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Asymptotic Formulas for the Eigenvalues of the Schrodinger Operator

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Abstract

In this paper, we obtain asymptotic formulas for the eigenvalues of the d-dimensional Schrodinger operator

$$L = -\Delta + q(x)$$

in d-dimensional parallelepiped F with Dirichlet and Neumann boundary conditions.

Let $\Omega = \{m_1 w_1 + m_2 w_2 + ... + m_d w_d : m_i \in Z, i = 1, 2, ..., d\}$ be a lattice in R^d with the reduced orthonormal basis

$$w_1 = (a_1, 0, ..., 0), w_2 = (0, a_2, 0, ..., 0), ..., w_d = (0, ..., 0, a_d)$$

and $\Gamma = \{m_1\gamma_1 + m_2\gamma_2 + ... + m_d\gamma_d : m_i \in Z, i = 1, 2, ..., d\}$ be the dual lattice of Ω, where the vectors $\{\gamma_i\}_{i=1}^d$ are biorthoganal to the vectors $\{w_i\}_{i=1}^d$. Denote by $F \equiv [0, a_1) \times [0, a_2) \times ... \times [0, a_d)$ the fundamental domain R^d/Ω of the lattice Ω.

We consider the Schrodinger operators $L_D(q(x))$ and $L_N(q(x))$, defined by the differential expression

$$Lu = -\Delta u + q(x)u \tag{1}$$

in $L_2(F)$ with the Dirichlet boundary condition

$$u|_{\partial F} = 0 \tag{2}$$

and the Neumann boundary condition

$$\frac{\partial u}{\partial n}|_{\partial F} = 0, \tag{3}$$

respectively.

Here ∂F denotes the boundary of F, $x = (x_1, x_2, ...x_d) \in R^d, d \geq 2, \Delta$ is the Laplace operator in R^d, $\frac{\partial}{\partial n}$ denotes differentiation along outward normal n and $q(x)$ is a real valued, periodic (with respect to lattice Ω) function of $W_2^l(F)$, where $l \geq \frac{(d+2)(d-1)}{2}+d+1$.

First asymptotic formula for the eigenvalue of Schrodinger operator in parallelpiped with quasiperiodic boundary condition is obtained in papers [6], [7], [8]. The other asymptotic formulas for quasiperiodic boundary conditions in two and three dimensional cases are obtained in [4], [5], [1], [2]. The asymptotic formula for Dirichlet boundary condition in two dimension is obtained in [3].

We use the method of papers [7], [8] to find the asymptotic formula for the eigenvalues of $L_D(q(x))$ and $L_N(q(x))$ in arbitrary dimension.

We denote the eigenfunctions and the eigenvalues of the operator $L_D(q(x))$ by Φ_n and μ_n, respectively and denote the eigenfunctions and the eigenvalues of the operator $L_N(q(x))$ by Ψ_n and Λ_n , respectively.

The eigenvalues of the operators $L_D(0)$ and $L_N(0)$ are $|\gamma|^2$ for $\gamma \in \frac{\Gamma}{2}$. The normalized eigenfunctions of the operators $L_D(0)$ and $L_N(0)$, corresponding to the eigenvalue $|\gamma|^2$ are $\sum_{\alpha \in A_\gamma}(\text{sign } \prod_{i=1}^d \alpha_i)e^{\langle \alpha, x\rangle}$ and $\sum_{\alpha \in A_\gamma} e^{\langle \alpha, x\rangle}$, respectively, where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_d) \in \frac{\Gamma}{2}$ and

$$A_\gamma = \{\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in R^d : |\alpha_i| = |\gamma_i|, i = 1, 2, ..., d\}.$$

The potential $q(x)$ in the expression (1) can be written in the form

$$q(x) = \sum_{\gamma \in \frac{\Gamma}{2}} q_\gamma \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}, \tag{4}$$

where $q_\gamma = \int_F q(x)\overline{\sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}}dx$ for $\gamma \in \frac{\Gamma}{2}$ (without loss of generality we can assume $q_0 = \int_F q(x)dx = 0$.) are the Fourier coefficients of the potential $q(x)$ with respect to the basis $\{\sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle} : \gamma \in \frac{\Gamma}{2}\}$. Since $q(x) \in W_2^l(F)$, one can write

$$q(x)=\sum_{\gamma\in\Gamma(\rho^{\alpha})}q_{\gamma}\sum_{\beta\in A_{\gamma}}e^{i\langle\beta,x\rangle}+O(\rho^{-p\alpha}),\tag{5}$$

where $p=l-d$, $\Gamma(\rho^{\alpha})=\{\gamma\in\frac{\Gamma}{2}:0<|\gamma|<\rho^{\alpha}\}$, $\alpha=1/(d+2)$ and ρ is a large parameter.

Let us introduce the following notations:

$$M\equiv\sum_{\gamma\in\frac{\Gamma}{2}}|q_{\gamma}|\tag{6}$$

$$V_b(\rho^{\alpha})\equiv\{x\in R^d:||x|^2-|x+b|^2|<\rho^{\alpha}\}$$

$$U(\rho^{\alpha},p)\equiv R^d\backslash\bigcup_{b\in\Gamma(p\rho^{\alpha})}V_b(\rho^{\alpha}).$$

The domain $U(\rho^{\alpha},p)$ is said to be non-resonance domain and the eigenvalues $|\gamma|^2$ are called non-resonance eigenvalues, if $\gamma\in U(\rho^{\alpha},p)$. The domains $V_b(\rho^{\alpha})$ for all $b\in\Gamma(p\rho^{\alpha})$ are called resonance domains and the eigenvalues $|\gamma|^2$ are called resonance eigenvalues, if $\gamma\in V_b(\rho^{\alpha})$. Note that the number of non-resonance eigenvalues is essentially greater than the number of resonance eigenvalues. Namely, if $N_n(\rho)$ and $N_r(\rho)$ denote the number of $\gamma\in U(\rho^{\alpha},p)\bigcap(R(2\rho)\backslash R(\rho))$ and $\gamma\in\bigcup_{b\in\Gamma(p\rho^{\alpha})}V_b(\rho^{\alpha})\bigcap(R(2\rho)\backslash R(\rho))$, respectively, then

$$\frac{N_r(\rho)}{N_n(\rho)}=O(\rho^{(d+1)\alpha-1})=o(1)\tag{7}$$

for $(d+1)\alpha<1$ where $R_{\rho}=\{x\in R^d:|x|\le\rho\}$ (see remark 1).

In this paper, we obtain asymptotic formulas for non-resonance eigenvalues by using the following well-known formulas:

$$(\Lambda_n-|\gamma|^2)(\Psi_n,\sum_{\alpha\in A_{\gamma}}e^{i\langle\alpha,x\rangle})=(\Psi_n,q(x)\sum_{\alpha\in A_{\gamma}}e^{i\langle\alpha,x\rangle})\tag{8}$$

$$(\mu_n-|\gamma|^2)(\Phi_n,\sum_{\alpha\in A_{\gamma}}(\text{sign}\prod_{i=1}^{n}\alpha_i)e^{i\langle\alpha,x\rangle})=(\Phi_n,q(x)\sum_{\alpha\in A_{\gamma}}(\text{sign}\prod_{i=1}^{d}\alpha_i)e^{i\langle\alpha,x\rangle})\tag{9}$$

where $(.,.)$ is the inner product in $L_2(F)$.

Note that (8) can be obtained from

$$-\Delta\Psi_n(x) + q(x)\Psi_n(x) = \Lambda_n\Psi_n(x) \tag{10}$$

by multiplying both sides of this equation by $\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}$.

The Formula (9) can be obtained in the same way.

We say that $|\gamma|^2$ is of the order of ρ^2 and write $|\gamma|^2 \sim \rho^2$, if $c_1\rho^2 < |\gamma|^2 < c_2\rho^2$, where by $c_i, i = 1, 2, ...$ we denote the positive, independent on ρ constants whose exact values are not important.

Lemma 1 *Let $|\gamma|^2$ be the eigenvalue of the operators $L_D(0)$ and $L_N(0)$ of the order of ρ^2 . Then there are n_1 and n_2 such that $|\Lambda_{n_1} - |\gamma|^2| < 2M$,*
$|\mu_{n_2} - |\gamma|^2| < 2M$,$|(\Psi_{n_1}, \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})| > c_3\rho^{\frac{-(d-1)}{2}}$ and
$|(\Phi_{n_2}, \sum_{\alpha\in A_\gamma}(sign\prod_{i=1}^{d}\alpha_i)e^{i\langle\alpha,x\rangle})| > c_4\rho^{\frac{-(d-1)}{2}}$, where M is the number defined in (6).

proof: It is well known that the set of eigenfunctions Ψ_n of the self-adjoint operator $L_N(q(x))$ is an orthonormal basis in $L_2(F)$. Using (8) and (6) we get

$$\sum_{n:|\Lambda_n-|\gamma|^2|>2M} |(\Psi_n(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})|^2 \leq \frac{1}{4}.$$

Hence by the Parsevals equality, we have

$$\sum_{n:|\Lambda_n-|\gamma|^2|\leq 2M} |(\Psi_n(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})|^2 > \frac{3}{4}. \tag{11}$$

On the other hand, it is well known that if $a \sim \rho$ then the number of $\gamma \in \frac{\Gamma}{2}$ satisfying $||\gamma| - a| < 1$ is less than $c_5\rho^{d-1}$. Therefore the number of eigenvalues of $L_N(0)$ lying in $(a^2 - \rho, a^2 + \rho)$ is less than $c_6\rho^{d-1}$. Since, by general perturbation theory, the n-th eigenvalue of $L_N(q(x))$ lies in M-neighborhood of the n-th eigenvalue of $L_N(0)$, the number of the eigenvalues Λ_n of the operator $L_N(q(x))$ in the interval $I = [|\gamma|^2 - 2M, |\gamma|^2 + 2M]$ is less than $c_7\rho^{d-1}$. By this fact and the inequality (11), there exists $n_1 \in I$ such that

$$|(\Psi_{n_1}(x), \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle})| > c_3 \rho^{-\frac{(d-1)}{2}}$$

Similarly, by using (9) for $\Phi_n(x)$, we get

$$|(\Phi_{n_2}, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^{d} \alpha_i) e^{i\langle \alpha, x\rangle})| > c_4 \rho^{-\frac{(d-1)}{2}}$$

The lemma is proved. □

Lemma 2 *Let $\gamma \in U(\rho^\alpha, p)$, i.e. $|\gamma|^2$ be the non-resonance eigenvalue of $L_D(0)$ and $L_N(0)$ and Λ_n and μ_n be the eigenvalues of $L_N(q(x))$ and $L_D(q(x))$, respectively, lying in the interval $I = [|\gamma|^2 - 2M, |\gamma|^2 + 2M]$, then $|\Lambda_n - |\gamma + b|^2| > \frac{1}{2}\rho^\alpha$ and $|\mu_n - |\gamma + b|^2| > \frac{1}{2}\rho^\alpha$ for all $b \in \Gamma(m\rho_\alpha)$.*

proof: If $\gamma \in U(\rho^\alpha, p)$, then for all $b \in \Gamma(m\rho_\alpha)$ we have the inequality

$$||\gamma|^2 - |\gamma + b|^2| \geq \rho^\alpha$$

which, together with the fact that $\Lambda_n \in I$, implies

$$|\Lambda_n - |\gamma + b|^2| = |\Lambda_n - |\gamma + b|^2 \mp |\gamma|^2| \geq ||| \gamma|^2 - |\gamma + b|^2| - |\Lambda_n - |\gamma|^2|| \geq |\rho^\alpha - 2M|,$$

where ρ^α is sufficiently large so the result follows. Similarly $|\mu_n - |\gamma + b|^2| > \frac{1}{2}\rho^\alpha$. □

Theorem 1 *Let $\gamma \in U(\rho^\alpha, p), |\gamma| \sim \rho$; i.e.,$|\gamma|^2$ be non-resonance eigenvalue of the operators $L_D(0)$ and $L_N(0)$. Then there exists an eigenvalue Λ_n of the operator $L_N(q(x))$ and an eigenvalue μ_n of the operator $L_D(q(x))$ satisfying the following formulas :*

$$\Lambda_n = |\gamma|^2 + O(\rho^{-\alpha}) \quad (12)$$

$$\mu_n = |\gamma|^2 + O(\rho^{-\alpha}). \quad (13)$$

proof: First, we prove the theorem for $L_N(q(x))$, i.e., we prove (12).

By Lemma 1, there is an index n such that $|\Lambda_n - |\gamma|^2| \leq 2M$ and $|(\Psi_n(x), \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle})| > c_3 \rho^{-\frac{(d-1)}{2}}$. We prove that this eigenvalue satisfies the Formula (12). Substituting the decomposition (5) of the potential $q(x)$ in the Formula (8) we have:

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1}(\Psi_n, \sum_{\beta_1 \in A_{\gamma_1}} e^{i\langle \beta_1, x\rangle} \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}) + O(\rho^{-p\alpha}).$$

Using the formula

$$\sum_{\beta_1 \in A_{\gamma_1}} e^{i\langle \beta_1, x\rangle} \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle} = \sum_{\beta_1 \in A_{\gamma_1}} \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x\rangle}, \tag{14}$$

which can be easily proved by direct calculation, we get

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1}(\Psi_n, \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x\rangle}) + O(\rho^{-p\alpha}).$$

Since $\gamma + \beta_1 \in \frac{\Gamma}{2}$, i.e.; $|\gamma + \beta_1|^2$ is an eigenvalue of the operator $L_N(0)$ with the corresponding eigenfunction $\sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x\rangle}$, we can use the Formula (8). Therefore using (8) in the last equation we obtain

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1} \frac{(\Psi_n, q(x) \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x\rangle}) + O(\rho^{-p\alpha})}{\Lambda_n - |\gamma + \beta_1|^2}$$

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1} \frac{(\Psi_n, q(x) \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x\rangle})}{\Lambda_n - |\gamma + \beta_1|^2} + O(\rho^{-p\alpha}) \tag{15}$$

Here, we use the fact(see Lemma 2) that the denominator of the fraction in (15) satisfies

$$|\Lambda_n - |\gamma + \beta_1|^2| > \frac{1}{2}\rho^\alpha$$

since $\beta_1 \in \Gamma(\rho^\alpha)$. Again, substituting the decomposition of $q(x)$ in Equation (15) and using the last inequality, we get

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle\alpha,x\rangle})$$

$$= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1} \frac{(\Psi_n, \sum_{\gamma_2 \in \Gamma(\rho^\alpha)} q_{\gamma_2} \sum_{\alpha \in A_{\gamma_2}} e^{i\langle\alpha,x\rangle} \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle\alpha,x\rangle}) + O(\rho^{-p\alpha})}{\Lambda_n - |\gamma+\beta_1|^2}$$

$$+O(\rho^{-p\alpha}).$$

Now using the Equation (14), we have

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle\alpha,x\rangle}) = \sum_{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}, \beta_2 \in A_{\gamma_2}} q_{\gamma_1} q_{\gamma_2} \frac{(\Psi_n, \sum_{\alpha \in A_{\gamma+\beta_1+\beta_2}} e^{i\langle\alpha,x\rangle})}{\Lambda_n - |\gamma+\beta_1|^2}$$

$$+O(\rho^{-p\alpha})$$

If the terms with coefficient $(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle\alpha,x\rangle})$ are isolated, we obtain

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle\alpha,x\rangle}) = \sum_{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha)} \sum_{\substack{\beta_2 = -\beta_1 \\ \beta_1 \in A_{\gamma_1} \\ \beta_2 \in A_{\gamma_2}}} q_{\gamma_1} q_{\gamma_2} \frac{(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle\alpha,x\rangle})}{\Lambda_n - |\gamma+\beta_1|^2}$$

$$+ \sum_{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha)} \sum_{\substack{\beta_2 \neq -\beta_1 \\ \beta_1 \in A_{\gamma_1} \\ \beta_2 \in A_{\gamma_2}}} q_{\gamma_1} q_{\gamma_2} \frac{(\Psi_n, \sum_{\alpha \in A_{\gamma+\beta_1+\beta_2}} e^{i\langle\alpha,x\rangle})}{\Lambda_n - |\gamma+\beta_1|^2} + O(\rho^{-p\alpha}) \quad (16)$$

By the same method as above, iterating p times the formula (16) and isolating each time the terms with multiplicant $(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle\alpha,x\rangle})$, we get

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle\alpha,x\rangle}) = (\sum_{i=1}^{p} S_i)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle\alpha,x\rangle}) + C_p + O(\rho^{-p\alpha}), \quad (17)$$

where

$$S_m(\Lambda_n) = \sum_{\gamma_1,\ldots\gamma_{m+1}\in\Gamma(\rho^\alpha)} \sum_{\substack{\beta_{m+1}=-(\beta_1+\ldots+\beta_m)\\ \beta_1\in A_{\gamma_1},\ldots,\beta_{m+1}\in A_{\gamma_{m+1}}}} \frac{q_{\gamma_1}\ldots q_{\gamma_{m+1}}}{(\Lambda_n-|\gamma+\beta_1|^2)\ldots(\Lambda_n-|\gamma+\beta_1+\ldots+\beta_m|^2)} \tag{18}$$

$$C_p = \sum_{\gamma_1,\ldots\gamma_{p+1}\in\Gamma(\rho^\alpha)} \sum_{\substack{\beta_{p+1}\neq-(\beta_1+\ldots+\beta_p)\\ \beta_1\in A_{\gamma_1},\ldots,\beta_{p+1}\in A_{\gamma_{p+1}}}} \frac{q_{\gamma_1}\ldots q_{\gamma_{p+1}}(\Psi_n,\sum_{\alpha\in A_{\gamma+\beta_1+\ldots+\beta_{p+1}}} e^{i\langle\alpha,x\rangle})}{(\Lambda_n-|\gamma+\beta_1|^2)\ldots(\Lambda_n-|\gamma+\beta_1+\ldots+\beta_p|^2)} \tag{19}$$

For all $m=1,2,\ldots,p,\gamma_m\in\Gamma(\rho^\alpha)$ and
$\beta_m\in A_{\gamma_m}\Rightarrow|\gamma_m|=|\beta_m|<\rho^\alpha$ and $|\beta_1+\beta_2+\ldots+\beta_m|<p\rho^\alpha$, hence we can use Lemma 2 and the Equation (6). Then we have

$$\sum_{m=1}^{p} S_m(\Lambda_n)=O(\rho^{-\alpha}), \qquad C_p=O(\rho^{-p\alpha}). \tag{20}$$

Taking into account that for Λ_n, we only used the condition $\Lambda_n\in I$, we have

$$\sum_{m=1}^{p} S_m(a)=O(\rho^{-\alpha}), \qquad \forall a\in I \tag{21}$$

If we substitute (20) into (17), we get

$$(\Lambda_n-|\gamma|^2)(\Psi_n,\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})=O(\rho^{-\alpha})(\Psi_n,\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})+O(\rho^{-p\alpha}) \tag{22}$$

dividing both sides of the Equation (22) by $(\Psi_n,\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})$, using Lemma 1 and the obvious inequality $p\alpha>\frac{d-1}{2}+\alpha$ (see definition of p and α), we get the proof for $L_N(q(x))$.

By the same way, we can prove the theorem for $L_D(q(x))$, i.e, for the non-resonance eigenvalue $|\gamma|^2$ of $L_D(0)$ ($\gamma\in U(\rho^\alpha,l)$), there is an eigenvalue μ_n of $L_D(q(x))$ such that the Formula (13) is satisfied. Indeed, to prove this, instead of (8), we use the Formula (9) with the same decomposition (5) of $q(x)$ and we get

$$(\mu_n - |\gamma|^2)(\Phi_n, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^{d} \alpha_i) e^{i\langle \alpha, x\rangle})$$

$$= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} (\Phi_n, \sum_{\beta_1 \in A_{\gamma_1}} e^{i\langle \beta_1, x\rangle} \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^{d} \alpha_i) e^{i\langle \alpha, x\rangle}) + O(\rho^{-p\alpha})$$

and instead of (14), using the following formula

$$\sum_{\beta_1 \in A_{\gamma_1}} e^{i\langle \beta_1, x\rangle} \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^{d} \alpha_i) e^{i\langle \alpha, x\rangle} = \sum_{\beta_1 \in A_{\gamma_1}} \sum_{\alpha \in A_{\gamma+\beta_1}} (\text{sign} \prod_{i=1}^{d} \alpha_i) e^{i\langle \alpha, x\rangle}$$

we get

$$(\mu_n - |\gamma|^2)(\Phi_n, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^{d} \alpha_i) e^{i\langle \alpha, x\rangle})$$

$$= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1} (\Phi_n, \sum_{\alpha \in A_{\gamma+\beta_1}} (\text{sign} \prod_{i=1}^{d} \alpha_i) e^{i\langle \alpha, x\rangle}) + O(\rho^{-p\alpha})$$

By the similar considerations, we can iterate the above formula p times and by isolating the coefficient of $(\Phi_n, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^{d} \alpha_i) e^{i\langle \alpha, x\rangle})$, we obtain the equation

$$(\mu_n - |\gamma|^2)(\Phi_n, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^{d} \alpha_i) e^{i\langle \alpha, x\rangle}) = (\sum_{m=1}^{p} S_m)(\Phi_n, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^{d} \alpha_i) e^{i\langle \alpha, x\rangle}) + C_p + O(\rho^{-p\alpha}), \quad (23)$$

instead of (17), where S_m is the same as Equation (18) and

$C_p =$

$$\sum_{\gamma_1,...\gamma_{p+1}\in\Gamma(\rho^\alpha)} \sum_{\substack{\beta_{p+1}\neq-(\beta_1+...+\beta_p) \\ \beta_1\in A_{\gamma_1},...,\beta_{p+1}\in A_{\gamma_{p+1}}}} \frac{q_{\gamma_1}...q_{\gamma_{p+1}}(\Phi_n, \sum_{\alpha\in A_{\gamma+\beta_1+...+\beta_{p+1}}}(\text{sign}\prod_{i=1}^d \alpha_i)e^{i\langle\alpha,x\rangle})}{(\mu_n-|\gamma+\beta_1|^2)...(\mu_n-|\gamma+\beta_1+...+\beta_p|^2)}.$$

Hence, by similar calculations, we get the proof. □

Theorem 2 *Let $\gamma \in U(\rho^\alpha, p)$, $|\gamma| \sim \rho$ then there is an eigenvalue Λ_n of the operator $L_N(q(x))$ and an eigenvalue μ_n of the operator $L_D(q(x))$ satisfying the formulas*

$$\Lambda_n = |\gamma|^2 + F_{k-1} + O(\rho^{-k\alpha}), \tag{24}$$

and

$$\mu_n = |\gamma|^2 + F_{k-1} + O(\rho^{-k\alpha}), \tag{25}$$

for all $k = 1, 2, ...p - z$ where

$$F_0 = 0, F_1 = \sum_{\gamma_1\in\Gamma(\rho^\alpha)} \sum_{\beta_1\in A_{\gamma_1}} \frac{|q_{\gamma_1}|^2}{|\gamma|^2 - |\gamma-\beta_1|^2},$$

$$F_s = \sum_{i=1}^{s} S_i(|\gamma|^2 + F_{s-1}), s = 2, 3, ..., p$$

and $z = [\frac{d-1}{2\alpha}] + 1$. ($[\frac{d-1}{2\alpha}]$ is the integer part of $\frac{d-1}{2\alpha}$.)

proof: We prove that for the eigenvalues Λ_n and μ_n satisfying the Formulas (12) and (13) the Formulas (24) and (25) hold, respectively. Let us prove it by mathematical induction on k :

for $k = 1$; by Theorem 1,Λ_n and μ_n satisfy the equations

$$\Lambda_n = |\gamma|^2 + F_0 + O(\rho^{-\alpha}),$$

$$\mu_n = |\gamma|^2 + F_0 + O(\rho^{-\alpha}),$$

where $F_0 = 0$,

for $k = j$; assume that it is true, i.e

$$\Lambda_n = |\gamma| + F_{j-1} + O(\rho^{-j\alpha}). \tag{26}$$

$$\mu_n = |\gamma| + F_{j-1} + O(\rho^{-j\alpha}). \tag{27}$$

For $k = j + 1$, we must prove that

$$\Lambda_n = |\gamma|^2 + F_j + O(\rho^{-(j+1)\alpha}), \tag{28}$$

$$\mu_n = |\gamma|^2 + F_j + O(\rho^{-(j+1)\alpha}). \tag{29}$$

To prove this we put Expression (26) into $S_m(\Lambda_n)$ and (27) into $S_m(\mu_n)$ and divide both sides of (17) by $(\Psi_n(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})$ and (23) by
$(\Phi_n(x), \sum_{\alpha\in A_\gamma}(\text{sign}\prod_{i=1}^{d}\alpha_i)e^{i\langle\alpha,x\rangle})$, we get

$$\Lambda_n = |\gamma|^2 + \sum_{m=1}^{p} S_m(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) + O(\rho^{-(p-z)\alpha})) \tag{30}$$

$$\mu_n = |\gamma|^2 + \sum_{m=1}^{p} S_m(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) + O(\rho^{-(p-z)\alpha})) \tag{31}$$

adding and subtracting the term $\sum_{m=1}^{p} S_m(|\gamma|^2 + F_{j-1})$ in (30) and (31), we have

$$\Lambda_n = |\gamma|^2 + [\sum_{m=1}^{p} S_m(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_m(|\gamma|^2 + F_{j-1})]$$
$$+ \sum_{m=1}^{p} S_m(|\gamma|^2 + F_{j-1}) + O(\rho^{-(p-z)\alpha}) \tag{32}$$

$$\mu_n = |\gamma|^2 + [\sum_{m=1}^{p} S_m(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_m(|\gamma|^2 + F_{j-1})]$$

$$+ \sum_{m=1}^{p} S_m(|\gamma|^2 + F_{j-1}) + O(\rho^{-(p-z)\alpha}) \quad (33)$$

$\sum_{m=1}^{p} S_m(|\gamma|^2 + F_{j-1}) = F_j$, so we need only to show that the expressions in the square brackets in (32) and (33) are equal to $O(\rho^{-(j+1)\alpha})$. First we prove that $F_j = O(\rho^{-\alpha})$ for all $j = 0, 1, 2, ..., p$ by induction. By Theorem 1 $F_0 = 0$. Suppose $F_{j-1} = O(\rho^{-\alpha})$ then by (21)$F_j = S_m(|\gamma|^2 + F_{j-1}) = O(\rho^{-\alpha})$.

Using this and Lemma 2, we have

$$||\gamma|^2 + F_{j-1} + O(\rho^{-(j)\alpha}) - |\gamma + \beta_1 + ... + \beta_m|^2| > \frac{1}{3}\rho^{\alpha}$$

$$||\gamma|^2 + F_{j-1} - |\gamma + \beta_1 + ... + \beta_m|^2| > \frac{1}{3}\rho^{\alpha}, m = 1, 2, ..., p$$

Hence, by direct calculations and using the above inequalities, it can be easily checked that the expressions in the square brackets are equal to $O(\rho^{-(j+1)\alpha})\Box$.

Remark 1 *It is clear that* $V_b(\rho^{\alpha}) \bigcap (R(2\rho)\backslash R(\rho))$ *is the part of* $(R(2\rho)\backslash R(\rho))$ *which is contained between two parallel hyperplanes* $\{x : |x|^2 - |x + b|^2 = -\rho^{\alpha}\}$ *and* $\{x : |x|^2 - |x + b|^2 = \rho^{\alpha}\}$*. The distance of this hyperplanes from the origin is* $\frac{\rho^{\alpha}}{|b|}$*. Therefore* $\mu(V_b(\rho^{\alpha}) \bigcap (R(2\rho)\backslash R(\rho))) = O(\rho^{d-1+\alpha})$*. Since the number of the vectors* γ *in* $\Gamma(p\rho^{\alpha})$ *is equal to* $\rho^{d\alpha}$ *and* $\mu(R(2\rho)\backslash R(\rho)) \sim \rho^d$*, we have* $\mu(\bigcup_{b\in\Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha}) \bigcap (R(2\rho)\backslash R(\rho))) = O(\rho^{d-1+(d+1)\alpha})$ *and* $\mu(U(\rho^{\alpha}, p) \bigcap (R(2\rho)\backslash R(\rho))) = \mu((R(2\rho)\backslash R(\rho)))(1 + O(\rho^{(d+1)\alpha-1}))$ *from which we get (7).*

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