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# Variations on Fintushel-Stern Knot Surgery on 4-manifolds

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**Abstract**

We discuss some consequences Fintushel-Stern 'knot surgery' operation coming from its handlebody description. We give some generalizations of this operation and give a counterexample to their conjecture.

## 1. Introduction

Let $X$ be a smooth 4-manifold and $K \subset S^3$ be a knot, In [4] among other things Fintushel and Stern had shown that the operation $K \rightsquigarrow X_K$ of replacing a tubular neighborhood of imbedded torus in $X$ by $(S^3 - K) \times S^1$ could results change of smooth structure of $X$. In [1] an algorithm of describing handlebody of $X_K$ in terms of the handlebody of $X$ was described. In this article we will give some corollaries of this construction, and present a counterexample to conjecture of Fintushel and Stern which was overlooked in [1]. First we need to recall the precise description of $X_K$: Recall that the first picture of Figure 1 is $T^2 \times D^2$, and the second one is the cusp $C$ (i.e. $B^4$ with a 2-handle attached along the trefoil knot with the zero framing). Clearly the cusp $C$ contains a copy of $T^2 \times D^2$.

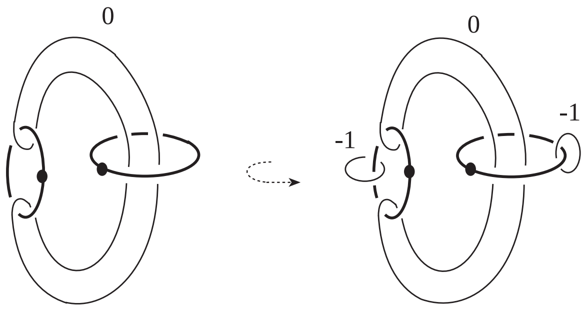


**Figure 1**

In [4] an imbedded torus $T^2 \subset X$ is called a *c-imbedded* torus if it has a cusp neighborhood in $X$, i.e. $T^2 \hookrightarrow C \hookrightarrow X$ as in Figure 1. Now let $N \approx K \times D^2$ be the trivialization of the open tubular neighborhood of the knot $K$ in $S^3$ given by the 0-framing. Let

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$\varphi : \partial(T^2 \times D^2) \to \partial(K \times D^2) \times S^1$ be any diffeomorphism with $\varphi(p \times \partial D^2) = K \times p$, where $p \in T^2$ is a point, then define:

$$X_K = (X - T^2 \times D^2) \smile_{\varphi} (S^3 - N) \times S^1$$

Let $Spin_c(X)$ be the set of $Spin_c$ structures on $X$, e.g. if $H_1(X)$ has no 2-torsion then.

$$Spin_c(X) = \{ \ a \in H^2(X; \mathbf{Z}) \mid a = w_2(TX) \ \mathrm{mod} 2 \ \}$$

Recall that Seiberg-Witten invariant $SW_X$ of $X$ is a symmetric function

$$SW_X : Spin_c(X) \to \mathbf{Z}$$

It is known that the function $SW_X$ is nonzero on the complement of a finite set $B = \{\pm\alpha_1, \pm\alpha_2, .., \pm a_n\}$ which is called *the set of basic homology classes classes*. By setting $\alpha_0 = 0$ and $t_j = exp(\alpha_j)$, the function $SW_X$ is usually written as a single polynomial

$$SW_X = \sum_{j=0}^{n} SW_X(\alpha_j) t_j$$

Now if $T$ is a $c$-imbedded torus in $X$ , and $[T]$ be the homology class in $H_2(X_K; \mathbf{Z})$ induced from $T^2 \subset X$, and $t = \exp(2[T])$, and $\Delta_K(t)$ the Alexander polynomial of the knot $K$ (as a symmetric Laurent polynomial), then Fintushel and Stern [4] theorem says:

**Theorem 1.1.** $SW_{X_K} = SW_X . \, \Delta_K(t)$

Recall that in [1] the algorithm of drawing the handlebody of $X_K$ from $X$ is described as follows: First we identify the *core circles* of the 1-handles of the handlebody of $S^3 - K$

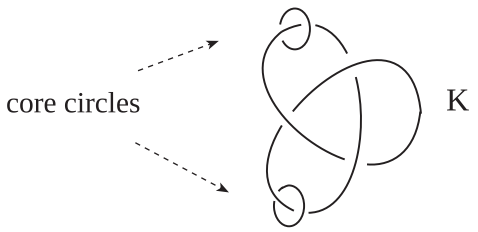


**Figure 2**

Then when we see an imbedded cusp $C$ in the handlebody of $X$ as in the first picture of Figure 3, we change it to the second picture $\tilde{C}$ of Figure 3. This means that we change one of the 1-handles of $T^2 \times D^2$ inside of $C$ to the "slice 1-handle" obtained from $K\#(-K)$ (i.e. remove the obvious slice disk which $K\#(-K)$ bounds from $B^4$), and connect the *core circles* of the knots $K$ and $-K$ by 2-handles as shown in Figure 3. More precisely, there is a diffeomorphism between the boundaries of manifolds $C$ and $\tilde{C}$ of Figure 3, and the operation $X \rightsquigarrow X_K$ corresponds to cutting out $T^2 \times D^2$ from $X$ and gluing the second manifold of Figure 3 by this diffeomorphism (in the figure $K$ is drawn as the trefoil not).

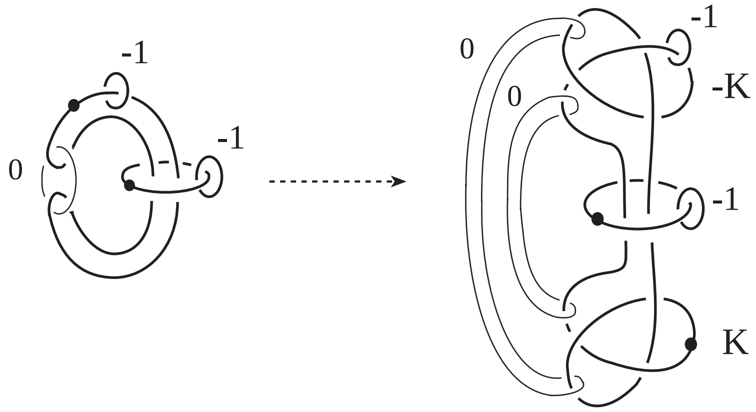


**Figure 3**

Since the attaching circles of the other 2-handles of $X$ could tangle to the boundary of $T^2 \times D^2$, it is important to indicate where the various linking circles of the boundary are thrown to by the diffeomorphism of Figure 3. This is indicated in Figure 6.

Recall that since 3- and 4- handles of four manifolds are attached in the canonical way, to describe a 4-manifold it suffices to describe its 1- and 2- handle structure. So, in order to visualize $(S^3 - K) \times S^1$, which is obtained by by identifying the two ends of $(S^3 - K) \times I$, it suffices to visualize $(B^3 - K_0) \times I$ with its ends identified, where $K_0 \subset B^3$ is a properly imbedded arc with the knot $K$ tied on it (the rest is a 3-handle). The second picture of Figure 4 gives the handlebody picture of $(B^3 - K_0) \times I$.

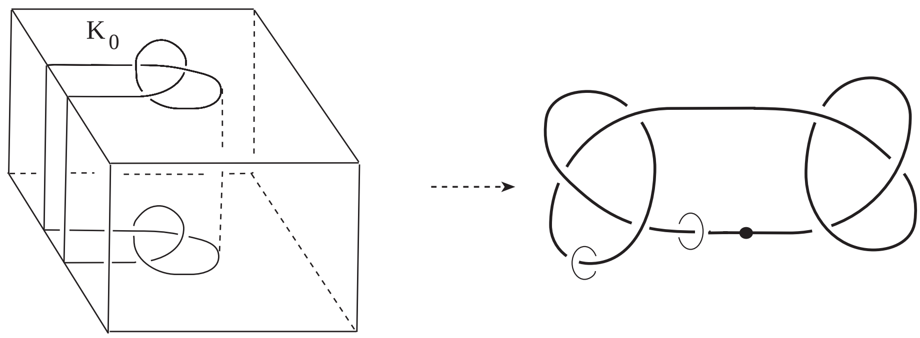


**Figure 4**

Identifying the ends of $(B^3 - K_0) \times I$ (up to 3-handles) corresponds to attaching a new 1-handle, and 2-handles, where the new 2-handles are attached along the 1-handles of the two boundary components of $(B^3 - K_0) \times I$ as indicated in Figure 5 (more specifically the 2-handles are attached along the loops connecting the *core circles* of the knot complements).

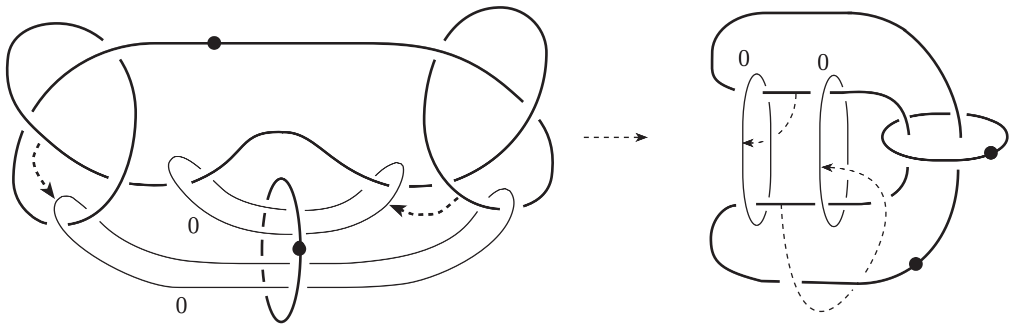


**Figure 5**

To see the diffeomorphism of Figure 3 (i.e. to see that the boundary of the first picture in Figure 5 is standard), we simply remove the dot on the "slice" 1-handle (i.e. turn it to a 2-handle) and slide it over the two 2-handles (as indicated by the arrows) in the first picture of Figure 5. This gives the second picture of Figure 5. After sliding 2-handles over each other of second picture of Figure 5, and cancelling the resulting $S^2 \times D^2$ with the 3-handle we obtain $T^2 \times D^2$. Also, to see the inverse boundary diffeomorphism from $T^2 \times D^2$ to the first picture in Figure 5, we remove the dot from the 1-handle of the second picture of Figure 5 and perform the handle slides indicated by the arrows.

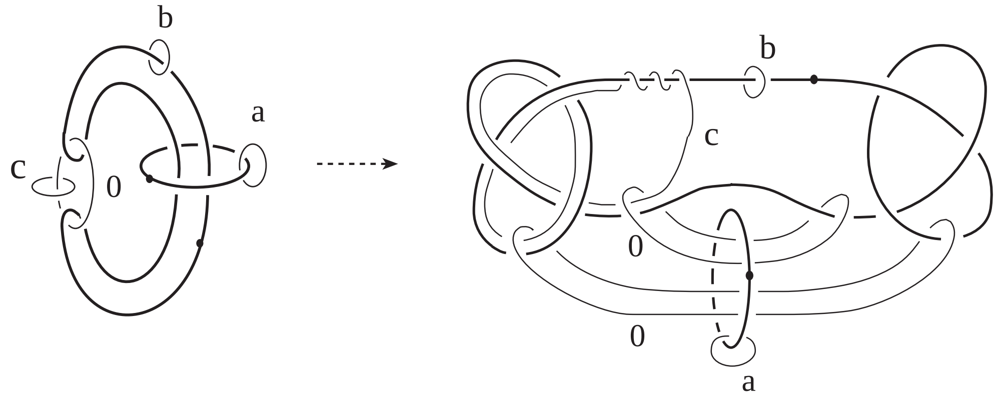


**Figure 6**

Now putting these together in Figure 6 we see where the boundary diffeomorphism takes various linking circles of $\partial(T^2 \times D^2)$. In particular the linking circle $c$ of the 2-handle is thrown to the loop which corresponds the zero push-off of $K$ in $K\#(-K)$.

Figure 7 is the same as the second picture of Figure 6 except that the slice disk complement, which $K\#(-K)$ bounds, is drawn more concretely. Also note that, though our discussion is for general $K$, for the sake of concreteness, we have drawn our figures by taking $K$ to be the trefoil knot.

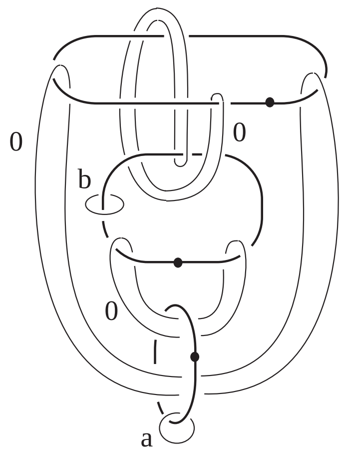


**Figure 7**

## 2. Applications

In [4] Fintushel and Stern conjectured that if $X$ is the Kummer surface $K3$, then the association $K \rightsquigarrow X_K$ gives an injective map from the isotopy classes of knots $K$ in $S^3$ to the set of diffeomorphism classes of smooth structures on $X$. The following theorem provides a counterexample to this conjecture. Let $-K$ be the mirror image of the knot $K$.

**Theorem 2.1.** $X_K = X_{-K}$

*Proof.* There is an obvious self-diffeomorphism of the second picture in Figure 3 (i.e. $(S^3 - K) \times S^1$) exchanging roles of $K$ and $-K$ ; i.e. the diffeomorphism induced by $180^o$ rotation of $\mathbf{R}^3$ around the $y$-axis. It is easily check that this diffeomorphism extends to the interior of $(S^3 - K) \times S^1$, implying the desired result. □

The following says that all smooth manifolds $X_K$ obtained from $X$ by using from different knots $K$ become standard after single stabilization. This result was independently observed by Auckly [2].

**Theorem 2.2.** $X_K \# (S^2 \times S^2) = X\#(S^2 \times S^2)$

*Proof.* $X_K \# (S^2 \times S^2)$ is obtained by surgering any homotopically trivial loop (with the correct framing). We choose to surger $X_K$ along the trivially linking circle of its slice 1-handle (the knot $K\#(-K)$ with a dot). This corresponds to turning the slice 1-handle to a 0-framed 2-handle (i.e. replace the dot with 0 framing), hence we are free to isotop the attaching circle of this 2-handle to the standard position as indicated in Figure 5. In particular, this makes the boundary diffeomorphism between the two handlebodies of Figure 5 extend to a 4-manifold diffeomorphism. So, Surgered $X_K$ is diffeomorphic to the surgered $X$ which is $X\#(S^2 \times S^2)$.

Note that though we indicated the argument for the trefoil knot $K$ in our pictures, the same applies for a general $K$ (i.e. in Figure 5 the knot $K\#(-K)$ unknots in the presence of the 2-handles) $\square$

Notice that $X_K$ can be viewed as $X_K = X_f = (X - T^2 \times D^2) \smile_{\varphi} (S^3 \times S^1 - U)$, where $U$ is an open tubular neighborhood of an imbedded torus $f : S^1 \times S^1 \to S^3 \times S^1$, with $Image(f) = K \times S^1$. The map $f$ is induced from the obvious imbedding $K \times I \to S^3 \times I$ by identifying the ends. More generally to any concordance $s$ from $K$ to itself, we can associate an imbedding of a torus $f_s : S^1 \times S^1 \hookrightarrow S^3 \times S^1$, hence getting map

$$C(K) \longrightarrow \left\{ \begin{array}{c} \text{Diffeomorphism classes of} \\ \text{smooth structures on } X \end{array} \right\}$$

defined by $s \to X_s$, where $C(K) = \{\ s : S^1 \times I \hookrightarrow B^3 \times I \mid s|_{S^1 \times 0} = s|_{S^1 \times 1} = K\ \}$. It is an interesting question that how the diffeomorphism class of $X_s$ depends on the concordance class $s$ of $K$? The following says that the above map is not injective.

**Theorem 2.3.** *If $K$ is the trefoil knot, there is $s \in C(K\#(-K))$ such that $X_s = X$*

*Proof.* Let $s$ be the concordance of $K\#(-K)$ to itself, given by connected summing the two obvious slice discs which two copies of $K\#(-K)$ bound in $B^4$ as in the second picture of Figure 8.

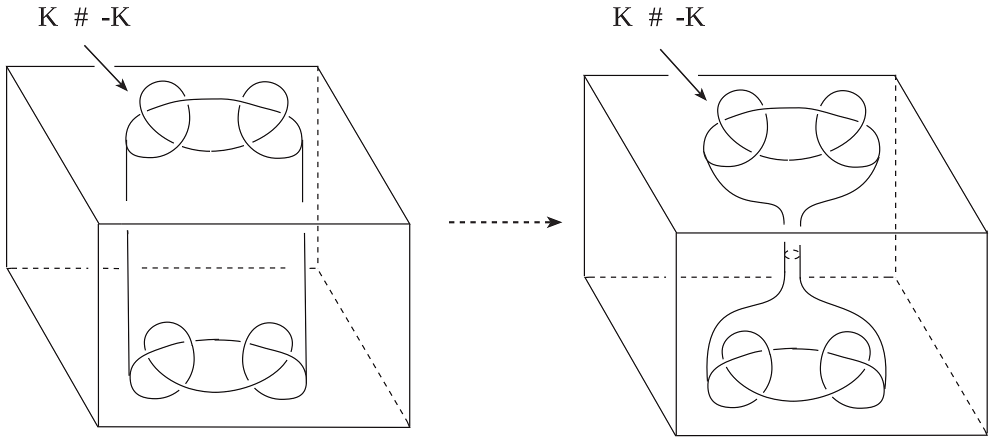


**Figure 8**

Now if we use the product concordance $s_0$ from $K\#(-K)$ to itself, i.e. the first picture of Figure 8, our algorithm says that changing the cusp neighborhood by $(S^3 - K\#(-K)) \times S^1$ is given by the handlebody of Figure 9, which is the same as Figure 10 (where the slice 1-handle is drawn as a usual handlebody). Whereas if we use the concordance $s$, described above, we get Figure 11. By an isotopy we see that Figure 11 is diffeomorphic to Figure 12 which is diffeomorphic to Figure 13, and Figure 13 is isotopic to Figure 14. By handle slides indicated in Figures 14 and 15 we obtain the second picture of Figure 15. By cancelling a 1-and 2- handle pairs we get the first picture of Figure 16. Then by a 2-handle slide, and cancelling an unknotted 0-framed 2-handle by a 3-handle, we obtain the last picture of 16 which is the cusp $C$. So we proved $C_s = C$, but since $C \subset X$ and every self diffeomorphism of $\partial C$ extends to $C$ we conclude $X_s = X$ □

**Remark 2.1.** This theorem says that taking different elements $s \in C(K)$ can result changing the smooth structure of $X_s$. For example, if take any c-imbedded torus in a smooth manifold $X$ with $SW_X \neq 0$, and $K$ is the trefoil knot, and if $s_0 \in C(K\#(-K))$ is the product concordance, then by Theorem 1.1

$$SW_{X_{s_0}} = SW_{X_{K\#(-K)}} = SW_X.\Delta_{K\#(-K)} \neq SW_X$$

hence $X_{s_0} \neq X$. But on the other hand Theorem 2.3 says that there is $s \in C(K\#(-K))$ with $X_s = X$, so $X_{s_0} \neq X_s$. In particular, this shows that the concordances $s_0$ and $s$ are different. This gives a hope the that hard to distinguish knot concordances might be distinguished by the Seiberg-Witten invariants of the associated manifolds $X_s$.

**Remark 2.2.** Let $s \in C(K)$, and $f_s : S^1 \times S^1 \hookrightarrow S^3 \times S^1$ be the corresponding imbedding **R** be One can ask whether Theorem 1.1 generalizes to $SW_{X_s} = SW_X.\Delta_s(t)$?, where $\Delta_s(t)$ is Alexander polynomial associated to this imbedding.

### 2.1. A twisted version of $X_K$

Another version of the operation $X \rightsquigarrow X_K$ that was previously introduced in [3], which, in a sense, is the square root of this operation: Let $K$ is an invertible knot, i.e. an orientation preserving involution $\tau : \mathbf{R}^3 \to \mathbf{R}^3$ (e.g. $180^0$ rotation) restricts to $K$ as an involution with two fixed points, and let $N$ be the open tubular neighborhood of $K$. Then we can form the following $S^3 - N$ bundle over $S^1$:

$$(S^3 - N)\tilde{\times}S^1 = (S^3 - N) \times [0,1]/(x,0) \sim (\tau(x), 1)$$

Define a $D^2$-bundle over the Klein bottle $\mathbf{K}^2$ by $C^* = S^1 \times D^2 \times [0,1]/(x,0) \sim (\tau(x), 1)$. Then $(S^3 - N)\tilde{\times}S^1$ and $C^*$ have the same boundaries, and so if $X$ is a smooth 4-manifold with $C^* \subset X$, we can construct

$$X^*_K = (X - C^*) \smile_{\varphi} (S^3 - N)\tilde{\times}S^1$$

where $\varphi : \partial C^* \to \partial(K \times S^1)\tilde{\times}S^1$ is a diffeomorphism with $\varphi(p \times \partial D^2) = K \times p$. The operation $X \rightsquigarrow X^*_K$, is a certain generalization of the Fintushel-Stern operation $X \rightsquigarrow X_K$ done using a 'Klein bottle' instead of a torus. By using the previous arguments one can see see that the handlebody picture of the operation $X \rightsquigarrow X^*_K$ is given by Figure 17. The first picture of Figure 17 is $C^*$ and the second is $(S^3 - N)\tilde{\times}S^1$. The rest of $X^*_K$ is obtained by simply by drawing the images of the additional handles under the diffeomorphism $\varphi : \partial C^* \to \partial(S^3 - N)\tilde{\times}S^1$. For convenience, in Figure 17 the images of the linking circles $a, b, c$ under $\varphi$ are indicated.

Now, call an imbedded Klein bottle $\mathbf{K}^2 \subset X$ *c-imbedded Klein bottle*, if

$$\mathbf{K}^2 \subset C^* \subset U \subset X$$

where U is either one of the manifolds of Figure 18, and $\pi_1(U) \to \pi_1(X)$ injects (notice $\pi_1(U) = \mathbf{Z}_2$). Then it is easy to see that the obvious 2-fold cover $\tilde{X} \to X$ contains a cusp $C$ (so it contains a *c-imbedded* $T^2$), and the operation $X \rightsquigarrow X_K^*$ lifts to the usual Fintushel-Stern knot surgery operation $\tilde{X} \rightsquigarrow \tilde{X}_K$ (done using this $T^2$). Hence if $SW_{\tilde{X}} \neq 0$ and $\Delta_K(t) \neq 0$, the operation $X \rightsquigarrow X_K^*$ changes the smooth structure of $X$, i.e. $X \neq X_K^*$. For example, $X$ can be a manifold with boundary, which is 2-fold covered by a Stein manifold $\tilde{X}$ (so $\tilde{X}$ compactifies into a closed symplectic manifold which Theorem 1.1 applies). It is easy to check that the first manifold of Figure 18 is such an example.

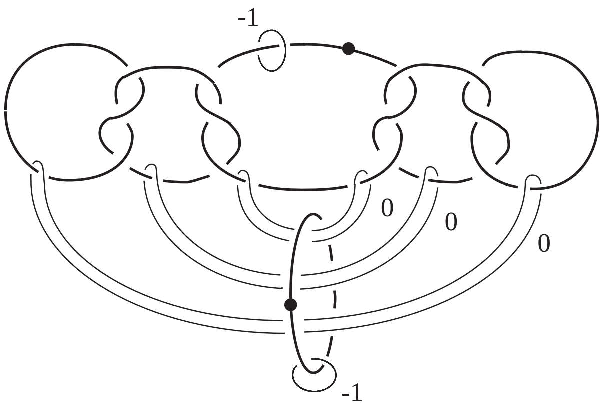


**Figure 9**

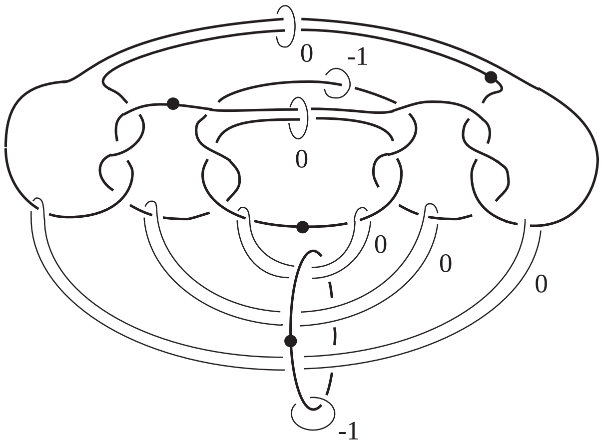


**Figure 10**

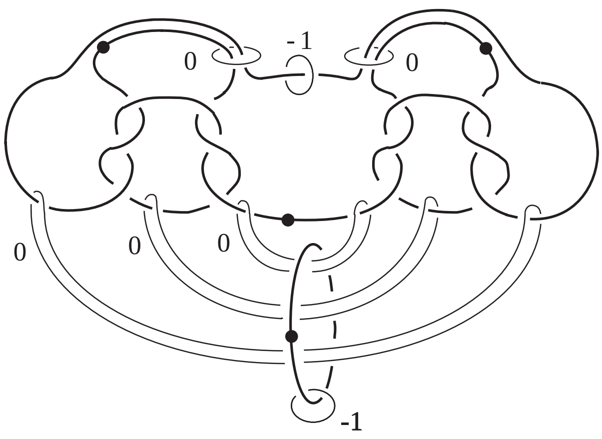


**Figure 11**

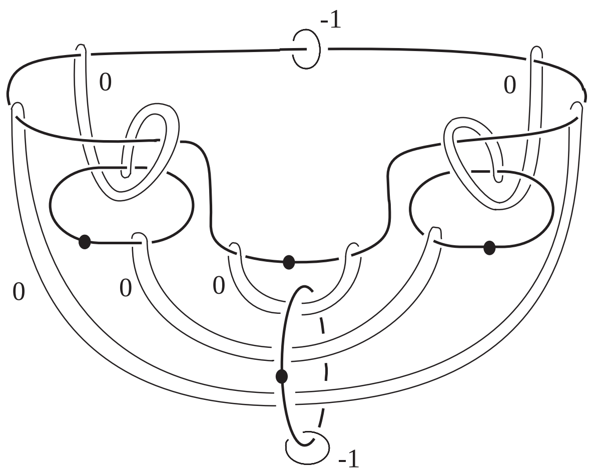


**Figure 12**

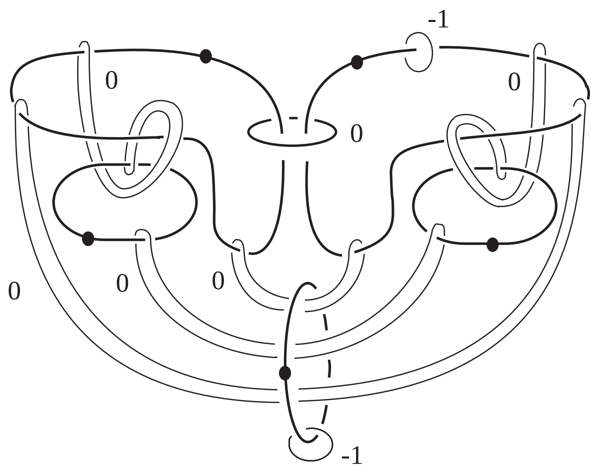


**Figure 13**

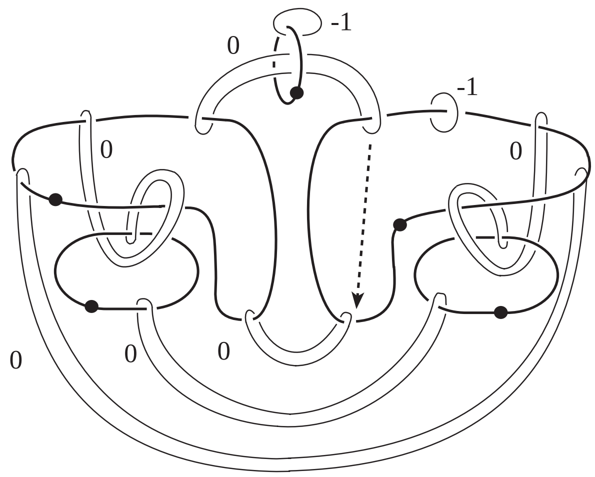


**Figure 14**

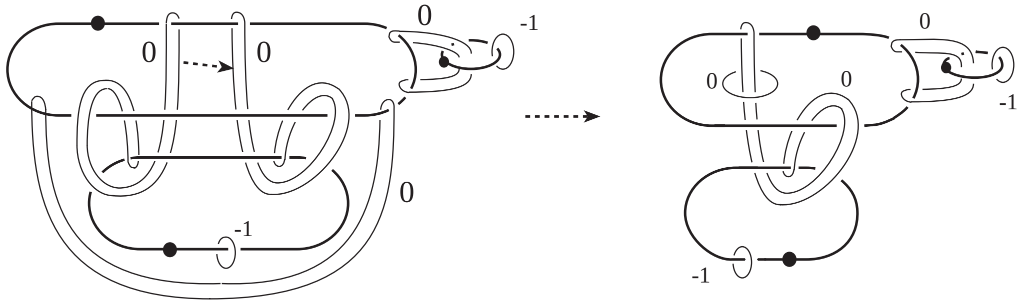


**Figure 15**

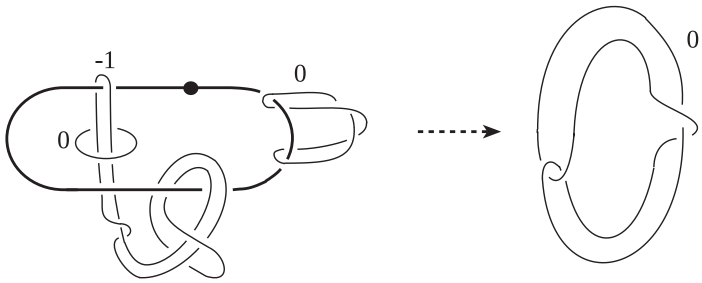


**Figure 16**

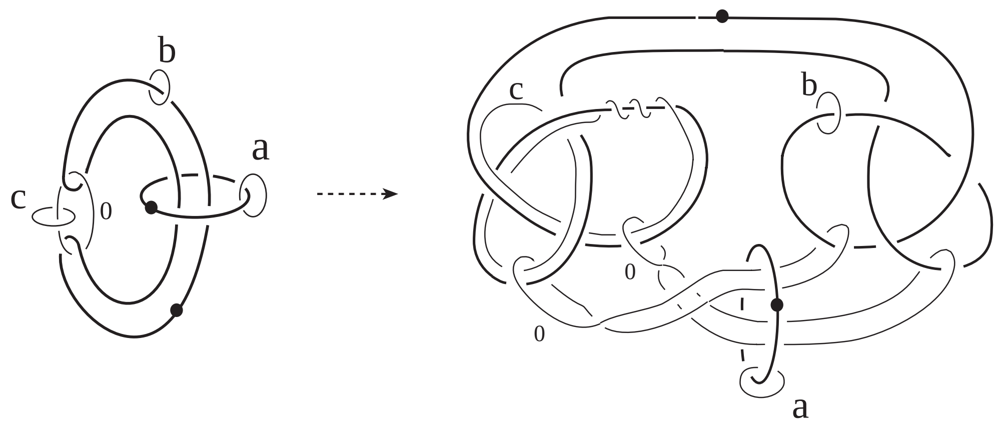


**Figure 17**

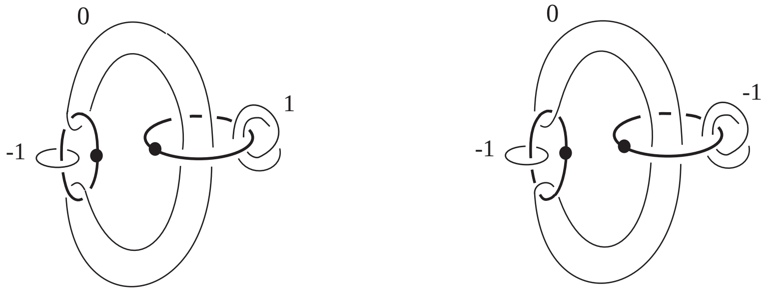


**Figure 18**

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