Turk J Math
25 (2001) , 427 – 432.

On the L^p Solutions of Dilation Equations

İbrahim Kırat

Abstract

Let $A \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det(A)| = q$ and let $K = \{k_1 \cdots k_q\} \subseteq \mathbb{R}^n$ be a digit set. The set $\mathcal{T} =: \mathcal{T}(A, K) = \{\sum_{i=1}^{\infty} A^{-i} k_{j_i} : k_{j_i} \in K\} \subset \mathbb{R}^n$ is called a *self-affine tile* if the Lebesgue measure of $\mathcal{T}$ is positive. In this note, we consider dilation equations of the form $f(x) = \sum_{j=1}^{q} c_j f(Ax - k_j)$ with $q = \sum_{j=1}^{q} c_j$, $c_j \in \mathbb{R}$, and prove that this equation has a nontrivial L^p solution $(1 \le p \le \infty)$ if and only if $c_j = 1$ $\forall j \in \{1, ..., q\}$ and $\mathcal{T}$ is a tile.

Key Words: Dilation equtions, tiles, wavelets, self-similar measures

1. Introduction

Let A be an expanding integral matrix in $M_n(\mathbb{Z})$, i.e., all its eigenvalues λ_i have modulus > 1. Let $|\det(A)| = q$ and let $K = \{k_1, \cdots, k_q\} \subseteq \mathbb{R}^n$ be a set of q distinct vectors. The affine maps S_j defined by

$$S_j(x) = A^{-1}(x + k_j), \quad 1 \le j \le q,$$

are all contractions under a suitable norm in $\mathbb{R}^n$ (see [8, pp. 29-30]). The family $\{S_j\}_{j=1}^{q}$ is called an *iterated function system* (IFS) and there is a unique non-empty compact set satisfying $\mathcal{T} = \bigcup_{j=1}^{q} S_j(\mathcal{T})$ ([1], [5]). $\mathcal{T}$ is called the *attractor* of the system and is given explicitly by

$$\mathcal{T} := \mathcal{T}(A, K) = \{\sum_{i=1}^{\infty} A^{-i} k_{j_i} : k_{j_i} \in K\}.$$

2000 *Mathematics Subject Classification.* 26A15, 26A18

We call $\mathcal{T}$ a *self-affine tile* if it has a positive Lebesgue measure. We use $m(\mathcal{T})$ to denote the Lebesgue measure of the set $\mathcal{T}$. The case that $m(\mathcal{T}) = 1$ is of particular interest and we call such $\mathcal{T}$ a *Haar tile*. Gröchenig and Madych [3] showed that for $A \in M_n(\mathbb{Z})$ and $K \subseteq \mathbb{Z}^n$, $\chi_{\mathcal{T}}$ generates a compactly supported orthonormal wavelet basis of $L^2(\mathbb{R}^n)$ if and only if $m(\mathcal{T}) = 1$.

Dilation equations play an important role in computer graphics and wavelet analysis. Recently, the existence of solutions of dilation equations has been studied by the mathematicians in approximation theory and wavelet analysis [2], [3], [4], [6]. The recent characterizations of L^p solutions of dilation equations in terms of p-norm joint spectral radius require considerable amount of computation.

Here we consider a special dilation equation and determine the c_j explicitly. In this paper, we study the L^p solutions of dilation equations of the following form

$$f(x) = \sum_{j=1}^{q} c_j f(Ax - k_j) \tag{1.1}$$

where $|\det(A)| = q = \sum_{j=1}^{q} c_j$, $k_j \in \mathbb{R}^n$, $c_j \in \mathbb{R}$. It is well known that there exists a unique compactly supported distribution f satisfying (1.1) subject to the condition $\widehat{f}(0) = 1$ (hat stands for the Fourier transform). This distribution is called the *normalized solution* to (1.1).

The main result of this note is that equation (1.1) has a nontrivial L^p-solution $(1 \leq p \leq \infty)$ if and only if $c_j = 1$ for every $j = 1, 2, ..., q$, and $m(\mathcal{T}) > 0$. Moreover, if (1.1) has nontrivial L^p-solutions, then $f = \chi_{\mathcal{T}}$ is such a solution.

2. Self-similar measures and dilation equations

A self-similar measure is defined to be a finite real measure on the Borel subsets of $\mathbb{R}^n$ with mass 1 which satisfies an identity

$$\mu = \sum_{j=1}^{q} w_j \mu \circ S_j^{-1}, \tag{2.1}$$

where $\sum_{j=1}^{q} w_j = 1,\ w_j \in \mathbb{R}$. It is known that there is a unique finite real measure with bounded support and mass 1 satisfying (2.1) [5]. Hutchinson introduced the important notion of *open set condition*: there exists a nonempty bounded open set U such that $S_jU \subseteq U$, $j = 1, ..., q$ and the sets S_jU are disjoint. We say that open set condition holds with *measure-theoretic disjointness* if there exists a nonempty bounded open set U such that $S_jU \subseteq U$, $j = 1, ..., q$ and $\mu(S_j\overline{U} \cap S_k\overline{U}) = 0$ for $j \neq k$ ($\overline{U}$ denotes the closure of U) [9].

Let $\mathcal{M}$ denote the set of finite real measures on the Borel subsets of $\mathbb{R}^n$ with bounded support and mass 1. For $\mu, \nu \in \mathcal{M}$, let

$$L(\mu, \nu) = \sup\{\mu(\phi) - \nu(\phi) : \phi : \mathbb{R}^n \to \mathbb{R}, \mathrm{Lip}\phi \leq 1\}$$

where $\mathrm{Lip}\phi = \sup\limits_{x \neq y} \frac{d(\phi(x), \phi(y))}{d(x,y)}$, $\mu(\phi) = \int \phi d\mu$. Then L is a metric on $\mathcal{M}$. We note that the metric L was originally defined by Hutchinson for probability measures [5]. ϕ is called a contraction if $\mathrm{Lip}\phi < 1$. Let $(S, w)(\nu)(E) = \sum_{j=1}^{q} w_j \nu \circ S_j^{-1}(E)$. Then $(S, w) : \mathcal{M} \to \mathcal{M}$ is a contraction map in the L metric [5].

LEMMA 2.1 *Let $m(\mathcal{T}) > 0$. Then for μ in (2.1), open set condition holds with measure-theoretic disjointness.*

Proof. Let m' denote the Lebesgue measure restricted to $\mathcal{T}$ and let m'' be the normalized m'. Then $(S, w)^n(m'') \to \mu$ in the L metric. Also $\mathcal{T}^\circ \neq \emptyset$ and $m(\partial\mathcal{T}) = 0$ [8] (Here we use the standard notation of topology). This implies that open set condition holds (choosing $U = \mathcal{T}^\circ$ as the nonempty bounded open set). Hence $(S, w)^n(m'')$ satisfies open set condition with measure-theoretic disjointness for each $n \in \mathbb{N}$. It follows that open set condition holds with measure theoretic disjointness. $\square$

We note that for self-similar probability measures, Strichartz conjectured that open set condition implies open set condition with measure-theoretic disjointness in general [9]. Lemma 2.1 shows that this conjecture holds true in our case here.

LEMMA 2.2 *Suppose that*

$$f(x) = \sum_{j=1}^{q} c_j f(Ax - k_j), \quad \text{where} \quad \sum_{j=1}^{q} c_j = q,$$

has a non-trivial L^p-solution ($1 \leq p \leq \infty$). Then $c_j \geq 0,\ 1 \leq j \leq q$.

Proof. Suppose that f is the nontrivial normalized solution. Then f is supported by $\mathcal{T}$ and $m(\mathcal{T}) > 0$. We can treat

$$f(x) = \sum_{j=1}^{q} c_j f(Ax - k_j)$$

as

$$\mu = \sum_{j=1}^{q} w_j \mu \circ S_j^{-1}$$

where $\mu(E) = \int_E f dm$.

Let $U = \mathcal{T}^\circ$ and $||\mu||$ denote the total variation of μ. We claim that $w_j \geq 0$. Otherwise, we see that $\sum |w_j| > 1$, let $\mu(U) = c$, then $\mu(S_j U) = cw_j$, $\mu(S_J U) = cw_J$, $J = (j_1, ..., j_m)$ is a multi-index, so that $||\mu|| \geq \sum_{|J|=n} |\mu(S_J U)| = c \sum_{|J|=n} |w_J| = c(\sum_{j=1}^{q} |w_j|)^n \to \infty$ as $n \to \infty$, a contradiction. $\square$

Now by Lemma 2.2, in order to have a non-trivial L^p-solution of (1.1), we must have $c_j \geq 0,\ 1 \leq j \leq q$. In order to solve the dilation equation (1.1), we can start with a nonnegative compactly supported function $\phi \in L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ and use the iteration scheme $\phi_n := T^n \phi$, $n = 0, 1, 2, ...$, where $T\phi := \sum_{j=1}^{q} c_j \phi(Ax - k_j)$. It follows that $f \geq 0$ a.e. and $\mu = \int f dm$ is a probability measure. Also note that $\text{supp}(\mu) \subseteq \text{supp}(f) \subseteq \mathcal{T}$.

Let Lip$S_j = r_j$ and let $\mu = \sum_{j=1}^{q} w_j \mu \circ S_j^{-1}$. We define the self similarity dimension of μ to be the unique positive α such that

$$\alpha = \frac{\sum_{j=1}^{q} w_j log w_j}{\sum_{j=1}^{q} w_j log r_j}.$$

There is a unique positive number s such that $\sum_{j=1}^{q} r_j^s = 1$. When $w_j = r_j^s$, which we call the natural weights, s is the self-similarity dimension of μ. In the case of natural weights, $w_j = \frac{1}{q}$ since the r_j are all equal. It is known that $\alpha \leq s$ and if the w_j are not the natural weights, then $\alpha < s$ [9].

THEOREM 2.3 *Let $A \in M_n(\mathbb{Z})$ be an expanding matrix. Then*

$$f(x) = \sum_{j=1}^{q} c_j f(Ax - k_j)$$

where $|\det(A)| = \sum_{j=1}^{q} c_i = q$, $k_j \in \mathbb{R}^n$, *has a nontrivial* L^p*-solution* $(1 \leq p \leq \infty)$ *if and only if* $c_j = 1\ \forall j \in \{1, ..., q\}$ *and* $m(\mathcal{T}) > 0$.

Proof. The sufficiency is easy since $f = \chi_{\mathcal{T}}$ when $c_j = 1\ \forall j \in \{1, ..., q\}$ and $m(\mathcal{T}) > 0$. We now prove the necessity. Suppose that f is the nontrivial normalized solution. Then the support of f, $\mathrm{supp}(f)$, has positive Lebesgue measure. As in Lemma 2.2, we can treat

$$f(x) = \sum_{j=1}^{q} c_j f(Ax - k_i)$$

as

$$\mu = \sum_{i=j}^{q} w_j \mu \circ S_j^{-1}$$

where $w_j = \frac{c_j}{q}$. By Lemma 2.2, we have $w_j \geq 0$. Let α be the self-similarity dimension of μ. Since the open set condition holds with measure-theoretic disjointness (Lemma 2.1), there exists a measurable set G which supports μ such that $\mu_\beta(G) = 0\ \forall \beta > \alpha$, where μ_β is the β-dimensional Hausdorff measure [Theorem 2.1, 9]. Suppose that the w_j are not the natural weights. It follows that $\alpha < s \leq n$. This yields that $m(G) = 0$. Let $\mathrm{supp}(f) = F$. Hence $\mu(F \setminus G) = 0$ since G supports μ and $m(F \setminus G) > 0$. This means that $\mu(F \setminus G) = \int_{F \setminus G} f dm = 0$. This is a contradiction since $m(F \setminus G) > 0$ and $f(x) > 0$ for $x \in F \setminus G$ a.e.. Hence the w_j, $j = 1, ..., q$, must be the natural weights and $f = \chi_{\mathcal{T}}$ where $\mathcal{T} = \mathcal{T}(A, K)$. □

We note that Theorem 2.1 in [9] was proved for similarities but the theorem is obviously true for expanding matrices. In general, it is difficult to determine if $m(\mathcal{T}) > 0$. However, for $K = \{0, k\} \subseteq \mathbb{R}^n$, we have the following simple corollary.

Corollary 2.4 *Suppose that* $A \in M_n(\mathbb{Z})$ *is an expanding matrix with* $|\det(A)| = 2$. *Then*

$$f(x) = c_1 f(Ax) + c_2 f(Ax - k),\ \ k \in \mathbb{R}^n$$

has a nontrivial L^p*-solution if and only if* $c_1 = c_2 = 1$ *and* $\{k, Ak, ..., A^{n-1}k\}$ *is a linearly independent set.*

Proof. The proof directly follows from Theorem 1.1 in [7] and Theorem 2.3. □

Acknowledgements. The author takes this opportunity to thank the mathematics department of Istanbul Technical University for its hospitality. He also thanks the referee for helpful suggestions.

References

[1] Falconer, K.: *Fractal Geometry, Mathematical Foundations and Applications*, Wiley, 1990.

[2] Flaherty, T. and Wang, Y.: Haar-type multiwavelet bases and self-affine multitiles, *Asian J. Math.* 3 (1999), no. 2, 387-400.

[3] Gröchenig, K. and Madych, W.: Multiresolution analysis, Haar bases, and self-similar tilings, *IEEE Trans. Info. Theory* 38 (1992), 556-568.

[4] Han, B. and Jia, R. Q.: Multivariate refinement equations and convergence of subdivision schemes, *SIAM J. Math. Anal.* 29 (1998), 1177-1199.

[5] Hutchinson, J.: Fractals and self-similarity, *Indiana Univ. Math. J.* 30 (1981), 713-747.

[6] Jia, R. Q., Lau, K. S. and Zhou D. X.: L_p solutions of vector refinement equations, *SIAM J. Math. Anal.* to appear.

[7] Kirat, I. and Lau, K. S.: On the connectedness of self-affine tiles, *J. London Math. Soc.*, (2) 62 (2000), 291-304.

[8] Lagarias, J. C. and Wang Y.: Self-affine tiles in $\mathbb{R}^n$, *Adv. in Math.* 121 (1996), 21-49.

[9] Strichartz, R. S.: Self-Similar Measures and Their Fourier Transforms I, *Indiana Univ. Math. J.* 39 (1990), 797-817.

İbrahim KIRAT
Department of Mathematics
Sakarya University
54140 Sakarya-TURKEY

Received 26.02.2001