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On an Application of the Hardy Classes to the Riemann Zeta-Function

K. Ilgar Eroğlu and Iossif V. Ostrovskii

Abstract

We show that the function

$$f(z) := \frac{z}{1-z}\zeta\left(\frac{1}{1-z}\right), \quad |z| < 1,$$

belongs to the Hardy class H_p if and only if $0 < p < 1$.

Key Words: Riemann zeta-function, Hardy class, Poisson representation

1. Introduction

In [1], some applications of the Hardy classes to the Riemann zeta-function ζ were considered. In particular, the following fact has been established.

Theorem ([1]). *The function*

$$f(z) := \frac{z}{1-z}\zeta\left(\frac{1}{1-z}\right), \quad |z| < 1, \tag{1}$$

belongs to the Hardy class $H_{\frac{1}{3}}$.

In this connection, the following question arises. What is the set of all values of the parameter p, $0 < p \leq \infty$, such that $f \in H_p$? We answer this question. Our result is the following:

Theorem. *The function f defined by* (1) *belongs to H_p if and only if* $0 < p < 1$.

We also present a simpler proof of the main result of [1] (formula (8) below). Our proof is independent of the theory of the Hardy classes.

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2. Required results from the theory of zeta-function

Theorem 1 ([5], p.95).

$$\zeta(s) = O(|s|), \quad |s| \to \infty,\ \Re s \geq \frac{1}{2}.$$

Theorem 2 ([5], p.143).

$$\int_0^T |\zeta(\frac{1}{2} \pm ix)|^2 dx = T \log T + O(T), \quad 0 < T \to \infty.$$

Theorem 3 ([5], p.310). *Let*

$$E_T = \left\{ t \in [-T, T] : \frac{\log|\zeta(\frac{1}{2} + it)|}{\sqrt{\frac{1}{2}\log\log(3 + |t|)}} \geq 1 \right\}. \tag{2}$$

Then

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} E_T > 0.$$

Note that the result (by A. Selberg and A. Ghosh) stated in [5] is much more precise and general, than Theorem 3 which follows if one puts $R = \{z \in \mathbf{C} : 1 \leq \Re z \leq 2,\ |\Im z| \leq 1\}$ in the formula on p.310 (line 4 from below) of [5].

3. Required results from the theory of the Hardy classes

We remind ([2], p.68) that the Hardy class H_p, $0 < p \leq \infty$, is the set of all functions g analytic in the unit disc $\{z : |z| < 1\}$ and satisfying the condition:

$$k_p(g) := \sup_{0\leq r<1} \int_{-\pi}^{\pi} |g(re^{i\theta})|^p d\theta < \infty, \quad \text{if} \quad 0 < p < \infty,$$

or, for $p = \infty$,

$$k_\infty(g) := \sup_{0\leq r<1} \max_{-\pi\leq\theta\leq\pi} |g(re^{i\theta})| < \infty.$$

Evidently, $H_q \supset H_p$ for $q < p$.

Theorem 4 ([2], p.70). *If $g \in H_p$, then, for almost all $\theta \in [-\pi, \pi]$, there exists*

$$\lim_{r \to 1} g(re^{i\theta}) =: g(e^{i\theta})$$

and

$$k_p(g) = \int_{-\pi}^{\pi} |g(e^{i\theta})|^p d\theta. \tag{3}$$

Theorem 5 ([2], p.74). *Let $g \in H_q$. If $q < p$ and*

$$\int_{-\pi}^{\pi} |g(e^{i\theta})|^p d\theta < \infty,$$

then $g \in H_p$.

4. Proof of Theorem

Step 1. *We show that $f \in H_q$ for $0 < q < \frac{1}{2}$.*

If $|z| < 1$, then $\Re(1/(1-z)) > 1/2$. Therefore, by Theorem 1,

$$|f(z)| = \frac{|z|}{|1-z|} \left| \zeta\left(\frac{1}{1-z}\right) \right| \leq \frac{K}{|1-z|^2}$$

for some positive constant K. Hence

$$|f(re^{i\theta})| \leq \frac{K}{|e^{-i\theta} - r|^2} \leq \frac{K}{\sin^2 \theta}$$

and

$$k_q(f) \leq \int_{-\pi}^{\pi} \frac{K^q d\theta}{\sin^{2q} \theta} < \infty, \quad \text{for} \quad q < 1/2.$$

Step 2. *We show that $f \in H_p$ for $0 < p < 1$.*

By virtue of Theorem 5, it suffices to prove that, for any $0 < p < 1$,

$$\int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta < \infty.$$

Putting $t = \frac{1}{2}\cot\frac{\theta}{2}$ and noting that

$$\frac{1}{1-e^{i\theta}} = \frac{1}{2} + it,$$

we get

$$\int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta = \int_{-\pi}^{\pi} \left| \frac{1}{1-e^{i\theta}} \zeta\left(\frac{1}{1-e^{i\theta}}\right) \right|^p d\theta =$$

$$\int_{-\infty}^{\infty} (t^2 + \frac{1}{4})^{\frac{p}{2}-1} |\zeta(\frac{1}{2} + it)|^p dt. \tag{4}$$

Integration by parts gives

$$\int_1^T t^{p-2} |\zeta(\frac{1}{2} \pm it)|^p dt =$$

$$T^{p-2} \int_1^T |\zeta(\frac{1}{2} \pm ix)|^p dx + (2-p) \int_1^T t^{p-3} \left(\int_1^t |\zeta(\frac{1}{2} \pm ix)|^p dx \right) dt. \tag{5}$$

Using the Hölder inequality and then Theorem 2, we get

$$\int_1^t |\zeta(\frac{1}{2} \pm ix)|^p dx \le \left(\int_1^t |\zeta(\frac{1}{2} \pm ix)|^2 dx \right)^{\frac{p}{2}} t^{\frac{2-p}{2}} = t(\log t)^{\frac{p}{2}} + O(t), \quad t \to +\infty.$$

It follows that the RHS of (5) is bounded as $T \to +\infty$ and therefore the integrals in (4) converge.

Step 3. *We show that $f \notin H_p$ for $p \ge 1$.*

Since $H_p \subset H_1$ for $p > 1$, it suffices to prove that $f \notin H_1$. By the formula (3) of Theorem 4, the latter is equivalent to

$$\int_{-\pi}^{\pi} |f(e^{i\theta})| d\theta = +\infty.$$

As in (4), the change $t = \frac{1}{2}\cot\frac{\theta}{2}$ shows that

$$\int_{-\pi}^{\pi} |f(e^{i\theta})| d\theta = \int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2} + it)|}{\sqrt{t^2 + \frac{1}{4}}} dt. \tag{6}$$

Let E_T be the set defined by (2). Then

$$\int_{-T}^{T} \frac{|\zeta(\frac{1}{2}+it)|}{\sqrt{t^2+\frac{1}{4}}} \geq \int_{E_T} \frac{\exp\sqrt{\frac{1}{2}\log\log(3+|t|)}}{\sqrt{t^2+\frac{1}{4}}} dt. \tag{7}$$

Observe that the integrand in the RHS of (7) is a decreasing function of $|t|$ for large enough $|t|$, say, $|t| \geq T_0$. Let $F_T := E_T \setminus [-T_0, T_0]$. By Theorem 3, we have

$$\operatorname{meas} F_T > 2\alpha T$$

for some constant $\alpha > 0$ and sufficiently large T. The RHS in (7) diminishes if we replace E_T with F_T. Using also the decrease of the integrand in $|t|$, we get for sufficiently large T:

$$\int_{-T}^{T} \frac{|\zeta(\frac{1}{2}+it)|}{\sqrt{t^2+\frac{1}{4}}} dt \geq \int_{F_T} \frac{\exp\sqrt{\frac{1}{2}\log\log(3+|t|)}}{\sqrt{t^2+\frac{1}{4}}} dt \geq$$

$$\int_{(1-\alpha)T<|t|<T} \frac{\exp\sqrt{\frac{1}{2}\log\log(3+|t|)}}{\sqrt{t^2+\frac{1}{4}}} dt \geq 2\alpha T \frac{\exp\sqrt{\frac{1}{2}\log\log(3+T)}}{\sqrt{T^2+\frac{1}{4}}} \to \infty$$

as $T \to +\infty$. Therefore the integral in the RHS diverges. □

5. Remark

The fact $f \in H_{\frac{1}{3}}$ was applied in [1] for the proof of the following formula:

$$\frac{1}{2\pi} \int_{\Re s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^2} |ds| = \sum_{\Re\rho_k > \frac{1}{2}} \log\left|\frac{\rho_k}{1-\rho_k}\right|, \tag{8}$$

where ρ_k's are zeros of the function ζ. We will show that the formula (8) can be proved in a simpler way that is independent of the theory of the Hardy classes. We use the following known result:

Theorem 6 ([4], p.105, or, an equivalent result in [3], p.52). *Let F be a function analytic in the half-plane $\{w : \Im w \geq 0\}$ and let $\log|F(w)|$ has a positive harmonic majorant in $\{w : \Im w > 0\}$. Then the following (Poisson) representation holds for $\Im w > 0$:*

$$\log|F(w)| = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Im w \log|F(u)|}{|w-u|^2} du + \sum_{\Im a_k > 0} \log\left|\frac{w-a_k}{w-\bar{a}_k}\right| + \sigma \Im w, \tag{9}$$

where a_k's are zeros of F and

$$\sigma = \limsup_{v\to+\infty} v^{-1} \log |F(iv)|.$$

The integral and series in the RHS of (9) *converge absolutely.*

Let us derive the formula (8). Theorem 1 shows that, for $\Re s \geq \frac{1}{2}$,

$$\log |(s-1)\zeta(s)| \leq K \log |s+2|$$

holds with some positive constant K. The RHS of this inequality is a positive harmonic function in the half-plane $\{s : \Re s \geq \frac{1}{2}\}$. The transformation

$$w = i(s - \frac{1}{2}) \tag{10}$$

takes $\{s : \Re s \geq \frac{1}{2}\}$ to $\{w : \Im w \geq 0\}$. Therefore Theorem 6 is applicable to the function

$$F(w) := (s-1)\zeta(s), \tag{11}$$

where w and s are connected by (10), and therefore the formula (9) holds for the function. For the function (11), the parameter σ in (9) equals 0 because $\zeta(s) \to 1$ as $0 < s \to +\infty$. Hence

$$\log |F(i/2)| = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log |F(u)| du}{u^2 + \frac{1}{4}} + \sum_{\Im a_k > 0} \log \left| \frac{i - 2a_k}{i - 2\bar{a}_k} \right|. \tag{12}$$

Taking into account that

$$F(i/2) = \lim_{s\to1} (s-1)\zeta(s) = 1, \quad 2a_k = 2i\rho_k - i,$$

we rewrite (12) in the form

$$\frac{1}{2\pi} \int_{\Re s = \frac{1}{2}} \frac{\log |(s-1)\zeta(s)|}{|s|^2} |ds| = \sum_{\Re \rho_k > \frac{1}{2}} \log \left| \frac{\rho_k}{1 - \rho_k} \right|.$$

It remains to note that

$$\int_{\Re s = \frac{1}{2}} \frac{\log |s-1|}{|s|^2} |ds| = \Re \int_{-\infty}^{\infty} \frac{\log(\frac{1}{2} - iu)}{\frac{1}{4} + u^2} du = 0$$

by the residue theorem. □

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K. Ilgar EROĞLU
Department of Mathematics,
Bilkent University,
06533 Bilkent, Ankara-TURKEY
Iossif V. OSTROVSKII
Department of Mathematics,
Bilkent University,
06533 Bilkent, Ankara-TURKEY

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