambda) = -f_m^*(\Lambda) = -m[K]\,, \tag{3}$$

where $\Lambda \in H^2(\mathbb{C}\mathrm{p}^2, \mathbb{Q})$ is the class of the projective line in $\mathbb{C}\mathrm{p}^2$.

The set of generic coverings f of $\mathbb{C}\mathrm{p}^2$ branched along a cuspidal curve C is in one-to-one correspondence with the set of epimorphisms from the fundamental group $\pi_1(\mathbb{C}\mathrm{p}^2 \setminus C)$ to the symmetric groups $S_{\deg f}$ (up to inner automorphisms) satisfying some additional properties (see [K]). By Theorem 3 in [K], for $C_{m,1}$ (respectively, $C_{m,2}$) there exists only one such an epimorphism $\varphi_m : \pi_1(\mathbb{C}\mathrm{p}^2 \setminus C_{m,1}) \to S_{\deg f_m}$ (respectively, $\overline{\varphi}_m$). Thus, if there exists an isotopy $F_t : \mathbb{C}\mathrm{p}^2 \to \mathbb{C}\mathrm{p}^2$ such that $F_0 = \mathrm{id}$ and $F_1(C_{m,1}) = C_{m,2}$, then the epimorphism

$$\varphi_{m,t} : \pi_1(\mathbb{C}\mathrm{p}^2 \times [0,1] \setminus \{(F_t(C_{m,1}), t)\}) \to S_{\deg f_m}$$

defines a locally trivial, with respect to t, family of generic coverings $f_{m,t} : Y \to \mathbb{C}\mathrm{p}^2, t \in [0,1]$, which provides in its turn a homeomorphism h of X such that $h^*[K] = -[K] \in H_*(X;\mathbb{Q})$, as it follows from (3). Thus, the isotopy F_t can not exist. □

C. Two inequivalent classes of symplectic structures. Here, we treat the equivalence relation defined in Introduction.

Let Y be a Kähler surface satisfying Lemma 2.2, or, more generaly, a Kähler manifold Y which has no diffeomorphisms $f : Y \to Y$ with $f^*[K] = -[K], [K] \in H^2(Y;\mathbb{Q})$. One can take, for example, the surfaces given in Section 2 or any manifold Y like in Propositions 3.9 or 3.11. Denote by ω the symplectic structure on Y which is the imaginary part of its Kähler structure.

Proposition 4.4. *The symplectic structures ω and $-\omega$ are not equivalent to each other. In particular, they are not symplectomorphic.*

Proof. The class $[K]$ is the canonical class of ω, while $-[K]$ is the canonical class of $-\omega$. So, the result follows from the invariance of the canonical class under deformations and the absence of diffeomorphisms transforming $[K]$ in $-[K]$. □

Remark. The above argument can be applied to deformations in the class of almost-complex structures. It shows that for any manifold $Y = (Y, J)$ as above the structures J and $-J$ are not equivalent, where similar to equivalence of symplectic structures, we call two almost-complex structures *equivalent* if they can be obtained one from another by deformation followed, if necessary, by a diffeomorphism.

The following proposition shows that between manifolds studied in Section 3 there are more inequivalent symplectic and almost-complex structures. Its proof repeats word by word the proof of Proposition 3.7 (iii).

Proposition 4.5. *Let Y be the common underlying oriented smooth manifold of complex manifolds $X_{p,q}$ from Proposition 3.7. Then,*

(i) *there are no homeomorphisms $h : Y \to Y$ transforming $c_1(X_{p,q})$ in $c_1(X_{s,r})$ except if $p = s$ and $q = r$;*
(ii) *symplectic (respectively, almost-complex) structures on $X_{p,q}$ and $X_{r,s}$ are non equivalent except if $p = s$ and $q = r$.* □

Appendix. Geometric genus calculation

The aim of this Appendix is to show that the irregularity of the surface constructed in Section 2 equals zero. We start from explaining a general algorithm we use for this calculation and only after that apply it to the particular case in question. In fact, we calculate instead the geometric genus, which is sufficient, since their difference is a topological invariant, due to Noether's formula. In the calculation we use permanently the invariance of the geometric genus under birational transformations, which allows us at each step to use that nonsingular birational model which is more convenient for calculation.

The result is by no means new. It is contained, for example, in the results stated in [I]. For completeness of the proof of Theorem 4.1, we present a straightforward calculation and with as much details as possible.

Algorithm of reduction to cyclic coverings. Let $f : X_G \to \mathbb{C}\mathrm{p}^2$, where X_G is supposed to be a normal surface, be a Galois covering with abelian Galois group $G = (\mathbb{Z}/p\mathbb{Z})^m$, where p is a prime number, and branched along curves $B_1, \dots, B_t \subset \mathbb{C}\mathrm{p}^2$. Such a covering is determined by an epimorphism $\phi : H_1(\mathbb{C}\mathrm{p}^2 \setminus \cup B_i) \to G$. Write it in a form

$$\phi(\gamma_i) = k_{1,i}\alpha_1 + \dots + k_{m,i}\alpha_m, \qquad i = 1, \dots, t,$$

where α_j are standard generators of $G = \oplus(\mathbb{Z}/p\mathbb{Z})^m$, γ_i are standard generators of $H_1(\mathbb{C}\mathrm{p}^2 \setminus \cup B_i)$ dual to B_i and $k_{i,j} \in \mathbb{Z}/\mathbb{Z}p$, $0 \le k_{i,j} < p$, are coordinates of $\phi(\gamma_i)$ with respect to α_j. In this notation, X_G is the normalization of the projective closure of the affine surface $Y_G \subset \mathbb{C}^{m+2}$ given by

$$z_j^p = \prod_{i=1}^{t} h_i^{k_{j,i}}(x, y), \qquad , j = 1, \dots, m,$$

where $h_i(x, y)$ are the equations of B_i in some chart $\mathbb{C}^2 \subset \mathbb{C}\mathrm{p}^2$.

Let $\widetilde{X}_G$ be the minimal desingularization of X_G. As is known, it exists, it is unique and the action of G lifts, in an unique way, to a regular action on $\widetilde{X}_G$.

Consider the action of G on the space $H^0(\widetilde{X}_G, \Omega^2_{\widetilde{X}_G})$ of regular 2-forms. It provides a decomposition

$$H^0(\widetilde{X}_G, \Omega^2_{\widetilde{X}_G}) = \oplus H_{(s_1,\dots,s_m)}$$

into the direct sum of eighenspaces $H_{(s_1,\dots,s_m)}$, where

$$\omega \in H_{(s_1,\dots,s_m)} \quad \text{iff} \quad \alpha_j(\omega) = e^{2\pi s_j i/p} \cdot \omega \quad \text{for any } j = 1, \dots, m.$$

Let $H \subset G$ be a subgroup and $G_1 = G/H$. We have the following commutative diagram

$$\begin{array}{ccc} X_G & \xrightarrow[h]{} & X_{G_1} \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_1}\downarrow \\ \mathbb{C}\mathrm{p}^2 & \xrightarrow[id]{} & \mathbb{C}\mathrm{p}^2 \end{array}$$

where $f_1 : X_{G_1} \to \mathbb{C}\mathrm{p}^2$ is the Galois covering corresponding to $\phi_1 = i \circ \phi$ with $i : G \to G_1 = G/H$ being the canonical epimorphism. The map h induces a rational dominant (i.e., whose image is everywhere dense) map $\widetilde{X}_G \to \widetilde{X}_{G_1}$, and the latter, as any rational dominant map between nonsingular varieties, transforms holomorphic p-forms in holomoprhic p-forms. Thus, the subspace $h^*(H^0(\widetilde{X}_{G_1}, \Omega^2_{\widetilde{X}_{G_1}})) \subset H^0(\widetilde{X}_G, \Omega^2_{\widetilde{X}_G})$ is well defined, and it coincides with the subspace $H^0(\widetilde{X}_G, \Omega^2_{\widetilde{X}_G})^H \subset H^0(\widetilde{X}_G, \Omega^2_{\widetilde{X}_G})$ of the elements invariant under the action of H. On the other hand, an eighenspace $H_{(s_1,\dots,s_m)}$ is

invariant under $x_1\alpha_1+\dots x_m\alpha_m$ if and only if $x_1s_1+\dots+x_ms_m=0\,(p)$. Hence, the sum $\oplus H_{(\theta s_1,\dots,\theta s_m)}$ taken over $\theta\in\mathbb{Z}/p\mathbb{Z}$ coincides with $H^0(\widetilde{X}_G,\Omega^2_{\widetilde{X}_G})^H$, where

$$H=\{\,x_1\alpha_1+\dots+x_m\alpha_m\mid x_1s_1+\dots x_ms_m=0\,(p)\}.$$

So, this sum is isomorphic to $H^0(\widetilde{X}_{G/H},\Omega^2_{\widetilde{X}_{G/H}})$. These considerations give rise to the following result.

Proposition A.1. *The geometric genus $p_g(\widetilde{X}_G)=\dim H^0(\widetilde{X}_G,\Omega^2_{\widetilde{X}_G})$ of $\widetilde{X}_G$ is equal to*

$$p_g(\widetilde{X}_G)=\sum_H p_g(\widetilde{X}_{G/H}),$$

where the sum is taken over all subgroups H of G of $rk\,H=rk\,G-1$.

Cyclic coverings. Now, let $G=\mathbb{Z}/\mathbb{Z}p$ be a cyclic group. To compute $p_g(\widetilde{X}_G)$, let us choose homogeneous coordinates $(x_0:x_1:x_2)$ in $\mathbb{C}\mathrm{p}^2$ such that the line $x_0=0$ does not belong to the branch locus of $f:X_G\to\mathbb{C}\mathrm{p}^2$. As above, X_G is the normalization of the projective closure of the hypersurface in $\mathbb{C}^3$ given by equation

$$z^p=h(x,y),$$

where $x=\frac{x_1}{x_0}$, $y=\frac{x_2}{x_0}$,

$$h(x,y)=\prod_{i=1}^t h_i^{k_i}(x,y),$$

$h_i(x,y)$ are irreducible equations in $\mathbb{C}^2\subset\mathbb{C}\mathrm{p}^2$ of the curves B_i constituting the branch locus, and $0<k_i<p$. Note that the degree

$$\deg h(x,y)=\sum k_i\deg h_i(x,y)=np$$

is divisible by p, since the line $x_0=0$ does not belong to the branch locus.

It is easy to see that over the chart $x_1\neq 0$ the variety X_G coincides with the normalization of the hypersurface in $\mathbb{C}^3$ given by equation

$$w^p=\widetilde{h}(u,v),$$

where $u=\frac{1}{x}$, $v=\frac{y}{x}$, $\widetilde{h}(u,v)=u^{np}h(\frac{1}{u},\frac{v}{u})$, and $w=zu^n$.

Regularity condition over a generic point of the base. Consider

$$\omega\in H^0(X_G\setminus\mathrm{Sing}X_G,\Omega^2_{X_G\setminus\mathrm{Sing}X_G})$$

and find a criterion of its regularity outside the ramification and singular loci.

Over the chart $x_0\neq 0$ the form ω can be written as

$$\omega=(\sum_{j=0}^{p-1}z^jg_j(x,y))\frac{dx\wedge dy}{z^{p-1}},\tag{4}$$

where $g_j(x,y)$ are rational functions in x and y. The form

$$\frac{dx \wedge dy}{z^{p-1}}$$

has neither poles or zeros outside of the preimage of the branch locus. Therefore, ω is regular at such a point iff all $g_j(x,y)$ are regular at each point $(a,b) \notin \sum B_i$.

In fact, if some $g_j(x,y)$ is not regular at (a,b), then the sum

$$\sum_{j=0}^{p-1} z^j g_j(x,y)$$

can be written as

$$\frac{\sum_{j=0}^{p-1} z^j P_j(x,y)}{P_p(x,y)},$$

where $P_j(x,y)$, $j = 0, \dots, p$, are polynomials such that $P_j(a,b) \neq 0$ for some $j < p$ and $P_p(a,b) = 0$. Therefore,

$$\sum_{j=0}^{p-1} z^j P_j(a,b) = 0$$

at all the p points belonging to $f^{-1}(a,b)$, since otherwise ω would not be regular over (a,b). On the other hand, it is impossible, since a non-trivial polynomial of degree less than p can not have p roots.

Regularity condition over the line at infinity. Consider the form ω over the chart $x_1 \neq 0$,

$$\omega = -(\sum_{j=0}^{p-1} w^j \frac{\widetilde{g}_j(u,v)}{u^{jn+\deg g_j}}) \frac{1}{u^{3-n(p-1)}} \frac{du \wedge dv}{w^{p-1}},$$

The similar arguments as above show that the regularity criterion is equivalent to the following bound on the degree of the rational functions g_j

$$\deg g_j(x,y) \leq (p-j-1)n - 3. \tag{5}$$

Regularity conditions over a nonsingular point of the branch curve. Consider our form

$$\omega = (\sum_{j=0}^{p-1} z^j g_j(x,y)) \frac{dx \wedge dy}{z^{p-1}}$$

over a nonsingular point (a,b) of one of the components, B_{i_0}, of the branch curve. Let r_j be the order of zero (or of the pole if $r_j < 0$) of the function g_j along the curve B_{i_0}, i.e., $g_j = \overline{g}_j \cdot h_{i_0}^{r_j}$ with $\overline{g}_j$ having neither poles nor zeros along B_{i_0}. Since (a,b) is a nonsingular point of B_{i_0}, we can assume that $h_{i_0}(x,y)$ and some function $g(x,y)$ are local analytic cootdinates in some neighborhood U of (a,b) (denote them by u and v). So, over U the surface X_G (after analytic change of variebles) is isomorphic to the normalization $X_{G,loc}$ of the surface in $\mathbb{C}^3$ given by

$$z^p = u^{k_{i_0}}.$$

There is an analytic function w in $X_{G,loc}$ such that $u = w^p$ and $z = w^{k_{i_0}}$, and such that w and y are analytic coordinates in $X_{G,loc}$. The differencial 2-form ω considered above has the following form in the new coordinates

$$\omega = (\sum_{j=0}^{p-1} w^{jk_{i_0}} \overline{g}_j(x,y) w^{pr_j}) \frac{pw^{p-1} dw \wedge dv}{w^{(p-1)k_{i_0}}}.$$

It is easy to see that

$$j_1 k_{i_0} + pr_{j_1} + p - 1 - (p-1)k_{i_0} \neq j_2 k_{i_0} + pr_{j_2} + p - 1 - (p-1)k_{i_0}$$

if $0 < k_{i_0} < p$, $0 \leq j_1, j_2 \leq p-1$, and $j_1 \neq j_2$. Therefore, all the rational functions $g_j(x,y)$ from (4) are regular functions over a nonsingular point (a,b) of B_{i_0} iff

$$jk_{i_0} + pr_j + p - 1 - (p-1)k_{i_0} \geq 0$$

Moreover, if ω is a regular form over B_{i_0} then r_j must be greater than 0, since for $0 < k_{i_0} < p$, $0 \leq j \leq p-1$, and $r_j \leq -1$, we obtain that

$$jk_{i_0} + pr_j + p - 1 - (p-1)k_{i_0} < 0.$$

From this it follows that if ω is a regular form then all the rational functions $g_j(x,y)$ are regular functions everywere in $\mathbb{C}^2$ outside codimension 2, and thus $g_j(x,y)$ should be polynomials in x and y. Moreover, the polinomials $g_j(x,y)$ must be divisible by $h_i^{r_j}(x,y)$, where r_j is the smallest integer satisfying the inequality

$$pr_j \geq (p-j-1)k_i - p + 1. \tag{6}$$

Regularity conditions over singular points of the branch curve. This is the only step where the singular points of X_G are concerned. Let $\nu : \widetilde{X}_G \to X_G$ be the minimal resolution of singularities of X_G and E be the exceptional divisor of ν. Pick a composition $\sigma : Y \to \mathbb{C}\mathrm{p}^2$ of σ-processes with centers at singular points of B (and their preimages) such that $\sigma^{-1} \circ f \circ \nu(E_i)$ is a curve for each irreducible component E_i of E. Let Z be the normalization of $Y \times_{\mathbb{C}\mathrm{p}^2} X_G$. Denote by $g : \widetilde{X}_G \to Z$ the birational map induced by ν and σ. It follows from the above choice of σ that for any $\omega \in H^0(Z \setminus \mathrm{Sing}Z, \Omega^2_{Z \setminus \mathrm{Sing}Z})$ its pull-back $g^*(\omega)$ is regular at generic points of E_i and, thus, extends to a regular form on the whole $\widetilde{X}_G$. Hence, $H^0(\widetilde{X}_G, \Omega^2_{\widetilde{X}_G})$ is isomorphic to $H^0(Z \setminus \mathrm{Sing}Z, \Omega^2_{Z \setminus \mathrm{Sing}Z})$.

Therefore, it remains to consider a 2-form ω written as in (4) and to find a criterion of its regularity on $Z \setminus \mathrm{Sing}Z$. It can be done by performing, step by step, the σ-processes chosen above. Let us accomplish only the first step, since it is sufficient for the calculation in our particular example which follows.

Represent, once more, X_G as normalization of the surface given by

$$z^p = h(x,y).$$

Denote by r the order of zero of $h(x,y)$ at the point $(0,0)$, $r = sp + q$, $0 \leq q < p$, and perform a σ-process with center at this point. In a suitable chart, this σ-process

$\sigma : \mathbb{C}^2_{(u,v)} \to \mathbb{C}^2_{(x,y)}$ is given by $x = u,\ y = uv$. The normalization Z_1 of $X_G \times_{\mathbb{C}^2_{(x,y)}} \mathbb{C}^2_{(u,v)}$ is birational to the normalization of the surface given by

$$w^p = u^q \overline{h}(u,v),$$

where $w = z/u^s$ and $\overline{h}(u,v) = h(u,uv)/u^r$. We have

$$\begin{aligned} \omega &= (\sum_{j=0}^{p-1} z^j g_j(x,y)) \frac{dx \wedge dy}{z^{p-1}} = \\ &= (\sum_{j=0}^{p-1} w^j \overline{g}_j(u,v) u^{sj+s_j+1-s(p-1)}) \frac{du \wedge dv}{w^{p-1}}, \end{aligned}$$

where s_j is the order of zero of $g_j(x,y)$ at $(0,0)$. Applying (6), we get necessary conditions for the regularity of the pull-back of ω at generic points of the exceptional divisor: the order of zero s_j of each $g_j(x,y)$ at singular point of the branch locus B of order r is the smallest integer satisfying the inequality

$$ps_j \geq (p-j-1)r - 2p + 1. \tag{7}$$

Principal calculation. Here, we consider the Galois $G = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ covering of $\mathbb{C}\mathrm{p}^2$ constructed in Section 2. In this special case, the minimal desingularization $\tilde{X}_G$ of X_G is the induced covering of $\mathbb{C}\mathrm{p}^2$ blown up in the twelve multiple points of the configuration.

According to the above algorithm, we should examine one by one the six subgroups $H_1 = ((0,1))$, $H_2 = ((1,0))$, $H_3 = ((4,1))$, $H_4 = ((3,1))$, $H_5 = ((2,1))$, and $H_6 = ((1,1))$ of G such that $G_i = G/H_i = \mathbb{Z}/5\mathbb{Z}$ (here we denote by (α) the cyclic subgroup of G generated by $\alpha \in G$). The cyclic Galois coverings $X_{G_i} \to \mathbb{C}\mathrm{p}^2$ can be represented as normalizations of the surfaces in $\mathbb{C}^3$ given respectively by

$$\begin{aligned} z^5 &= l_1 l_2 l_3 l_4^3 l_5^3 l_9, \\ z^5 &= l_1 l_3 l_4^3 l_6 l_7 l_8^2 l_9, \\ z^5 &= l_1^2 l_2 l_3^2 l_4 l_5^3 l_6 l_7 l_8^2 l_9^2, \\ z^5 &= l_1 l_2^2 l_3 l_4^3 l_5 l_6^4 l_7^4 l_8^3 l_9, \\ z^5 &= l_1 l_2^4 l_3 l_4^3 l_5^2 l_6^2 l_7^2 l_8^4 l_9, \\ z^5 &= l_2^2 l_5 l_6^3 l_7^3 l_8, \end{aligned}$$

To calculate the geometric genus of each of $\tilde{X}_{G_i}$ we find explicitly all the regular 2-forms, which we write as in (4). We use the criteria (5) – (7) (for convenience, we reproduce in Tables 2 - 4 below, using notations involved in (5) – (7), the exact values of these bounds evaluated in the case of $\mathbb{Z}/5$-coverings). For G_1 we get the forms

$$c l_4 l_5 l_9 z \frac{dx \wedge dy}{z^4}, c \in \mathbb{C};$$

for G_2 we get

$$(P_2 l_4^2 l_8 + P_1 l_4 l_8 z + c l_4 z^2)\frac{dx \wedge dy}{z^4}$$

where P_i are polynomials in x, y of degree i, $c \in \mathbb{C}$, and

$$p_{349}, p_{789}, p_{168}, p_{147} \in \{P_2 = 0\},\ p_{147} \in \{P_1 = 0\};$$

for G_3 we get

$$(P_1 l_1 l_3 l_5^2 l_6 l_8 l_9 + Q_1 l_1 l_3 l_4 l_5 l_8 l_9 z + c_1 l_3 l_8 z^2 + c_2 z^3)\frac{dx \wedge dy}{z^4}$$

where P_i, Q_i are polynomials of degree i, $c_k \in \mathbb{C}$, and

$$p_{123}, p_{789} \in \{P_1 = 0\}, p_{456} \in \{Q_1 = 0\};$$

for G_4 we get

$$(P_2 l_2 l_4^2 l_6^3 l_7^3 l_8^2 + Q_2 l_2 l_4 l_6^2 l_7^2 l_8 z + P_1 l_4 l_6 l_7 l_8 z^2 + Q_1 z^3)\frac{dx \wedge dy}{z^4}$$

with

$$p_{159}, p_{123} \in \{P_2 = 0\}, p_{348} \in \{Q_2 = 0\}, p_{267} \in \{P_1 = 0\}, p_{267} \in \{P_1 = 0\};$$

for G_5 we get

$$(P_2 l_2^3 l_4^2 l_5 l_6 l_7 l_8^3 + P_1 l_2^2 l_4 l_5 l_6 l_7 l_8^2 z + Q_2 l_2 l_4 l_8 z^2 + Q_1 z^3)\frac{dx \wedge dy}{z^4}$$

with

$$p_{159}, p_{369}, p_{357} \in \{P_2 = 0\}, p_{357}, p_{267}, p_{258} \in \{Q_2 = 0\}, p_{258} \in \{Q_1 = 0\};$$

and, finally, for G_6,

$$c l_2 l_6 l_7 z \frac{dx \wedge dy}{z^4}.$$

Therefore, $p_g(\tilde{X}_{G_1}) = 1$, $p_g(\tilde{X}_{G_2}) = 5$, $p_g(\tilde{X}_{G_3}) = 5$, $p_g(\tilde{X}_{G_4}) = 13$, $p_g(\tilde{X}_{G_5}) = 11$, and $p_g(\tilde{X}_{G_6}) = 1$.

$\deg g_j \leqslant$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
headers $j = 0$	1	5	8	13
$j = 1$	0	3	6	9
$j = 2$	0	1	3	5
$j = 3$	0	0	0	1
$j = 4$	0	0	0	0

Table 2

$r_j \geqslant$	$k_i = 1$	$k_i = 2$	$k_i = 3$	$k_i = 4$
$j = 0$	0	1	2	3
$j = 1$	0	1	1	2
$j = 2$	0	0	1	1
$j = 3$	0	0	0	0
$j = 4$	0	0	0	0

Table 3

$s_j \geqslant$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$	$r = 9$	$r = 10$	$r = 11$	$r = 12$
$j = 0$	0	1	2	3	3	4	5	6	7	7	8
$j = 1$	0	0	1	2	2	3	3	4	5	5	6
$j = 2$	0	0	0	1	1	1	2	2	3	3	3
$j = 3$	0	0	0	0	0	0	0	0	1	1	1
$j = 4$	0	0	0	0	0	0	0	0	0	0	0

Table 4

Now, from Proposition A.1. it follows that $p_g(\tilde{X}_G) = 36$. On the other hand, by Noether formula, $(c_1^2 + c_2)/12 = 37$ and, thus,

$$q(\tilde{X}_G) = \dim H^0(\tilde{X}_G, \Omega^1_{\tilde{X}_G}) = (c_1^2 + c_2)/12 - 1 - p_g = 0,$$

i.e., $\tilde{X}_G$ is a regular surface.

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References

[BPV] W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, Springer-Verlag, 1984.

[Ca] F. Catanese, *Moduli spaces of surfaces and real structures*, preprint math. AG/0103071.

[FM] R. Friedman, J.W. Morgan, *Algebraic surfaces and* 4*-manifolds*, Bull. A.M.S. **18** (1988), 1–19

[F] A. Fujiki, *Coarse Moduli Space for Polarized Compact Kähler Manifolds*, Publ. RIMS (Kyoto Univ.) **20** (1984), 977–1005.

[G] H. Grauert, *Ein Theorem der analytischen Garbentjeorie, und due Modulraüme komplexer Strukturen,* Publ. IHES **5** (1960).

[I] M.-N. Ishida, *The Irregularities of Hirzebruch's examples, of Surfaces of General Type with* $C_1^2 = 3C_2$*,* Math. Ann. **262** (1983), 407–420.

[H] F. Hirzebruch, *Arrangements of lines and algebraic surfaces Arithmetics and Geometry* II, Prog. Math. **36** Birkháuser 1983, pp. 113-140.

[KK] V. Kharlamov and Vik.S. Kulikov, *On real structures of rigid surfaces*, preprint AG/0101098, to appear in Izvestiya RAN: Mathematics.

[KK2] V. Kharlamov and Vik.S. Kulikov, *Diffeomorphisms, isotopies and braid monodromy factorizations of cuspidal curves*, preprint AG/0104021, to appear in C.R.A.S.

[K] Vik.S. Kulikov, On Chisini's Conjecture, Izvestiya Math. **63** (1999), 1139–1170.

[KN] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, Interscience publishers, 1963.

[Ko] D. Kotschick, *Orientations and geometrisations of compact complex surfaces*, Bull. London Math. Soc. **29** (1997), no. 2, 145–149.

[LeB] C. LeBrun, *Diffeomorphisms, symplectic forms and Kodaira fibrations*, Geometry & Topology **4** (2000), 451–456.

[LL] T.-J. Li, A.-K. Liu, *Uniqueness of symplectic canonical class, surface cone and symplectic cone of* 4*-manifolds with* $b^+ = 1$, preprint math. SG/ 0012048.

[LS] D. Lieberman, E. Sernesi, *Semicontinuity of L-dimension.*, Math. Ann. **225** no. 1, 77–88.

[Ma] M. Manetti, *On the moduli space of diffeomorphic algebraic surfaces*, Invent. Math. **143 (1)** (2001), 29–76.

[MT] C.T. McMullen, C.H. Taubes, 4*-manifolds with inequivalent symplectic forms and* 3*-manifolds with inequivalent fibrations,* Math. Research Let. **6** (1999), 681–696.

[Mu] D. Mumford, An algebraic surface with K ample $K^2 = 9$, $p_g = q = 0$, Amer. J. Math. **101** (1979), 233–244.

[R] Y.B. Ruan, *Symplectic topology on algebraic* 3*-manifolds,* J. Diff. Geom. **39** (1994), 215–227.

[Shi] G. Shimura, *On the real points of an arithmetic quotient of a bounded, symmetric domain* Math. Ann. **215** (1975), 135–164.

[S] Y.-T. Siu, *The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds,* Ann. Math **112** (1980), 73–111.

[Sm] I. Smith, *On moduli spaces of symplectic forms,* preprint math. SG/0012096.

[W] C.T.C. Wall, *On simply-connected 4-manifolds,* J. London Math. Soc. **39** (1964), 141–149.

[Z] A. V. Zubr, *Classification of simply connected six-dimensional spin manifolds,* Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 4, 839–859.

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