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On Non-Existence of Korovkin's Theorem in the Space of L_p-locally Integrable Functions

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Abstract

It is shown that a Korovkin-type theorem does not hold in the weighted space of L_p-locally integrable functions on the whole real axis.

Key words and phrases: Linear positive operators, Korovkin-type theorem, Weighted $L_p(loc)$ space

1. The problem of convergence of sequences of linear positive operators in the space of functions, which are continuous on a finite interval $[a, b]$ and bounded on the whole real axis, was systematically investigated in Korovkin's monograph [1]. Many generalizations and extensions of Korovkin's classical theorem are known (we refer to monograph [2] for a bibliography). In particular, it was shown in papers [3] and [4]* that Korovkin's theorem does not hold in the weighted spaces of functions f,which are continuous on the whole axis and satisfy the inequality $|f(x)| \leq M_f \rho(x)$, where M_f is a positive constant depending on the function f and $\rho(x) \geq 1$ is a continuous and increasing function on $(-\infty, \infty)$. This space is a linear normed space endowed with the norm

$$\|f\|_\rho = \sup_{-\infty<x<\infty} \frac{|f(x)|}{\rho(x)}.$$

The aim of this paper is to investigate the existence of Korovkin-type theorems in the space of L_p-locally integrable functions. Note that the problem of convergence of

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sequences of linear positive operators, acting from $L_p(a,b)$ to $L_p(a,b)$, has been studied by many authors. We refer the reader to the papers $[5]-[10]$. Note that all results mentioned are restricted to the case of the finite interval $[a,b]$.

We will consider the problem of convergence of sequences of linear positive operators in the space of locally integrable functions on the whole real axis.

Let $w(x)=1+x^2$, $-\infty<x<\infty$, and denote by $L_{p,w}(loc)$ the space of measurable functions f satisfying the inequality

$$\left(\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} |f(t)|^p\,dt\right)^{\frac{1}{p}} \le M_f w(x) \ , \quad -\infty<x<\infty,$$

where $p\ge 1$ and M_f is a constant depending on the function f. Setting

$$\|f\|_{p,w}=\sup_{-\infty<x<\infty}\frac{\left(\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} |f(t)|^p\,dt\right)^{\frac{1}{p}}}{w(x)},$$

we see that $L_{p,w}(loc)$ is a linear normed space with this norm.

We will deal with the following problem.

Let L_n, $n=1,2,...$, be a sequence of linear positive operators, acting from $L_{p,w}(loc)$ to $L_{p,w}(loc)$ and satisfying the following two conditions:

i) The norms of these operators are uniformly bounded;

ii) For $m=0,1,2$

$$\lim_{n\to\infty}\|L_n(t^m;x)-x^m\|_{p,w}=0. \tag{1.1}$$

Is it possible to assert then that for each function $f\in L_{p,w}(loc)$

$$\lim_{n\to\infty}\|L_nf-f\|_{p,w}=0\ ?$$

An affirmative solution to this problem would lead to a Korovkin-type theorem in $L_{p,w}(loc)$.

However, we are going to show that the answer is negative.

2. Main result

Our main result is the following.

Theorem 1. There exists a sequence of linear positive operators L_n, acting from $L_{p,w}(loc)$ to $L_{p,w}(loc)$ and satisfying conditions i), ii), and there exists a function $f^* \in L_{p,w}(loc)$ for which

$$\overline{\lim_{n\to\infty}} \|L_n f^* - f^*\|_{p,w} \geq 2^{1-\frac{1}{p}} .$$

Proof. We define a sequence of operators L_n by the formulas

$$L_n(f,x) = \begin{cases} \frac{x^2}{\left(x+\frac{1}{2}\right)^2} f(x+\frac{1}{2}), & \text{if } \ (n-\frac{1}{2}) \leq x \leq n \\ \\ f(x), & \text{otherwise.} \end{cases}$$

Obviously that L_n are linear positive operators, acting from $L_{p,w}(loc)$ to $L_{p,w}(loc)$ and

$$\|L_n f\|_{p,w} \leq 4 \|f\|_{p,w} .$$

Since

$$\|L_n(y^m,t) - t^m\|_{p,w} \leq \sup_{(n-\frac{1}{2})\leq x\leq n} \frac{\left(x+\frac{1}{2}\right)^m}{1+x^2} \leq \frac{\left(n+\frac{1}{2}\right)^m}{1+\left(n-\frac{1}{2}\right)^2}$$

for $m = 0,1$ and $L_n\left(t^2,x\right) = x^2$, conditions (i) holds.

Consider the function

$$f^*(x) = \begin{cases} x^2, & \text{if } \ x \in \bigcup\limits_{k=1}^{\infty} \left[k-\frac{1}{2},k\right) \\ -x^2, & \text{if } \ x \in \bigcup\limits_{k=0}^{\infty} \left(k,k+\frac{1}{2}\right] \\ 0\ , & \text{if } \ x<0 \end{cases}$$

which obviously belongs to $L_{p,w}(loc)$. For $n-\frac{1}{2} \leq y \leq n$ obviously $f^*(y) = y^2$, $f^*(y+\frac{1}{2}) = -(y+\frac{1}{2})^2$ and therefore

$$\begin{aligned} \|L_n f^* - f^*\|_{p,w} &\geq \frac{1}{w(n)} \left(\int\limits_{n-\frac{1}{2}}^{n} \left| \frac{y^2}{(y+\frac{1}{2})^2} f^*(y+\frac{1}{2}) - f^*(y) \right|^p dy \right)^{\frac{1}{p}} \\ &= \frac{1}{w(n)} \left(\int\limits_{n-\frac{1}{2}}^{n} \left| \frac{y^2}{(y+\frac{1}{2})^2} (y+\frac{1}{2})^2 + y^2 \right|^p dy \right)^{\frac{1}{p}} \\ &\geq 2^{1-\frac{1}{p}} \frac{\left(n-\frac{1}{2}\right)^2}{1+n^2} \end{aligned}$$

by the definition of $w(x)$. The theorem is proved.

3. In this section we will give an affirmative statement on approximation in $L_{p,w}(loc)$.

First of all, let $w_\alpha(x) = 1 + |x|^{2+\alpha}$, $\alpha > 0$, and let $L_{p,w_\alpha}(loc)$ be the space of measurable functions f with the finite norm

$$\|f\|_{p,w_\alpha} = \sup_{-\infty<x<\infty} \frac{1}{w_\alpha(x)} \left(\int\limits_{x-\frac{1}{2}}^{x+\frac{1}{2}} |f(t)|^p \, dt \right)^{\frac{1}{p}} .$$

Obviously, for any numbers a, b $(a < b)$

$$L_p(-\infty,\infty) \subset L_{p,w_\alpha}(loc) \subset L_{p,w}(loc) \subset L_p(a,b).$$

Let also $CB(-\infty,\infty)$ be the space of all continuous and bounded functions f on the whole real axis with the norm

$$\|f\|_{CB} = \sup_{-\infty<x<\infty} |f(x)|.$$

Lemma 1. Let $f\in L_{p,w}(loc)$. Then given $\varepsilon>0$ there exists a function $g\in CB\left(-\infty,\infty\right)$ such that

$$\left\|f-g\right\|_{p,w_\alpha}<\varepsilon$$

for any $\alpha>0$.

Proof. Using the inequality

$$\sup_{|x|\le x_0}\frac{1}{w_\alpha\left(x\right)}\left(\int\limits_{x-\frac{1}{2}}^{x+\frac{1}{2}}\left|f(t)\right|^p dt\right)^{\frac{1}{p}}\le\left(\int\limits_{-\left(x_0+\frac{1}{2}\right)}^{\left(x_0+\frac{1}{2}\right)}\left|f(t)\right|^p dt\right)^{\frac{1}{p}},$$

and the well known Lusin Theorem, we can find a continuous function g_1 such that

$$\sup_{|x|\le x_0}\frac{1}{w_\alpha\left(x\right)}\left(\int\limits_{x-\frac{1}{2}}^{x+\frac{1}{2}}\left|f(t)-g_1\left(t\right)\right|^p dt\right)^{\frac{1}{p}}<\varepsilon \tag{3.2}$$

holds for any $\varepsilon>0$.

Since by the definition of $L_{p,w}(loc)$

$$\sup_{|x|>x_0}\frac{1}{w_\alpha\left(x\right)}\left(\int\limits_{x-\frac{1}{2}}^{x+\frac{1}{2}}\left|f(t)\right|^p dt\right)^{\frac{1}{p}}\le M_f\sup_{|x|>x_0}\frac{w\left(x\right)}{w_\alpha\left(x\right)}, \tag{3.3}$$

we can choose $x_0>0$ so large that the inequality

$$\sup_{|x|>x_0}\frac{w\left(x\right)}{w_\alpha\left(x\right)}<\varepsilon \tag{3.4}$$

holds for any $\varepsilon>0$.

Therefore, denoting by g a continuous and bounded function on the whole real axis, which coincides with g_1 on $\left(-x_0-\frac{1}{2}\,,\,x_0+\frac{1}{2}\right)$, we complete the proof by using (3.2), (3.3) and (3.4).

Lemma 2. Let L_n be a sequence of linear positive operators acting from $L_{p,w}(loc)$ to $L_{p,w}(loc)$ and satisfying conditions (i) and (ii). Then for any $f\in CB\left(-\infty,\infty\right)$

$$\lim_{n\to\infty}\left\|L_nf-f\right\|_{p,w_\alpha}=0\ .$$

Proof. We have

$$\lim_{n\to\infty} \|L_n f - f\|_{p,w_\alpha} \leq \|L_n(|f(y) - f(t)|, t)\|_{p,w_\alpha} + \| f \|_{CB} \|L_n 1 - 1\|_{p,w}$$

and the last term tends to zero by (1.1).

Consider the first term on the right hand side. Since f is continuous and bounded we can write the inequality [1] as

$$|f(y) - f(t)| < \varepsilon + \frac{2\| f \|_{CB}}{\delta^2}(y-t)^2$$

and for x_0 satisfying (3.4) the following inequality holds:

$$\begin{aligned}\|L_n(|f(y) - f(t)|, t)\|_{p,w_\alpha} &\leq (2\| f \|_{CB} + 1)\|L_n 1\|_{p,w}\,\varepsilon \\ &+ \frac{2\| f \|_{CB}}{\delta^2} \sup_{|x|\leq x_0} \frac{1}{w(x)} \left(\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} L_n^p\left((y-t)^2, t\right) dt \right)^{\frac{1}{p}}\end{aligned}$$

It remains to note that by condition (i) the last term tends to zero as $n \to \infty$ and the $\|L_n 1\|_{p,w}$ are uniformly bounded.

Theorem 2. Let L_n be a sequence of linear positive operators acting from $L_{p,w}(loc)$ to $L_{p,w}(loc)$ as well as from $L_{p,w_\alpha}(loc)$ to $L_{p,w_\alpha}(loc)$ and satisfying conditions (i) and (ii). Then for any function $f \in L_{p,w}(loc)$

$$\lim_{n\to\infty} \|L_n f - f\|_{p,w_\alpha} = 0$$

and the result fails to be true for $\alpha = 0$.

Proof. Using Lemma 1 and the uniform boundedness of $\|L_n\|$ we have $\|L_n\| \leq M$ and

$$\begin{aligned}\|L_n f - f\|_{p,w_\alpha} &\leq \|L_n(f - g, t)\|_{p,w_\alpha} + \|L_n g - g\|_{p,w_\alpha} + \|f - g\|_{p,w_\alpha} \\ &\leq (M+1)\|f - g\|_{p,w_\alpha} + \|L_n g - g\|_{p,w_\alpha}.\end{aligned}$$

The proof now follows from the Lemma 1 and Lemma 2. The last assertion of the theorem follows from Theorem 1.

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