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Tangent Lines of Generalized Regular Curves Parametrized by Time Scales

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Abstract

In this paper a generalization of the notion of regular curve is introduced. For such curves the concept of tangent line is investigated.

Key Words: Time scale, delta derivative, generalized regular curve,tangent line.

1. Introduction

In order to unify continuous and discrete analysis, by Aulbach and Hilger [5,10] was introduced the concept of time scale (or measure chain) and the theory of calculus on time scales. This theory has recently received a lot of attention and has proved to be useful in the mathematical modeling of several important dynamic processes. As a result the theory of dynamic systems on time scales is developed [7,12].

The general idea in this paper is to study curves where in the parametric equations the parameter varies in a so –called time scale, which may be an arbitrary closed subset of the set of all real numbers. So our intention is to use as the "differential" part of classical Differential Geometry the time scales calculus.

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2. Preliminaries from the Time Scales Calculus

For an introduction to the theory of calculus on time scales we refer to the original works by Aulbach and Hilger [5,10] and to the recently appeared works [1-4,6,7,9,11,12]. To meet the requirements in the next sections here we introduce the basic notions and notations connected to time scales analysis.

A *time scale* (or *measure chain*) $\mathbf{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbf{R}$. The time scale $\mathbf{T}$ is a complete metric space with the metric

$$d(t_1, t_2) = |t_1 - t_2|.$$

For $t \in \mathbf{T}$ we define the *forward jump operator* $\sigma : \mathbf{T} \longrightarrow \mathbf{T}$ by

$$\sigma(t) = inf\{s \in \mathbf{T} : s > t\}$$

while the *backward jump operator* $\rho : \mathbf{T} \longrightarrow \mathbf{T}$ is defined by

$$\rho(t) = sup\{s \in \mathbf{T} : s < t\}.$$

In this definition we put in addition $\sigma(\max \mathbf{T}) = \max \mathbf{T}$ if there exists a finite $\max\mathbf{T}$, and $\rho(\min \mathbf{T}) = \min \mathbf{T}$ if there exists a finite $\min\mathbf{T}$. Of course both $\sigma(t)$ and $\rho(t)$are in $\mathbf{T}$ when $t \in \mathbf{T}$. This is because of our assumption that $\mathbf{T}$ is a closed subset of $\mathbf{R}$.

If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Also, if $\sigma(t)= t$, then t is called *right-dense*, and if $\rho(t)= t$, then t is called *left-dense.*

We introduce the set $\mathbf{T}^k$ which is derived from the time scale $\mathbf{T}$ as follows. If $\mathbf{T}$ has a left-scattered maximum M , then $\mathbf{T}^k = \mathbf{T} - \{\mathrm{M}\}$,otherwise $\mathbf{T}^k = \mathbf{T}$.

For $a, b \in \mathbf{T}$ with $a \leq b$ we define the interval $[a, b]$ in $\mathbf{T}$ by

$$[a, b] = \{t \in \mathbf{T}: a \leq t \leq b\} .$$

Open intervals and half-open intervals etc. are defined accordingly. We will let $[a, b]^k$ denote $[a, \rho(b)]$ if b is left-scattered and denote $[a, b]$ if b is left-dense.

If $f : \mathbf{T} \to \mathbf{R}$ is a function and $t \in \mathbf{T}^k$, then the *delta derivative* of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)\,[\sigma(t) - s] \right| \leq \varepsilon \,|\, \sigma(t) - s \,| \quad \text{for all} \quad s \in U.$$

The following theorems either are in the references [1-7,9-12] or are not difficult to verify.

Theorem 2.1 *For f : $\boldsymbol{T}\to\boldsymbol{R}$ and $t\in \boldsymbol{T}^k$ the following hold:*

(i) If f is Δ- differentiable at t , then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is Δ- differentiable at t and

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) If t is right-dense , then f is Δ-differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case $f^{\Delta}(t)$ is equal to this limit.

(iv) If f is Δ - differentiable at t , then

$$f(\sigma(t)) = f(t) + [\sigma(t) - t]\, f^{\Delta}(t).$$

Theorem 2.2 *If f,g : $\boldsymbol{T} \to \boldsymbol{R}$ are Δ-differentiable at $t \in \boldsymbol{T}^k$, then*

(i) f+g is Δ-differentiable at t and

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) For any constant c , cf is Δ-differentiable at t and

$$(cf)^{\Delta}(t) \;=\; c\, f^{\Delta}(t).$$

(iii) f.g is Δ-differentiable at t and

$$(f.g)^{\Delta}(t) \;=\; f^{\Delta}(t)\, g(t) \;+\; f(\sigma(t))\, g^{\Delta}(t) \;=\; f(t)\, g^{\Delta}(t) + f^{\Delta}(t) g(\sigma(t)).$$

(iv) If $g(t).g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is Δ-differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta}(t) \;=\; \frac{f^{\Delta}(t)\, g(t) - g^{\Delta}(t)\, f(t)}{g(t)\, g(\sigma(t))}.$$

A function $F:\, \mathbf{T}\rightarrow\mathbf{R}$ is called a Δ-*antiderivative* of $f:\, \mathbf{T}\rightarrow\mathbf{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in\mathbf{T}^{k}$. Then, the Cauchy Δ- *integral* from a to b of f is defined by

$$\int_{a}^{b} f(t)\, \Delta t = \; F(b) \;-\; F(a) \text{ for all } a, b \in \mathbf{T}.$$

In order to find a class of functions which possess an antiderivative, we introduce now the set of rd-continuous functions.

Let f: $\mathbf{T}\rightarrow$ $\mathbf{R}$ be a function. We say that f is *rd-continuous* if it is continuous at each right-dense point in $\mathbf{T}$ and $\underset{s\rightarrow t^{-}}{lim} f(s)$exists as a finite number for all left-dense points t$\in\mathbf{T}$.The set of rd-continuous functions on a time scale $\mathbf{T}$ will be denoted by C$_{rd}$. The set of functions that are Δ-differentiable and whose Δ-derivative is rd-continuous is denoted by C^{Δ}_{rd}.

Theorem 2.3 *Rd-continuous functions possess a Δ- antiderivative.*

Let $\mathbf{T}$ be a time scale and $\upsilon\, :\, \mathbf{T}\, \rightarrow\, \mathbf{R}$ be a strictly increasing function such that $\overline{T} = \upsilon(T)$ is also a time scale. By $\overline{\sigma}$ and $\overline{\rho}$ we denote the jump functions on $\overline{T}$, and by $\overline{\Delta}$ we denote the derivative on $\overline{T}$. Then

$$\upsilon\, o\, \sigma = \overline{\sigma}\, o\, \upsilon \text{ and } \upsilon\, o\, \rho = \overline{\rho}\, o\, \upsilon.$$

Theorem 2.4 *(Chain Rule). Assume* $\upsilon : \boldsymbol{T} \to \boldsymbol{R}$ *is strictly increasing and* $\overline{\boldsymbol{T}} = \upsilon(\boldsymbol{T})$ *is a time scale. Let* $w : \overline{\boldsymbol{T}} \to \boldsymbol{R}$. *If* $\upsilon^{\Delta}(t)$ *and* $w^{\bar{\Delta}}(\upsilon(t))$ *exist for* $t \in \boldsymbol{T}^k$ *, then* $(wo\upsilon)^{\Delta}$ *exists at t and satisfies the chain rule*

$$(w \circ \upsilon)^{\Delta} = (w^{\bar{\Delta}} \circ \upsilon)\upsilon^{\Delta} \ \text{at } t.$$

Theorem 2.5 *(Substitution). Assume* $\upsilon : \boldsymbol{T} \to \boldsymbol{R}$ *is strictly increasing and* $\overline{\boldsymbol{T}} = \upsilon(\boldsymbol{T})$ *is a time scale. If* $f : \boldsymbol{T} \to \boldsymbol{R}$ *is an rd-continuous function and* υ *is* Δ*- differentiable with rd-continuous* Δ*-derivative , then if* $a, b \in \boldsymbol{T}$

$$\int_a^b f(t)\upsilon^{\Delta}(t)\,\Delta t = \int_{\upsilon(a)}^{\upsilon(b)} (f \circ \upsilon^{-1})(s)\bar{\Delta}s.$$

3. Generalized Regular Curves. Tangent Lines of a Curve

The *Euclidean scalar product* of two real vectors $\xi = (\xi_1, \xi_2, ..., \xi_n)$ and $\zeta = (\zeta_1, \zeta_2, ..., \zeta_n)$ is the number

$$< \xi, \zeta >= \sum_{i=1}^{n} \xi_i \zeta_i.$$

The length (or norm) of a vector ξ, which we denote by $\|\xi\|$, is given by

$$\|\xi\| = \sqrt{< \xi, \xi >} = \sqrt{\sum_{i=1}^{n} \xi_i^2}.$$

Let **T** be a time scale.

Definition 3.1 *A* Δ*-regular curve (or an arc of a* Δ*-regular curve)* γ *is defined as a mapping*

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t), \quad t \in [a, b] \tag{3.1}$$

of the segment $[a,b] \subset \boldsymbol{T}$ *,* $a < b$ *, to the space* $\boldsymbol{R}^3$ *, where* f_1, f_2, f_3 *are real-valued functions defined on* $[a,b]$ *that are* Δ*-differentiable on* $[a,b]^k$ *with rd-continuous* Δ*-derivatives and*

$$\left|f_1^{\Delta}(t)\right|^2 + \left|f_2^{\Delta}(t)\right|^2 + \left|f_3^{\Delta}(t)\right|^2 \neq 0, t \in [a,b]^k. \tag{3.2}$$

Setting

$$r = (x,y,z), f(t) = (f_1(t), f_2(t), f_3(t))$$

we can rewrite the equations (3.1) in the vector form

$$r = f(t), t \in [a,b], \tag{3.3}$$

and the condition (3.2) in the form

$$\left\|f^{\Delta}(t)\right\| \neq 0, t \in [a,b]^k. \tag{3.4}$$

Definition 3.2 *Let* γ *be a curve given in the parametric form (3.1),*

$P_0 = (f_1(t_0), f_2(t_0), f_3(t_0)), t_0 \in [a,b]^k$ *, be a point on* γ*, and* L *be a line through* P_0^{σ}*, where*

$$P_0^{\sigma} = (f_1(\sigma(t_0)), f_2(\sigma(t_0)), f_3(\sigma(t_0))).$$

Take on γ *any point* P *. Denote by* d *the distance of the point* P *from the point* P_0^{σ}*, and by* δ *the distance of* P *from the line* L*. If* $\frac{\delta}{d} \to 0$ *as* $P \to P_0, P \neq P_0^{\sigma}$*, then we say that* L *is the forward tangent line to the curve* γ *at the point* P_0*.*

It is no difficult to see that if the curve γ has the forward tangent line L at the point P_0 , then the line PP_0^{σ} will converge to L as $P \to P_0$, $P \neq P_0^{\sigma}$. Conversely, if the line PP_0^{σ} converges to some line as $P \to P_0$, $P \neq P_0^{\sigma}$, then this limiting line will be the forward tangent line at P_0. For the proof it is sufficient to note that if α is the angle between lines L and PP_0^{σ}, then $\frac{\delta}{d} = \sin\alpha$.

Theorem 3.1 *Every $\Delta-$regular curve γ given by (3.3) has at any point*
$P_0 = (f_1(t_0), f_2(t_0), f_3(t_0)), t_0 \in [a,b]^k$ *, the forward tangent line that has the vector $f^{\Delta}(t_0)$as its direction vector.*

Proof. Suppose the curve γ has the forward tangent line L at the point $P_0(t_0)$. Let τ be a unit vector on the line L. The distance d of the point $P(t)$ from the point P_0^{σ} is equal to $\|f(t) - f(\sigma(t_0))\|$. Further, the distance δ of the point $P(t)$ from the line L is equal to $\|[f(t) - f(\sigma(t_0)] \wedge \tau\|$, where $\wedge$ denotes the vector product. By the definition of the forward tangent line, we have

$$\frac{\delta}{d} = \frac{\| [f(t) - f(\sigma(t_0)] \wedge \tau\|}{\|f(t) - f(\sigma(t_0)\|} \to 0 \quad (t \to t_0, t \neq \sigma(t_0)).$$

On the other hand

$$\frac{\| [f(t) - f(\sigma(t_0))] \wedge \tau\|}{\|f(t) - f(\sigma(t_0))\|} = \frac{\left\| \frac{f(t)-f(\sigma(t_0))}{t-\sigma(t_0)} \wedge \tau \right\|}{\left\| \frac{f(t)-f(\sigma(t_0))}{t-\sigma(t_0)} \right\|} \to \frac{\left\| f^{\Delta}(t_0) \wedge \tau \right\|}{\|f^{\Delta}(t_0)\|} \quad (t \to t_0, t \neq \sigma(t_0)).$$

Therefore $f^{\Delta}(t_0) \wedge \tau = 0$. Since $f^{\Delta}(t_0) \neq 0$, it follows that the vectors $f^{\Delta}(t_0)$ and $\tau$are collinear. Thus, if the forward tangent line exists, then it has $f^{\Delta}(t_0)$ as a direction vector and therefore it is unique. □

Conversely, let L be a line through the point P_0^{σ} and have the vector $f^{\Delta}(t_0)$ as its direction vector. Then the line L will be the forward tangent line at the point P_0. Indeed, as above, we have

$$\frac{\delta}{d} = \frac{\left\| [f(t) - f(\sigma(t_0)] \wedge \frac{f^{\Delta}(t_0)}{\|f^{\Delta}(t_0)\|} \right\|}{\|f(t) - f(\sigma(t_0)\|} \quad \to \quad \frac{\left\| f^{\Delta}(t_0) \wedge f^{\Delta}(t_0) \right\|}{\|f^{\Delta}(t_0)\|^2} = 0$$

The theorem is proved.

Remark 1 *It is easy to see that in the case $P_0 \neq P_0^{\sigma}$ the forward tangent line at the point P_0 to the curve γ will coincide with the line through the points P_0 and P_0^{σ}.*

From Theorem 3.1 it follows that the equation of the forward tangent line at the point $P_0(t_0)$, $t_0 \in [a,b]^k$, to the $\Delta-$regular curve γ given by the equation (3.1), will be

$$\frac{X - f_1(t_0)}{f_1^{\Delta}(t_0)} = \frac{Y - f_2(t_0)}{f_2^{\Delta}(t_0)} = \frac{Z - f_3(t_0)}{f_3^{\Delta}(t_0)}.$$

In the case of plane curve the equation of the forward tangent line is

$$\frac{X - f_1(t_0)}{f_1^{\Delta}(t_0)} = \frac{Y - f_2(t_0)}{f_2^{\Delta}(t_0)}.$$

If a Δ-regular plane curve is given by the equation

$$y = f(x), x \in [a,b] \subset \mathbf{T},$$

then the equation of the forward tangent line at the point $P_0(x_0)$, $x_0 \in [a,b]^k$, is

$$Y - f(x_0) = f^{\Delta}(x_0)\,(X - x_0).$$

4. Natural Parametrizations of a Curve

Let γ be a $\Delta-$regular curve given by the equation

$$r = f(t), t \in [a,b] \subset \mathbf{T}. \tag{4.5}$$

Introduce the function p(t) by the formula

$$p(t) = \int_a^t \left\| r^{\Delta}(s) \right\| \Delta s, t \in [a,b]. \tag{4.6}$$

Note that rd-continuity of $r^{\Delta}(t)$ implies rd- continuity of $\left\| r^{\Delta}(t) \right\|$ and therefore the integral (4.2) is well defined by Theorem 2.3.

The function $p(t)$ is strictly increasing and continuous (in the topology of $\mathbf{T}$). Therefore $p(\ [a,b]\ )$, the image of $[a,b]$ under the map p , will be a time scale. The forward jump function and the delta derivative on this time scale we will denote by $\tilde{\sigma}$ and $\tilde{\Delta}$, respectively.

The variable p can be used as a parameter for the curve γ. Such a parametrization of a curve we call *natural* (or *intrinsic*) Δ- *parametrization*.

By (4.2) the variable t becomes a function of $p : t = t(p)$. Therefore we have $r(t) = r(t(p))$.

For causing no ambiguity sometimes in writing the derivative of a function, we will indicate in subscript explicitly the parameter with respect to which the derivative is calculated.

Theorem 4.1 *In the case of natural* Δ- *parametrization of the curve* γ *the forward tangent vector* $r_p^{\tilde{\Delta}}$ *is a unit vector, i.e.,* $\left\| r_p^{\tilde{\Delta}} \right\| = 1$.

Proof. From (4.2) we have

$$p^{\Delta}(t) = \left\| r^{\Delta}(t) \right\| > 0, \quad t \in [a, b).$$

Therefore by chain rule (Theorem 2.4) we get

$$r_t^{\Delta} = r_p^{\overline{\Delta}} p_t^{\Delta} = r_p^{\overline{\Delta}} \left\| r_t^{\Delta} \right\|$$

and hence

$$r_p^{\overline{\Delta}} = \frac{r_t^{\Delta}}{\left\| r_t^{\Delta} \right\|}.$$

Consequently $\left\| r_p^{\tilde{\Delta}} \right\| = 1$. The theorem is proved. □

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