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On the Lebesgue Measure of Self-Affine Sets

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Abstract

Flaherty and Wang studied Haar-type multiwavelets and multi-tiles. The information on what digit sets give multi-attractors with positive Lebesgue measure is very limited. In this note, we give a few classes of digit sets leading to multi-attractors with positive measure. The attractors we obtain include the Haar-type multi-tiles of Flaherty and Wang.

Key Words: Multi-attractors, self-affine tiles, iterated function systems.

1. Introduction

Unless otherwise stated, we assume that B (or B_i) is an expanding integral matrix in $M_n(\mathbb{Z})$, i.e., all its eigenvalues λ_i have modulus > 1. Let $|\det B| = q$ and let $D \subseteq \mathbb{Z}^n$ be a set of q distinct vectors, called a *q-digit set.* The affine maps w_j defined by

$$w_j(x) = B^{-1}(x + d_j),\ d_j \in D,\quad 1 \leq j \leq q,$$

are all contractions under a suitable norm in $\mathbb{R}^n$ (see [**8**, pp. 29-30]). The family $\{w_j\}_{j=1}^q$ is called an *iterated function system* (IFS) and there is a unique non-empty compact set satisfying $T = \bigcup_{j=1}^q w_j(T)$ ([**4**], [**7**]). T is called the *attractor* of the system and is given explicitly by

$$T := T(B, D) = \{\sum_{i=1}^{\infty} B^{-i} d_{j_i} : d_{j_i} \in D\}.$$

We use $\mu(T)$ to denote the Lebesgue measure of the set T. We call T an *integral self-affine tile* if it has a positive measure. In general, the classification of all digit sets D

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with $\mu(T) > 0$ is a complicated problem. The case that $\mu(T) = 1$ is of particular interest and we call such T a *Haar tile*. Gröchenig and Madych [**6**] showed that χ_T generates a compactly supported orthonormal wavelet basis of $L^2(\mathbb{R}^n)$ if and only if $\mu(T) = 1$.

We now consider generalized iterated function systems [**2**], [**3**], [**5**]. Following the notation and terminology of Flaherty and Wang [**5**], let $\mathcal{C}_n$ denote the the space of all non-empty compact subsets of $\mathbb{R}^n$. Let $||\cdot||$ be a norm on $\mathbb{R}^n$. We define the *Hausdorff metric* on $\mathcal{C}_n$ with respect to $||\cdot||$ by

$$d_H(D, D') := max\{\sup_{x\in D}\inf_{x'\in D'}\|x - x'\|, \sup_{y'\in D'}\inf_{y\in D}\|y - y'\|\}$$

It is well known that $(\mathcal{C}_n, d_H)$ is a complete metric space. Now, let d_H^r be the metric defined on $\mathcal{C}_n^r$, the space of all r-tuples of non-empty compact subsets of $\mathbb{R}^n$, given by

$$d_H^r(\mathbf{D}, \mathbf{D'}) := \max_{1\le i\le r}\{d_H(D_i, D_i')\}$$

Then one can show that $(\mathcal{C}_n^r, d_H^r)$ is also a complete space. Let S denote a finite group of matrices (in $M_n(\mathbb{Z})$) under matrix multiplication such that $SB_i = B_iS$ and let $s, s_i \in S$. Let $A_i = s_iB_i$, $1 \le i \le r$, and let $\mathcal{D}_{ij}$, $1 \le i, j \le r$, be finite subsets of $\mathbb{R}^n$, with $\cup_{j=1}^r \mathcal{D}_{ij}$ non-empty. Then there exist unique sets $\mathbf{Q} = (Q_1, ..., Q_r) \in \mathcal{C}_n^r$ and $\mathbf{Q'} = (Q_1', ..., Q_r') \in \mathcal{C}_n^r$ such that

$$A_i(Q_i) = \cup_{j=1}^r (Q_j + \mathcal{D}_{ij}), \quad 1 \le i \le r, \tag{1.1}$$

$$Q_i' = \cup_{j=1}^r A_j^{-1}(Q_j' + \mathcal{D}_{ij}), \quad 1 \le i \le r. \tag{1.2}$$

We call $\mathbf{Q}$ or $\mathbf{Q'}$ self-affine multi-attractors (or multi-attractors). Let $Q = \cup_{i=1}^r Q_i$ and the Q_i are defined by (1.1). In this note we prove the following theorem for the multi-attractor case in which we assume that $|\det s_i| = 1$, i.e. S is symmetry group of $\{B_i\}$.

Theorem 1.1 *Let $\mathcal{D}_{ij}$, $1 \le i, j \le r$, be finite subsets of $\mathbb{Z}^n$ with $\cup_{j=1}^r \mathcal{D}_{ij}$ non-empty. Then $\mu(Q) > 0$ in either of the following cases.*

(i) Suppose that $A_i = s_iB_i$, $1 \le i \le r$, B_i are expanding matrices in $M_n(\mathbb{Z})$ and for every $i \in \{1, ..., r\}$, there exists $j \in \{1, ..., r\}$ such that $\mathcal{D}_{ij} + A_i\mathbb{Z}^n = \mathbb{Z}^n$.

(ii) Suppose that $A_i = A = sB$, $1 \le i \le r$, where $B \in M_n(\mathbb{Z})$ is an expanding matrix and $\cup_{i=1}^r \mathcal{D}_{ij} + A\mathbb{Z}^n = \mathbb{Z}^n$ for every $j \in \{1, ..., r\}$.

We note that Theorem 1.1 is a generalization of a result of Bandt [**1**]. One can see that the digit sets of the support of Haar-type scaling functions meet the conditions of Theorem 1.1 (ii) (See Example in Section 2).

2. The Proof of the Main Theorem

We first state the following theorem without proof and refer to [**2**], [**3**], [**5**] for the proof.

Theorem 2.1 *Suppose that $A_i = s_i B_i$, $s_i \in S$, $1 \le i \le r$, where the B_i are as above and let $\mathcal{D}_{ij}$, $1 \le i, j \le r$, be finite subsets of $\mathbb{R}^n$, with $\cup_{j=1}^r \mathcal{D}_{ij}$ non-empty. Then there exist unique $\mathbf{Q} = (Q_1, ..., Q_r) \in \mathcal{C}_n^r$ and $\mathbf{Q'} = (Q'_1, ..., Q'_r) \in \mathcal{C}_n^r$ such that*

(i)

$$A_i(Q_i) = \cup_{j=1}^r (Q_j + \mathcal{D}_{ij}), \quad 1 \le i \le r. \tag{2.1}$$

(ii)

$$Q'_i = \cup_{j=1}^r A_j^{-1}(Q'_j + \mathcal{D}_{ij}), \quad 1 \le i \le r. \tag{2.2}$$

We now introduce some terminology. We call the attractors satisfying (2.1) and (2.2) an *attractor of the first type* and an *attractor of the second type* respectively. Let $Q^0 \in \mathcal{C}_n^r$ be arbitrary. If we define $Q^N = \cup_{i=1}^r Q_i^N$, where $Q_i^{N+1} = A_i^{-1} \cup_{j=1}^r (Q_j^N + \mathcal{D}_{ij})$ for $N \ge 0$, one can see that Q_i^N converges to Q_i in the Hausdorff metric d_H.

Note that in (i) of Theorem 2.3, if we take $A_i = A$, $1 \le i \le r$, we get the attractors considered in [**5**]. Let $\mathcal{D}_{ij} = \mathcal{D}_{i'j}$, $1 \le i', i, j \le r$. Additionally, if we let $\mathcal{D}_{ij} = 1$, $1 \le i, j \le r$, we get attractors considered in [**6**]. Then we can make the following definitions. Consider the maps $\Phi, \Psi : \mathcal{C}_n^r \to \mathcal{C}_n^r$ whose $i-th$ components are defined by

$$\Phi_i(\mathbf{D}) = A_i^{-1} \cup_{j=1}^r (D_j + \mathcal{D}_{ij}), \tag{2.3}$$

$$\Psi_i(\mathbf{D}) = \cup_{j=1}^r A_j^{-1}(D_j + \mathcal{D}_{ij}). \tag{2.4}$$

Then we can call $\{\Phi_i\} := \{\Phi_i : 1 \le i \le r\}$ or $\{\Psi_i\} := \{\Psi_i : 1 \le i \le r\}$ a *generalized iterated function system* (GIFS). Now we can define the generalized open set condition for a GIFS $\{\Phi_i\}$. Let $\mathbb{R}_n^r$ denote the space of all r-tuples of subsets of $\mathbb{R}^n$. We say that $\{\Phi_i\}$ ($\{\Psi_i\}$) satisfies the *generalized open set condition* (GOSC) if there exists a set $V = (V_1, ..., V_r) \in \mathbb{R}_n^r$, where $V_i \in \mathbb{R}^n$, $1 \le i \le r$, are non-empty open sets, such that $\Phi_i(V) \subseteq V_i$ ($\Psi_i(V) \subseteq V_i$) $\forall i \in \{1, ..., r\}$ with the union in (2.3) ((2.4)) is disjoint.

Let $\mathbf{Q}$ be the attractor of a GIFS $\{\Phi_i\}$. If $\mu(\mathbf{Q}) > 0$ and $\mu((Q_j + \mathcal{D}_{ij}) \cap (Q'_j + \mathcal{D}_{ij'})) = 0$ for $j \neq j'$, $i, j \in \{1, ..., r\}$, then $\{\Phi_i\}$ satisfies the GOSC since we can take $V = (\overset{\circ}{Q}_1, ..., \overset{\circ}{Q}_r)$, where $\circ$ stands for interior. Self-affine tiles always satisfy the open set condition (OSC), see [**6**]. The support of a Haar-type scaling function is a multi-tile and satisfies GOSC, see [**5**].

Lemma 2.2 *Suppose that* $Q \in \mathbb{R}^n$ *is a compact set such that* $Q + \mathbb{Z}^n = \mathbb{R}^n$. *Then* $\mu(Q) \ge 1$.

Proof. Let $f(x) := \sum_{k \in \mathbb{Z}^n} \chi_Q(x + k)$, and let $Q^0 := [0, 1]^n$. Then

$$\mu(Q) = \int_{\mathbb{R}^n} \chi_Q(x)dx = \int_{\bigcup_{k \in \mathbb{Z}^n} Q^0 + k} \chi_Q(x)dx = \sum_{k \in \mathbb{Z}^n} \int_{Q^0 + k} \chi_Q(x)dx = \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \chi_{Q \cap (Q^0 + k)}(x)dx.$$

But we have

$$\begin{aligned} \int_{\mathbb{R}^n} \chi_{Q \cap (Q^0 + k)}(x)dx &= \int_{\mathbb{R}^n} \chi_{Q \cap (Q^0 + k)}(x + k)dx \\ &= \int_{\mathbb{R}^n} \chi_Q(x + k)\chi_{Q^0 + k}(x + k)dx \\ &= \int_{\mathbb{R}^n} \chi_Q(x + k)\chi_{Q^0}(x)dx. \end{aligned}$$

Then

$$\begin{aligned}\mu(Q)=\int_{\mathbb{R}^n}\chi_Q(x)dx &= \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\chi_{Q\cap(Q^0+k)}(x)dx\\ &= \int_{\mathbb{R}^n}\chi_{Q^0}(x)\sum_{k\in\mathbb{Z}^n}\chi_Q(x+k)dx\\ &= \int_{Q^0}\sum_{k\in\mathbb{Z}^n}\chi_Q(x+k)dx\\ &= \int_{Q^0}f(x)dx.\end{aligned}$$

Note that $f(x)\geq 1$ since $Q+\mathbb{Z}^n=\mathbb{R}^n$. Then, $\mu(Q)=\int_{Q^0}f(x)dx\geq\int_{Q^0}1dx=1$. □

Using a technique in [**6**], we give the following proof.

Proof of Theorem 1.1. (i) Let $Q_i^0=[0,1]^n$, $1\leq i\leq r$, and $Q^0=\cup_{i=1}^r Q_i=[0,1]^n$. Note that $[0,1]^n+\mathbb{Z}^n=\mathbb{R}^n$. Let us define $Q^{N+1}=\cup_{i=1}^r Q_i^{N+1}$, where $Q_i^{N+1}=A_i^{-1}\cup_{j=1}^r(Q_j^N+\mathcal{D}_{ij})$. Now suppose that $Q_i^k+\mathbb{Z}^n=\mathbb{R}^n$ for $k\leq N$ and $1\leq i\leq r$. We will show that $Q_i^{N+1}+\mathbb{Z}^n=\mathbb{R}^n$ for $1\leq i\leq r$. $Q^{N+1}+\mathbb{Z}^n=\cup_{i=1}^r(Q_i^{N+1}+\mathbb{Z}^n)$.

$$\begin{aligned}Q_i^{N+1}+\mathbb{Z}^n &= A_i^{-1}\cup_{j=1}^r(Q_j^N+\mathcal{D}_{ij})+\mathbb{Z}^n\\ &= A_i^{-1}\cup_{j=1}^r(Q_j^N+\mathcal{D}_{ij}+A_i\mathbb{Z}^n).\end{aligned}$$

Now by hypothesis, $\mathcal{D}_{ij}+A_i\mathbb{Z}^n=\mathbb{Z}^n$ for some j. By induction hypothesis, we get $Q_j^N+\mathcal{D}_{ij}+A_i\mathbb{Z}^n=\mathbb{R}^n$ for some j. Then, we get $Q_i^{N+1}+\mathbb{Z}^n=\mathbb{R}^n$ and hence $Q^{N+1}+\mathbb{Z}^n=\mathbb{R}^n$.

Note that $Q^N, N\geq 0$, converges to Q in the Hausdorff metric. We want to show that $Q+\mathbb{Z}^n=\mathbb{R}^n$. Let $x\in\mathbb{R}^n=Q^N+\mathbb{Z}^n$. Then there exists a sequence of lattice points m_N such that $x-m_N\in Q^N$. We can find a ball $B_R(0)$ containing all the Q^N. Thus $||x-m_N||<R$ implies that $||m_N||<||x||+R$. Thus m_N is a bounded sequence. Hence it has a constant subsequence $m_{N_j}=m$. Therefore, $d(x-m,Q)\leq d(x-m,Q^{N_j})+d_H(Q^{N_j},Q)=d_H(Q^{N_j},Q)$ implies that $d(x-m,Q)\leq\lim_{j\to\infty}d_H(Q^{N_j},Q)=0$. Thus $x-m\in Q$ and it follows that $Q+\mathbb{Z}^n=\mathbb{R}^n$. Then Lemma 2.2 concludes the proof of (i).

(ii) The proof is similar to that of (i). As in the proof of (i), let $Q^0 = [0,1]^n$ so that $Q^0 + \mathbb{Z}^n = \mathbb{R}^n$. Suppose that $Q^k + \mathbb{Z}^n = \mathbb{R}^n$ for $k \leq N$. By induction, we will show that $Q^N + \mathbb{Z}^n = \mathbb{R}^n \quad \forall N \in \mathbb{N}$ and for this we only need to observe that

$$\begin{aligned} Q^{N+1} + \mathbb{Z}^n = \bigcup_{i=1}^{r} (Q_i^{N+1} + \mathbb{Z}^n) &= \bigcup_{i=1}^{r} A^{-1}(\cup_{j=1}^{r}(Q_j^N + \mathcal{D}_{ij})) + \mathbb{Z}^n \\ &= \bigcup_{j=1}^{r} A^{-1}(\cup_{i=1}^{r}(Q_j^N + \mathcal{D}_{ij})) + \mathbb{Z}^n \\ &= \bigcup_{j=1}^{r} A^{-1}(Q_j^N + (\cup_{i=1}^{r}\mathcal{D}_{ij} + A\mathbb{Z}^n)) \\ &= A^{-1}(\cup_{j=1}^{r} Q_j^N + \mathbb{Z}^n) \\ &= A^{-1}(Q^N + \mathbb{Z}^n) \\ &= A^{-1}\mathbb{R}^n \\ &= \mathbb{R}^n. \end{aligned}$$

The rest of the proof is the same as that of (i). □

Example. Let $A \in M_n(\mathbb{Z})$ be an expanding matrix and let $c_k \in M_r(\mathbb{Z})$. Suppose that $\chi_{\mathbf{Q}} := [\chi_{Q_1}, \chi_{Q_2}, ..., \chi_{Q_r}]^T$, where $\mathbf{Q} = (Q_1, ..., Q_r) \in \mathcal{C}_n^r$, is a scaling function vector satisfying the vector refinement equation

$$\chi_{\mathbf{Q}} = \sum_{k \in \mathbb{Z}^n} c_k \chi_{\mathbf{Q}}(Ax - k).$$

Let $\mathcal{D}_{ij} = \{k : \ (c_k)_{ij} = 1\}$. We note that for each $k \in \mathbb{Z}^n$,

(i) the matrix c_k is a zero-one matrix

(ii) the matrix

$$b_k = \sum_{l \in \mathbb{Z}^n} c_{k+Al}$$

is a zero-one matrix and contains exactly one entry of 1 in each column, see [Theorem 1.1, **5**]. Thus one can see that for every $j \in \{1, .., r\}$, $\cup_{i=1}^{r} \mathcal{D}_{ij} + A\mathbb{Z}^n = \mathbb{Z}^n$. □

We also note that similar results to Theorem 1.1 can be obtained for attractors of the second type, see Proposition 3.1. Now we wish to state the properties of the attractors obtained in Theorem 1.1.

Corollary 2.3 *The attractor* $\mathbf{Q}$ *of Theorem 1.1 has the following properties:*

(i) $Q + \mathbb{Z}^n = \mathbb{R}^n$ *and* $\mu(Q) \geq 1$,

(ii) Q_i, $1 \leq i \leq r$, *has a nonempty interior.*

Proof. (i) is clear from Lemma 2.2 and the proof of Theorem 1.1. It is known that (ii) is a consequence of (i). However, we prove it here for completeness. Let U be the closed unit ball in $\mathbb{R}^n$. Then $Q + \mathbb{Z}^n = \mathbb{R}^n$ implies that $U = \bigcup_{l \in \mathbb{Z}^n} ((Q + l) \cap U)$. Since U is bounded, $(Q + l) \cap U \neq \emptyset$ for a finite number of $l \in \mathbb{Z}^n$. Hence, $U = \cup_{i=1}^m U_i$, where $U_i = (Q + l_i) \cap U$. By Baire's Category Theorem at least one of the U_i's has a nonempty interior. This concludes the proof. □

We shall be concerned with the attractors in (i) of Theorem 2.1. In [**9**], GOSC played an important role in the computation of the Hausdorff dimension of the boundary of the tiles obtained using expanding similarities. In general, GOSC is hard to check. In regard to GOSC, we have the following corollary where # denotes the cardinality.

Corollary 2.4 *Suppose that the* $\mathcal{D}_{ij}$ *in Theorem 1.1 satisfy* $\# \cup_{j=1}^r \mathcal{D}_{ij} = |det\, A_i| := r_i$. *Then,* $\mu(Q_1) = \mu(Q_2) = \ldots = \mu(Q_r)$ *and* $\mu((Q_j + \mathcal{D}_{ij}) \cap (Q_{j'} + \mathcal{D}_{ij'})) = 0$ *for* $j \neq j'$ *and each fixed* i. *Thus* $\{\Phi_i\}$ *in (2.3) satisfies the GOSC.*

Proof. Let i_0 be such that $\mu(Q_{i_0}) = \max \mu(Q_i)$. To prove the first claim of the corollary, we note that $A_{i_0} Q_{i_0} = \cup_{j=1}^r (Q_j + \mathcal{D}_{i_0 j}))$ implies

$$r_{i_0} \mu(Q_{i_0}) = \mu(\cup_{j=1}^r (Q_j + \mathcal{D}_{i_0 j})) \leq \sum_{j=1}^r \mu(Q_j + \mathcal{D}_{i_0 j}).$$

Thus if there were $j \neq j'$ such that $\mu(Q_j) \neq \mu(Q_{j'})$, we would have $r_{i_0} \mu(Q_{i_0}) < r_{i_0} \mu(Q_{i_0})$ by the above inequality.

To prove the second claim of the corollary, we just need to observe that if $\mu((Q_j + \mathcal{D}_{ij}) \cap (Q_{j'} + \mathcal{D}_{ij'})) > 0$ for some $j \neq j'$, then $r_i \mu(Q_i) = \mu(\cup_{j=1}^r (Q_j + \mathcal{D}_{ij})) < r_i \mu(Q_i)$, which is a contradiction. This concludes the proof. □

3. Remarks

We note that attractors of the second type are quite difficult to handle. As far as we know, there are a few results on them (see e.g. [**10**]). However, by using the technique in Section 2, we can prove the following proposition for the attractors of the second type.

Proposition 3.1 *Suppose that* $A_j = s_j B_j$, $1 \le j \le r$, B_j *are expanding matrices in* $M_n(\mathbb{Z})$ *and let* $\mathcal{D}_{ij}$, $1 \le i, j \le r$, *be finite subsets of* $\mathbb{Z}^n$, *with* $\cup_{j=1}^r \mathcal{D}_{ij}$ *non-empty. Suppose that, for every* $i \in \{1, ..., r\}$, *there exists a* $j_i \in \{1, ..., r\}$ *such that* $\mathcal{D}_{ij_i} + A_{j_i}\mathbb{Z}^n = \mathbb{Z}^n$. *Then* $\mu(Q) > 0$, *where* $Q = \cup_{i=1}^r Q_i$ *and the* Q_i *are defined by (1.2)*

The following proposition shows that there is a connection between the attractors of the first type and the attractors of the second type for some special matrices.

Proposition 3.2 *Let* $A \in M_n(\mathbb{Z})$ *be as in Theorem 1.1(ii) and let* $A_j = A^{N_j} \in M_n(\mathbb{Z})$, $1 \le j \le r$, $N_j \in \mathbb{N}$, *in (2.2). Then studying the measure properties of the attractor of the second type*

$$Q'_i = \cup_{j=1}^r A_j^{-1}(Q'_j + \mathcal{D}'_{ij}), \quad 1 \le i \le r,$$

is equivalent to studying the attractor of the first type

$$B_i(Q_i) = \cup_{j=1}^r (Q_j + \mathcal{D}_{ij}), \quad 1 \le i \le r.$$

for suitable B_i *and* $\mathcal{D}_{ij}$.

Proof. Let $N = \max\{N_1, ..., N_r\}$. Then let $B_i = A^{N_i}$ and $\mathcal{D}_{ij} = A^{N-N_j}\mathcal{D}'_{ij}$. Thus, by the uniqueness of attractors, $Q_i = A^{N-N_i}Q'_i$. This concludes the proof. □

As a direct application of Theorem 1.1 and Proposition 3.2, we get the following result.

Corollary 3.3 *Let* A_j *be as in Proposition 3.2. Suppose that for every* $i \in \{1, ..., r\}$, *there exists a* $j \in \{1, ..., r\}$ *such that* $A^{N-N_j}\mathcal{D}_{ij} + A^{N_i}\mathbb{Z}^n = \mathbb{Z}^n$. *Then* $\mu(Q') > 0$, *where* $Q' = \cup_{i=1}^r Q'_i$ *and the* Q'_i *are defined by (1.2). Furthermore,* Q' *has a non-empty interior.*

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