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Minimality of certain normal connected sums

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Abstract

We show that the normal connected sum of two minimal symplectic 4-manifolds (neither of them rational or ruled) is a minimal symplectic 4-manifold. In the proof we use a symplectic sum formula for Gromov-Witten invariants.

1. Introduction

In 1994 a very effective method of constructing symplectic manifolds has been introduced by Gompf [1] and McCarthy-Wolfson [7]. The *normal connected sum* M of two symplectic 4-manifolds (M_1, ω_1) and (M_2, ω_2) along the codimension-2 symplectic submanifolds $Z_1 \subset M_1$ and $Z_2 \subset M_2$ (where Z_1 is diffeomorphic to Z_2 and $[Z_1]^2 + [Z_2]^2 = 0$) is $M = (M_1 - \text{int } \nu Z_1) \cup_\varphi (M_2 - \text{int } \nu Z_2)$, where νZ_i is a tubular neighborhood of Z_i and φ is an orientation-reversing lift of a diffeomorphism $Z_1 \to Z_2$ to the unit normal circle bundle. In [1, 7] it has been proved that M supports a symplectic structure which can be constructed from the symplectic structures ω_1 and ω_2 (by possibly scaling ω_1 such that $\int_{Z_1} \omega_1 = \int_{Z_2} \omega_2$). A smooth 4-manifold is said to be smoothly minimal if it does not contain any smoothly embedded sphere with square -1. When studying topological properties of M it is often helpful to know whether it is smoothly minimal or not. In many cases *ad hoc* computations of certain gauge theoretic invariants show that M is minimal. Below we prove a general statement, namely we show

Theorem 1.1. *Suppose that M_1 and M_2 are two symplectic 4-manifolds which are neither rational nor ruled and $Z_i \subset M_i$. If M_1 is minimal and $M_2 - Z_2$ does not contain any smoothly embedded -1 sphere, then the normal connected sum M of M_1 and M_2 along Z_i is minimal.*

Corollary 1.2. *Suppose that M_1 and M_2 are two minimal symplectic 4-manifolds which are neither rational nor ruled and $Z_i \subset M_i$. Then the normal connected sum M of M_1 and M_2 along Z_i is minimal.*

Recall that a symplectic 4-manifold is said to be rational or ruled if it is an S^2-bundle or $\mathbb{CP}^2$ or their blow up. Notice that the requirement that M_1 is minimal and not rational or ruled can be substituted by the equivalent condition that any embedded sphere in M_1 has square ≤ -2.

Remark 1.3. We believe that (using arguments from [5]) the condition that M_i is not rational or ruled can be removed, and the condition that M_1 is minimal can be weakened

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to that there are no (-1)-spheres in the complement of Z_1 in M_1. We hope to return to this generalization in the future. We are content with the version here since it is strong enough for many applications.

The proof of Theorem 1.1 is an application of the symplectic sum formula for the genus-0 Gromov-Witten invariants proved in [3] and [6].

2. The proof

Let us first introduce the relevant genus-0 Gromov-Witten invariants. Let M be a closed 4-manifold with a symplectic form ω and Z a symplectic surface of positive genus in M. Choose an $\omega-$compatible almost complex structure J such that Z is $J-$holomorphic. In our case, we only need to consider the simplest genus zero invariant. Since Z is a surface of positive genus, the only $J-$holomorphic maps from $\mathbb{CP}^1$ to Z are the constant maps. So, for any homology class $A \in H_2(M;\mathbb{Z})$, we can define the genus-0 relative Gromov-Witten invariant $\psi_A^{(M,Z)}$ by counting the number of stable genus-0 $J-$holomorphic maps in the class A and intersecting Z at finitely many points with prescribed tangency. In order to give the definition of $\psi_A^{(M,Z)}$, we need to first fix a set of v positive integers $\mathbf{K} = \{k_1, \cdots, k_v\}$. Now consider the moduli space $\mathfrak{M}_A^{M,Z}(\mathbf{K})$ of $J-$holomorphic maps $f\colon \mathbb{CP}^1 \to M$ with marked points $y_1, \cdots, y_v$ such that $[f(\mathbb{CP}^1)] = A$, the set of intersection points of $f(\mathbb{CP}^1)$ and Z is $\{f(y_1), ..., f(y_v)\}$ and f is tangent to Z at $y_1, \cdots, y_v$ of order $k_1, \cdots, k_v$. Let us denote $(y_1, \cdots, y_v)$ by $\mathbf{y}$ and define the degree of $\mathbf{K}$ to be $\deg \mathbf{K} = \sum_{j=1}^v k_j$. Notice that $\deg \mathbf{K} = A \cdot Z$. The moduli space $\mathfrak{M}_A^{M,Z}(\mathbf{K})$ admits a compactification $\overline{\mathfrak{M}}_A^{M,Z}(\mathbf{K})$ by considering relative stable maps, and the compactified space carries a fundamental class $[\overline{\mathfrak{M}}_A^{M,Z}(\mathbf{K})]$ — for details see [6]. The compactified space also admits v evaluation maps $e_i\colon \overline{\mathfrak{M}}_A^{M,Z}(\mathbf{K}) \to Z$, $i = 1, ..., v$, defined by

$$(f, \mathbb{CP}^1, \mathbf{y}, \mathbf{K}) \longmapsto f(y_i).$$

The formal dimension of $\mathfrak{M}_A^{M,Z}(\mathbf{K})$ is given as

$$\mathrm{fdim}(\mathfrak{M}_A^{M,Z}(\mathbf{K})) = 2c_1(M) \cdot A - 2 - 2\sum_{i=1}^{v} k_i + 2v. \tag{1}$$

The relative Gromov-Witten invariants are defined through pulling back cohomology classes on Z via the evaluation maps e_i.

Definition 2.1. The genus-0 relative Gromov-Witten invariant $\psi_A^{(M,Z)}$ is a map from $\oplus_{v=1}^{\infty} H_2(Z;\mathbb{Z})^v \times \mathbb{Z}^v$ to $\mathbb{Z}$. More precisely, given a set $\beta = \{\beta^1, \cdots, \beta^v\}$ with $\beta^i \in H^*(Z;\mathbb{Z})$ and a set of v positive intergers $\mathbf{K} = \{k_1, \cdots, k_v\}$, define $\psi_A^{M,Z}(\beta, \mathbf{K})$ as the integral

$$\psi_A^{M,Z}(\beta, \mathbf{K}) = \int_{[\overline{\mathfrak{M}}_A^{M,Z}(\mathbf{K})]} \cup_{i=1}^{v} e_i^* \beta^i$$

once $\sum_{i=1}^v \deg \beta^i = \mathrm{fdim} \mathfrak{M}_A^{M,Z}(\mathbf{K})$ — and zero otherwise.

If Z is empty, then both $\mathbf{K}$ and β are necessarily empty sets and the corresponding invariant (which is the ordinary Gromov-Witten invariant) will be simply denoted by ψ_A^M. It is shown in [6] that $\psi_A^{(M,Z)}$ is independent of J and therefore an invariant of the pair (M, Z) of symplectic manifolds.

Remark 2.2. More general genus-0 relative Gromov-Witten invariants also allow some of k_i to be zero and the corresponding β_i to be cohomology classes of M.

Let N be the circle bundle over Z which splits M into M_1 and M_2. Consider the singular space $M_1 \cup_{Z_1=Z_2} M_2$, the map π collapsing the circle fibers

$$\pi\colon M \longrightarrow M_1 \cup_{Z_1=Z_2} M_2,$$

and the induced map π_* on H_2. $\ker(\pi_*)$ is generated by classes which are represented by tori of the form $\eta \times \tau$ where η is a curve in Z and τ is a fiber of $N \longrightarrow Z$. (These tori are frequently called *rim tori*.) It is easy to see that a rim torus $e = \eta \times \tau$ is Lagrangian (hence $\omega(e) = 0$) and it has vanishing self-intersection, that is, $e \cdot e = 0$. For a class $A \in H^2(M;\mathbb{Z})$, define $\langle A \rangle = \{A + e | e \in \ker(\pi_*)\}$ and consider

$$\Psi_{\langle A \rangle}^M = \sum_{B \in \langle A \rangle} \psi_B^M.$$

The following lemma plays a crucial role in our proof of Theorem 1.1.

Lemma 2.3. *Suppose that (X, ω) is a minimal symplectic 4-manifold which is not rational or ruled, and U is a symplectic surface in X. Then all relative genus-0 Gromov-Witten invariants $\psi_C^{X,U}(\beta, \mathbf{K})$ of (X, U) vanish.*

Proof. Let C be a class in $H_2(X;\mathbb{Z})$. If C is represented by a $J-$holomorphic sphere for some $\omega-$compatible almost complex structure, we claim that $c_1(X) \cdot C \leq 0$. Then for any $\mathbf{K}$ the formal dimension of $\mathfrak{M}_C^{X,U}(\mathbf{K})$ is negative by (1). This implies that the moduli spaces are empty, therefore all relative genus-0 Gromov-Witten invariants of (X, U) vanish. Now let us prove the claim that $c_1(X) \cdot C \leq 0$. According to [8], by possibly perturbing J, we can assume that the pseudo-holomorphic sphere representing C is immersed. If the number of double points (all necessarily positive) is l then the adjunction formula shows

$$2l - 2 = -c_1(X) \cdot C + C \cdot C;$$

equivalently $c_1(X) \cdot C = C \cdot C - (2l - 2)$. If $C \cdot C \geq 0$ then X is rational or ruled by [8]. If $C \cdot C \leq -2$, then $c_1(X) \cdot C \leq 0$. The only case remaining to consider is when $C \cdot C = -1$. Then $c_1(X) \cdot C > 0$ only if $l = 0$, hence C has square -1 and is represented by an embedded pseudo-holomorphic sphere. The existence of such class, however, contradicts the minimality of X therefore the proof is complete. □

Remark 2.4. Notice that the above lemma does not hold for higher genus Gromov-Witten invariants.

Now the symplectic sum formulae in [3, 6] compute the genus-0 absolute invariant $\Psi^M_{\langle A\rangle}$ of M in terms of the genus-0 relative Gromov-Witten invariants of (M_i, Z_i). More precisely, if A cannot be represented by a (not necessrily embedded) pseudo-holomorphic sphere in $M_i - Z_i$ for $i = 1$ or 2, then $\Psi^M_{\langle A\rangle}$ can be expressed as a sum of products of the form $\psi^{(M_1,Z_1)}_{A_1}(\beta_1, \mathbf{K}_1) \cdot \psi^{(M_2,Z_2)}_{A_2}(\beta_2, \mathbf{K}_2)$ with $\deg \mathbf{K}_1 = \deg \mathbf{K}_2 > 0$. This implies

Proposition 2.5. *If $\Psi^M_{\langle A\rangle}$ does not vanish and A cannot be represented by a pseudo-holomorphic sphere in $M_i - Z_i$ for $i = 1$ or 2, then some of the relative genus-0 Gromov-Witten invariants of (M_1, Z_1) with non-empty $\mathbf{K}_1$ and some of the relative genus-0 Gromov-Witten invariants of (M_2, Z_2) with non-empty $\mathbf{K}_2$ are non-zero.* □

The following lemma is proved in [5]; here we only sketch the argument proving it:

Lemma 2.6 ([5]). *Let (M, ω) be the normal connected sum of (M_i, ω_i) $(i = 1, 2)$. Suppose that A is a class represented by a symplectic sphere with square -1. Then for any $e \in Ker(\pi_*)$ and for the absolute Gromov-Witten invariants ψ^M_{A+e} we have $\psi^M_{A+e} = 0$ unless $e = 0$.*

Proof (sketch). If $b_2^+(M) = 1$ then $\omega(e) = 0$ implies that e is in a negative definite subspace, therefore $e^2 = 0$ shows that $e = 0$. In the case $b_2^+(M) > 1$ we will appeal to the equivalence between Seiberg-Witten and Gromov-Witten invariants proved by Taubes [12]. (In the following we will identify homology and cohomology classes through Poincaré duality.) The class $-c_1(M)$ is a Seiberg-Witten basic class [11], and if $A \in H_2(M; \mathbb{Z})$ can be represented by a (-1)-sphere then so is $-c_1(M) + 2A$. The adjunction inequality implies that if a class $a \in H_2(M; \mathbb{Z})$ can be represented by a torus and $a^2 = 0$ then for any basic class K we have $K \cdot a = 0$. Since $\ker \pi^*$ is generated by such tori, we have $K \cdot e = 0$ for all basic classes K and homology elements $e \in \ker \pi^*$. Consequently $-c_1(M)\cdot e = (-c_1(M)+2A)\cdot e = 0$, implying $A \cdot e = 0$. Now suppose that $\psi^M_{A+e} \neq 0$. The equivalence between Seiberg-Witten and Gromov-Witten invariants implies that $\psi^M_A \neq 0$. Represent A and $A+e$ by the $J-$holomorphic curves C and D. Since $A\cdot(A+e) = A^2 = -1$, the two curves must share components, therefore C must be contained in D. Since the $J-$holomorphic curve $D \setminus C$ represents e and $\omega(e) = 0$, we get that $D \setminus C$ is the empty curve, hence $e = 0$. □

Proof of Theorem 1.1. Suppose M is not minimal, i.e., there is a smoothly embedded (-1)-sphere in M. Let ω be a symplectic form on M. By [4] we know that, in fact, there must be an embedded $\omega-$symplectic (-1)-sphere. Let A be the homology class of this (symplectic) sphere; this sphere can be made $J-$holomorphic for some $\omega-$compatible almost complex structure J. For such a J, this sphere is the only pseudo-holomorphic sphere representing A, therefore $\psi^M_A = 1$ (see [8] for example). Now Lemma 2.6 implies that $\Psi^M_{\langle A\rangle} = 1$ as well. Our assumptions on M_1 and M_2 imply that A is not represented by a pseudo-holomorphic sphere in the complement of Z_i in M_i for $i = 1$ and 2. On the other hand, according to Lemma 2.3 all genus-0 relative Gromov-Witten invariants of (M_1, Z_1) are zero. Therefore Proposition 2.5 and Lemma 2.6 imply that $\Psi^M_{\langle A\rangle}$ is trivial as well. This contradiction finishes the proof. □

3. An easy application

In many cases our theorem is sufficient for proving minimality of certain symplectic 4-manifolds constructed by the symplectic normal connected sum operation. For example, the construction of symplectic 4-manifolds with various (c_1^2, c_2)-invariants given by Gompf in [1] can be modified to avoid the usage of rational surfaces such that the resulting construction provides essentially the same result. Meanwhile, minimality of the resulting 4-manifolds in these cases will be guaranteed by Theorem 1.1. Here we give a modification of the construction of simply connected symplectic 4-manifolds which are near to the Bogomolov-Miyaoka-Yau line; the original construction is described in [9] and we will only indicate the steps which are covered there. Recall that there exist complex surfaces $H(n^2)$ with Euler characteristic $\chi(H(n)) = 75n^2$ and $c_1^2(H(n)) = 225n^2$ which admit genus-$(15n+1)$ Lefschetz fibrations over Σ_{n+1}; moreover these Lefschetz fibrations admits sections with self-intersection $-n$ (see Proposition 2.2 in [9]). Fiber summing these with certain genus-$(15n+1)$ Lefschetz fibrations over the torus T^2 we get a sequence X_n of relatively minimal genus-$(15n+1)$ Lefschetz fibrations over Σ_{n+2} with $\chi(X_n) = 75n^2 + 180n+12$ and $c_1^2(X_n) = 225n^2+180n$. These fibrations contain sections T_n of genus $(n+2)$ and self-intersection $-(n+1)$. Since $X_n \to \Sigma_{n+2}$ is relatively minimal, it follows that X_n is a minimal symplectic 4-manifold (see [10]), which is not rational or ruled. Moreover, the symplectic structure can be chosen such that T_n is a symplectic submanifold. Define $(E(n+3), U_n)$ to be the appropriate elliptic surface with the symplectic submanifold U_n we get by smoothing the union of $(n+2)$ copies of the fiber and a section. Notice that U_n is a surface of genus $(n+2)$ with self-intersection $[U_n]^2 = 2(n+2) - n - 3 = n+1$. Now Theorem 1.1 (together with Lemma 3.3 in [10]) implies

Proposition 3.1. *The symplectic normal sum D_n of (X_n, T_n) and $(E(n+3), U_n)$ is a minimal, simply connected symplectic 4-manifold with $\chi(D_n) = 75n^2 + 188n + 44$ and $c_1^2(D_n) = 225n^2 + 196n - 64$. In particular, $c_1^2(D_n)/\chi(D_n)$ converges to 3 as $n \to \infty$.*

Remark 3.2. Instead of $(E(n+3), U_n)$ we might have used $(E(2)\#(n+1)\overline{\mathbb{CP}}^2, V_n)$ where $E(2)$ is the K3-surface and V_n is given as the $(n+1)$-fold blow-up of the symplectic submanifold we get by smoothing the union of $(n+2)$ disjoint (regular) fibers and a section. The details of the computation are left for the reader.

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