Turk J Math
26 (2002) , 199 – 205.
© TÜBİTAK

Characterizations of Artinian and Noetherian Gamma-Rings in Terms of Fuzzy Ideals

Mehmet Ali Öztürk, Mustafa Uckun and Young Bae Jun

Abstract

Using fuzzy ideals, characterizations of Noetherian Γ-rings are given, and a condition for a Γ-ring to be Artinian is also given.

Key words and phrases: (Artinian, Noetherian) Γ-ring, fuzzy left (right) ideal, Γ-residue class ring.

1. Introduction

The notion of a fuzzy set in a set was introduced by L. A. Zadeh [6], and since then this concept has been applied to various algebraic structures. N. Nobusawa [5] introduced the notion of a Γ-ring, a concept more general than a ring. W. E. Barnes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. W. E. Barnes [1], S. Kyuno [3] and J. Luh [4] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. Y. B. Jun and C. Y. Lee [2] applied the concept of fuzzy sets to the theory of Γ-rings. In this paper, using fuzzy ideals, we discuss characterizations of Noetherian Γ-rings, and we give a condition for a Γ-ring to be Artinian.

2000 *Mathematics Subject Classification:* 16P99, 03E72.

2. Preliminaries

Let M and Γ be two abelian groups. If for all $x, y, z \in M$ and all α, $\beta \in \Gamma$ the conditions

- $x\alpha y \in M$,
- $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z$,
- $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied, then we call M a Γ*-ring.* By a *right* (resp. *left*) *ideal* of a Γ-ring M we mean an additive subgroup U of M such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U is both a right and a left ideal, then we say that U is an *ideal* of M. Let U be an ideal of a Γ-ring M. If for each $a+U$, $b+U$ in the factor group M/U, and each $\gamma \in \Gamma$, we define $(a+U)\gamma(b+U) = a\gamma b + U$; then M/U is a Γ-ring which is called the Γ*-residue class ring* of M with respect to U (see [3]). For any subsets A and B of a Γ-ring M, by $A \subset B$ we exclude the possibility that $A = B$. A Γ-ring M is said to satisfy the *left* (*right*) *ascending chain condition* of left (right) ideals (or to be *left* (*right*) *Noetherian*) if every strictly increasing sequence $U_1 \subset U_2 \subset U_3 \subset \cdots$ of left (right) ideals of M is of finite length. A Γ-ring M is said to satisfy the *left* (*right*) *descending chain condition* of left (right) ideals (or to be *left* (*right*) *Artinian*) if every strictly decreasing sequence $V_1 \supset V_2 \supset V_3 \supset \cdots$ of left (right) ideals of M is of finite length. A Γ-ring M is *left* (resp. *right*) *Noetherian* if M satisfies the left (right) ascending chain condition on left (resp. right) ideals. M is said to be *Noetherian* if M is both left and right Noetherian. A Γ-ring M is *left* (resp. *right*) *Artinian* if M satisfies the left (right) descending chain condition on left (resp. right) ideals. M is said to be *Artinian* if M is both left and right Artinian.

We now review some fuzzy logic concepts. A fuzzy set μ in a Γ-ring M is called a *fuzzy left* (resp. *right*) *ideal* of M ([2]) if it satisfies

(FI1) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$

(FI2) $\mu(x\gamma y) \geq \mu(y)$ (resp. $\mu(x\gamma y) \geq \mu(x)$)

for all $x, y \in M$ and $\gamma \in \Gamma$. A fuzzy set μ in a Γ-ring M is called a *fuzzy ideal* of M if μ is both a fuzzy left and a fuzzy right ideal of M. We note from [2] that if μ is a fuzzy left (right) ideal of a Γ-ring M then $\mu(0) \geq \mu(x)$ for all $x \in M$, and μ is a fuzzy ideal of a Γ-ring M if and only if it satisfies (FI1) and

(FI3) $\mu(x\gamma y) \geq \max\{\mu(x), \mu(y)\}$ for all $x, y \in M$ and $\gamma \in \Gamma$.

3. Main results

Theorem 3.1. *Let U be an ideal of a Γ-ring M. If μ is a fuzzy left (right) ideal of M, then the fuzzy set $\bar{\mu}$ of M/U defined by*

$$\bar{\mu}(a+U) = \sup_{x \in U} \mu(a+x)$$

is a fuzzy left (right) ideal of the Γ-residue class ring M/U of M with respect to U.

Proof. Let $a, b \in M$ be such that $a + U = b + U$. Then $b = a + y$ for some $y \in U$, and so

$$\bar{\mu}(b+U) = \sup_{x \in U} \mu(b+x) = \sup_{x \in U} \mu(a+y+x) = \sup_{x+y=z \in U} \mu(a+z) = \bar{\mu}(a+U).$$

Hence $\bar{\mu}$ is well-defined. For any $x + U, y + U \in M/U$ and $\gamma \in \Gamma$, we have

$$\begin{aligned}
\bar{\mu}((x+U)-(y+U)) &= \bar{\mu}((x-y)+U) = \sup_{z \in U} \mu((x-y)+z) \\
&= \sup_{z=u-v \in U} \mu((x-y)+(u-v)) \\
&= \sup_{u,v \in U} \mu((x+u)-(y+v)) \\
&\geq \sup_{u,v \in U} \min\{\mu(x+u), \mu(y+v)\} \\
&= \min\Big\{\sup_{u \in U} \mu(x+u), \sup_{v \in U} \mu(y+v)\Big\} \\
&= \min\Big\{\bar{\mu}(x+U), \bar{\mu}(y+U)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mu}((x+U)\gamma(y+U)) &= \bar{\mu}(x\gamma y+U) = \sup_{z \in U} \mu(x\gamma y+z) \\
&\geq \sup_{z \in U} \mu(x\gamma y + x\gamma z) \qquad \text{because} \quad x\gamma z \in U \\
&= \sup_{z \in U} \mu(x\gamma(y+z)) \geq \sup_{z \in U} \mu(y+z) \\
&= \bar{\mu}(y+U).
\end{aligned}$$

Similarly, $\bar{\mu}((x+U)\gamma(y+U)) \geq \bar{\mu}(x+U)$. Hence $\bar{\mu}$ is a fuzzy left (right) ideal of M/U. □

Theorem 3.2. *Let U be an ideal of a Γ-ring M. Then there is a one-to-one correspondence between the set of fuzzy left ideals μ of M such that $\mu(0) = \mu(u)$ for all $u \in U$ and the set of all fuzzy left ideals $\bar{\mu}$ of M/U.*

Proof. Let μ be a fuzzy left ideal of M. Using Theorem 3.1, we find that $\bar{\mu}$ defined by $\bar{\mu}(a+U) = \sup_{x\in U} \mu(a+x)$ is a fuzzy left ideal of M/U. Since $\mu(0) = \mu(u)$ for all $u \in U$, we get

$$\mu(a+u) \geq \min\{\mu(a),\, \mu(u)\} = \mu(a).$$

Again, $\mu(a) = \mu(a+u-u) \geq \min\{\mu(a+u),\, \mu(u)\} = \mu(a+u)$. Hence $\mu(a+u) = \mu(a)$ for all $u \in U$, that is, $\bar{\mu}(a+U) = \mu(a)$. Therefore the correspondence $\mu \mapsto \bar{\mu}$ is injective. Now let $\bar{\mu}$ be any fuzzy left ideal of M/U and define a fuzzy set μ in M by $\mu(a) = \bar{\mu}(a+U)$ for all $a \in M$. For every $x, y \in M$ and $\gamma \in \Gamma$, we have

$$\begin{aligned} \mu(x-y) &= \bar{\mu}((x-y)+U) = \bar{\mu}((x+U)-(y+U)) \\ &\geq \min\{\bar{\mu}(x+U),\, \bar{\mu}(y+U)\} = \min\{\mu(x),\, \mu(y)\}, \end{aligned}$$

and $\mu(x\gamma y) = \bar{\mu}(x\gamma y+U) = \bar{\mu}((x+U)\gamma(y+U)) \geq \bar{\mu}(y+U) = \mu(y)$. Thus μ is a fuzzy left ideal of M. Note that $\mu(z) = \bar{\mu}(z+U) = \bar{\mu}(U)$ for all $z \in U$, which shows that $\mu(z) = \mu(0)$ for all $z \in U$. This completes the proof. □

Theorem 3.3. *If every fuzzy left ideal of a Γ-ring M has finite number of values, then M is left Artinian.*

Proof. Suppose that every fuzzy left ideal of a Γ-ring M has finite number of values and M is not left Artinian. Then there exists strictly descending chain $U_0 \supset U_1 \supset U_2 \supset \cdots$ of left ideals of M. Define a fuzzy set μ in M by

$$\mu(x) = \begin{cases} \frac{n}{n+1} & \text{if } x \in U_n \setminus U_{n+1},\ n = 0, 1, 2, \cdots, \\ 1 & \text{if } x \in \bigcap_{n=0}^{\infty} U_n, \end{cases}$$

where U_0 stands for M. Let us prove that $\mu(x-y) \geq \min\{\mu(x),\, \mu(y)\}$ for all $x, y \in M$. Let $x, y \in M$. Then $x-y \in U_n \setminus U_{n+1}$ for some n ($n = 0, 1, 2, \cdots$), and so either $x \notin U_{n+1}$

or $y \notin U_{n+1}$. So for definiteness, let $y \in U_k \setminus U_{k+1}$ for $k \leq n$. It follows that

$$\mu(x-y) = \frac{n}{n+1} \geq \frac{k}{k+1} \geq \min\{\mu(x),\, \mu(y)\}.$$

Next, let us show that $\mu(x\gamma y) \geq \mu(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$. There exists a non-negative integer n such that $x\gamma y \in U_n \setminus U_{n+1}$. Then $y \notin U_{n+1}$, and hence $y \in U_k \setminus U_{k+1}$ for $k \leq n$. Hence

$$\mu(x\gamma y) = \frac{n}{n+1} \geq \frac{k}{k+1} = \mu(y).$$

Therefore μ is a fuzzy left ideal of M and μ has infinite number of different values. This contradiction proves that M is a left Artinian Γ-ring. □

Theorem 3.4. *A Γ-ring M is left Noetherian if and only if the set of values of any fuzzy left ideal of M is a well ordered subset of* $[0, 1]$.

Proof. Suppose that μ is a fuzzy left ideal of M whose set of values is not a well ordered subset of $[0, 1]$. Then there exists a strictly decreasing sequence $\{\lambda_n\}$ such that $\mu(x_n) = \lambda_n$. Denote by U_n the set $\{x \in M \mid \mu(x) \geq \lambda_n\}$. Then $U_1 \subset U_2 \subset U_3 \subset \cdots$ is a strictly ascending chain of left ideals of M, which contradicts that M is left Noetherian.

Conversely, assume that the set of values of any fuzzy left ideal of M is a well ordered subset of $[0, 1]$ and M is not a left Noetherian Γ-ring. Then there exists a strictly ascending chain

$$U_1 \subset U_2 \subset U_3 \subset \cdots \tag{3.1}$$

of left ideals of M. Note that $U := \bigcup_{i \in \mathbf{N}} U_i$ is a left ideal of M, where $\mathbf{N}$ is the set of all natural numbers. Define a fuzzy set μ in M by

$$\mu(x) = \begin{cases} 0 & \text{if } \ x \notin U_i, \\ \frac{1}{k} & \text{where } \ k = \min\{i \in \mathbf{N} \mid x \in U_i\}. \end{cases}$$

It can be easily seen that μ is a fuzzy left ideal of M. Since the chain (3.1) is not terminating, μ has a strictly descending sequence of values, contradicting that the value set of any fuzzy left ideal is well ordered. Consequently, M is left Noetherian. □

Lemma 3.5. ([2, Theorem 3]) *A fuzzy set μ in a Γ-ring M is a fuzzy left (right) ideal of M if and only if for every $\lambda \in [0,1]$, the set $U(\mu;\lambda) := \{x \in M \mid \mu(x) \geq \lambda\}$ is a left (right) ideal of M when it is nonempty.*

Lemma 3.6. *Let $S = \{\lambda_n \in (0,1) \mid n \in \mathbf{N}\} \cup \{0\}$, where $\lambda_i > \lambda_j$ whenever $i < j$. Let $\{U_n \mid n \in \mathbf{N}\}$ be a family of left ideals of a Γ-ring M such that $U_1 \subset U_2 \subset U_3 \subset \cdots$. Then a fuzzy set μ in M defined by*

$$\mu(x) = \begin{cases} \lambda_1 & \text{if } x \in U_1, \\ \lambda_n & \text{if } x \in U_n \setminus U_{n-1},\ n = 2,3,\cdots, \\ 0 & \text{if } x \in M \setminus \bigcup_{n=1}^{\infty} U_n, \end{cases}$$

is a fuzzy left ideal of M.

Proof. Using Lemma 3.5, the proof is straightforward.

Theorem 3.7. *Let $S = \{\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots\} \cup \{0\}$ where $\{\lambda_n\}$ is a fixed sequence, strictly decreasing to* 0 *and* $0 < \lambda_n < 1$. *Then a Γ-ring M is left Noetherian if and only if for each fuzzy left ideal μ of M, Im(μ) $\subset S$ implies that there exists $n_0 \in \mathbf{N}$ such that Im(μ) $\subset \{\lambda_1, \lambda_2, \cdots, \lambda_{n_0}\} \cup \{0\}$.*

Proof. If M is left Noetherian, then Im(μ) is a well ordered subset of $[0,1]$ by Theorem 3.4 and so the condition is necessary by noticing that a set is well ordered if and only if it does not contain any infinite descending sequence. Conversely, if possible let M be not left Noetherian. Then there exists a strictly ascending chain of left ideals of M $U_1 \subset U_2 \subset U_3 \subset \cdots$. Define a fuzzy set μ in M by

$$\mu(x) = \begin{cases} \lambda_1 & \text{if } x \in U_1, \\ \lambda_n & \text{if } x \in U_n \setminus U_{n-1},\ n = 2,3,\cdots, \\ 0 & \text{if } x \in M \setminus \bigcup_{n=1}^{\infty} U_n. \end{cases}$$

Then, by Lemma 3.6, μ is a fuzzy left ideal of M. This contradicts our assumption. Hence M is left Noetherian. □

References

[1] W. E. Barnes, On the Γ-rings of Nobusawa, *Pacific J. Math.* **18** (1966), 411-422.

[2] Y. B. Jun and C. Y. Lee, Fuzzy Γ-rings, *Pusan Kyongnam Math. J. (presently, East Asian Math. J.)* **8(2)** (1992), 163-170.

[3] S. Kyuno, On prime gamma rings, *Pacific J. Math.* **75(1)** (1978), 185-190.

[4] J. Luh, On the theory of simple Γ-rings, *Michigan Math. J.* **16** (1969), 65-75.

[5] N. Nobusawa, On a generalization of the ring theory, *Osaka J. Math.* **1** (1964), 81-89.

[6] L. A. Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965), 338-353.

M. A. ÖZTÜRK
Received 27.07.2001
Department of Mathematics
Faculty of Arts and Sciences
Cumhuriyet University
58140-Sivas-TURKEY
e-mail: maozturk@cumhuriyet.edu.tr
M. UCKUN
Department of Mathematics
Faculty of Arts and Sciences
İnönü University
44069-Malatya-TURKEY
e-mail: muckun@inonu.edu.tr
Y. B. JUN
Department of Mathematics Education
Gyeongsang National University
Chinju 660-701-KOREA
e-mail : ybjun@nongae.gsnu.ac.kr