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Induced Cat^1-groups

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Abstract

In this paper we define the pullback cat^1-group and show that this Pullback has a right adjoint which is the induced cat^1-group. Later we show that this right adjoint is a pushout of category of cat^1-groups. We calculate the Peiffer subgroups to find a finite group of the source of induced cat^1-groups. The generating set of Peiffer subgroups are also given in this paper. All results are corrected by a GAP[13] program package in [4]. This paper also contains the some computational examples which are the calculation-induced cat^1-group and comparative times between the induced crossed modules and induced cat^1-groups.

Key Words: Pullback, Crossed module, Cat1-group, induced crossed modules, induced cat^1-groups, Peiffer commutators, cocomplete, GAP.

1. Introduction

We begin by considering the possibility of implementing functions for doing calculations with crossed modules, derivations, actor crossed modules, cat^1-groups, sections, induced crossed modules and induced cat^1-groups in GAP[13].

To this end, we first explain the importance of crossed modules. The general points are:

- crossed modules may be thought of as 2-dimensional groups;
- a number of phenomena in group theory are better seen from a crossed module point of view;
- crossed modules occur geometrically as $\pi_2(X, A) \to \pi_1 A$ when A is a subspace of X or as $\pi_1 F \to \pi_1 E$ where $F \to E \to B$ is a fibration;

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- crossed modules are usefully related to forms of double groupoids.

Particular constructions, such as induced crossed modules, are important for the applications of the 2-dimensional Van-Kampen Theorem of Brown and Higgins [6], and so for the computation of homotopy 2-types.

For all these reasons, the facilitation of the computations with crossed modules should be advantageous. It should help to solve specific problems, and it should make it easier to construct examples and see relations with better known theories.

The powerful computer algebra system GAP[13] provides a high level programming language with several advantages for the coding of new mathematical structures. The GAP system has been developed over the last 15 years at RWTH in Aachen. Some of its most exciting features are:

- it has a highly developed, easy to understand programming language incorporated;
- it is especially powerful for group theory;
- it is portable to a wide variety of operating systems on many hardware platforms.
- it is public domain and it has a lively forum, with open discussion. These make it increasingly used by the mathematical community.

On the other hand, GAP has some disadvantages, too:

- the built in programming language is an interpreted language, which makes GAP programs relatively slow compared to compiled languages such as C or Pascal. GAP source can not be compiled.
- the demands on system resources are quite high for serious calculations.

However, the advantages outweigh the disadvantages, and so GAP was chosen.

Our aim in this paper is to describe a share package XMOD [4] for the GAP group theory language which enables computations with the equivalent notions of finite, permutation *crossed modules* and *cat1-groups*.

The term crossed module was introduced by J. H. C. Whitehead in [15]. Most of crossed modules references state the axioms of a crossed module using left actions, but we shall use right actions since this is the convention used by most computational group packages.

A crossed module $\mathcal{X} = (\partial : S \to R)$ consists of a group homomorphism ∂, called the *boundary* of $\mathcal{X}$, together with an action $\alpha : R \to \mathrm{Aut}(S)$ satisfying, for all $s, s' \in S$ and $r \in R$,

$$\begin{array}{lrcl} \textbf{XMod 1:} & \partial(s^r) & = & r^{-1}(\partial s)r \\ \textbf{XMod 2:} & s^{\partial s'} & = & {s'}^{-1}ss'. \end{array}$$

The kernel of ∂ is abelian.

Standard constructions for crossed modules include the following:

1. A *conjugation crossed module* is an inclusion of a normal subgroup $S \trianglelefteq R$, where R acts on S by conjugation.

2. A *central extension crossed module* has as boundary a surjection $\partial : S \to R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$.

3. An *automorphism crossed module* has as range a subgroup R of the automorphism group $\mathrm{Aut}(S)$ of S which contains the inner automorphism group of S. The boundary maps $s \in S$ to the inner automorphism of S by s.

4. An *R-Module crossed module* has an R-module as source and ∂ is the zero map.

5. The direct product $\mathcal{X}_1 \times \mathcal{X}_2$ of two crossed modules has source $S_1 \times S_2$, range $R_1 \times R_2$ and boundary $\partial_1 \times \partial_2$, with $R_1,\ R_2$ acting trivially on $S_2,\ S_1$, respectively.

A morphism between two crossed modules $\mathcal{X}_1$ and $\mathcal{X}_2$ is a pair (σ, ρ), where $\sigma : S_1 \to S_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$\partial_2 \sigma = \rho \partial_1, \quad \sigma(s^r) = (\sigma s)^{\rho r}.$$

When $\mathcal{X}_2 = \mathcal{X}_1$ and σ, ρ are automorphisms then (σ, ρ) is an automorphism of $\mathcal{X}_1$. The group of automorphisms is denoted by $\mathrm{Aut}(\mathcal{X}_1)$.

In [11] Loday reformulated the notion of a crossed module as a cat1-group, namely a group G with a pair of homomorphisms $t, h : G \to G$ having a common image R and satisfying certain axioms. We find it convenient to define a cat1-group $\mathcal{C} = (e; t, h : G \to R)$ as having source group G, range group R, and three homomorphisms: two surjections $t, h : G \to R$ and an embedding $e : R \to G$ satisfying:

$$\begin{array}{ll} \textbf{Cat 1:} & te = he = \mathrm{id}_R, \\ \textbf{Cat 2:} & [\ker t, \ker h] = \{1_G\}. \end{array}$$

The maps t, h are usually referred to as the *source* and *target*, but we choose to call them the *tail* and *head* of $\mathcal{C}$, because *source* is the GAP term for the domain of a function.

A morphism $\mathcal{C}_1 \to \mathcal{C}_2$ of cat1-groups is a pair (γ, ρ) where $\gamma : G_1 \to G_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$h_2\gamma = \rho h_1, \quad t_2\gamma = \rho t_1, \quad e_2\rho = \gamma e_1. \tag{1}$$

Induced crossed modules were introduced by Brown and Higgins in [6]. Later, induced cat^n-group structures were defined by Brown and Loday in [7]. In this paper, they gave some applications of cat^n-groups. They also showed that the H-module f_*M induced from a G-module M by a morphism $f : G \to H$ is given by $f_*M = M \otimes_{ZG} ZH$. Thus f_* is a left adjoint to the pull-back functor

$$f^* : (H - \text{modules}) \to (G - \text{modules}).$$

In the case of crossed modules, the inducing functor

$$f_* : (\text{crossed H} - \text{modules}) \to (\text{crossed G} - \text{modules})$$

is the left adjoint of the pull-back functor

$$f^* : (\text{crossed } H - \text{modules}) \to (\text{crossed G} - \text{modules}).$$

In addition, they used the notion of induced crossed square and the corresponding notion for cat^n-groups. See the historical background given by Brown in [5] for further details. The pull-back crossed modules were defined in [9] and [10]. In section 2, we describe the construction of pullback crossed modules and induced crossed modules. We also define pullback and induced cat^1-groups. Section 3 contains an outline algorithm for computing induced cat^1-groups and a table of sample execution times.

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2. Pull-back crossed modules and Pull-back Cat^1-groups

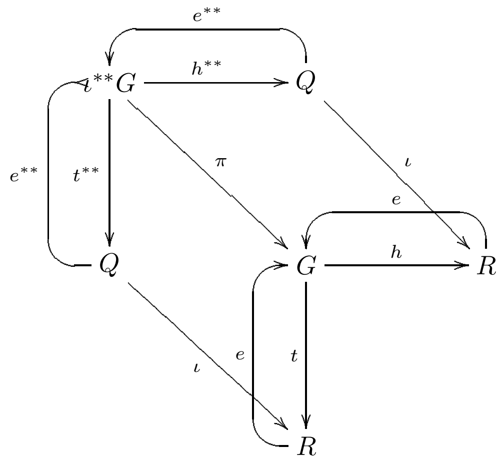
Let $\mathcal{X} = (\partial : S \to R)$ be a crossed R-module and $\iota : Q \to R$ be a morphism of groups. Then $\iota^{**}\mathcal{X} = (\partial^\bullet : \iota^{**}S \to Q)$ is the pullback of $\mathcal{X}$ by ι, where $\iota^{**}S = \{(q, s) \in Q \times S \mid \iota q = \partial s\}$ and $\partial^\bullet(q, s) = q$. The action of Q on $\iota^{**}S$ is given by

$$(q_1, s)^q = (q^{-1}q_1q, s^{\iota q}). \tag{2}$$

Proposition 2.1 *[6]* The functor $\iota^{**} : \mathcal{XM}/R \to \mathcal{XM}/Q$ has a right adjoint ι_{**}.

The universal property of induced crossed modules and the proof of above proposition can be found in [9].

A pullback cat^1-group is defined as follows.



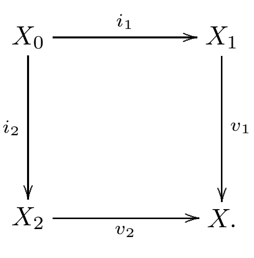
Let $\mathcal{C} = (e; t, h : G \to R)$ be a cat^1-group and let $\iota : Q \to R$ be a group homomorphism. Define $\iota^{**}\mathcal{C} = (e^{**}; t^{**}, h^{**} : \iota^{**}G \to Q)$ to be the pullback of G where

$$\iota^{**}G = \{(q_1, g, q_2) \in Q \times G \times Q \mid \iota q_1 = tg, \iota q_2 = hg\},$$

$t^{**}(q_1, g, q_2) = q_1$, $h^{**}(q_1, g, q_2) = q_2$ and $e^{**}(q) = (q, e\iota q, q)$. Multiplication in $\iota^{**}G$ is componentwise. The pair (π, ι) is a morphism of cat^1-groups where $\pi : \iota^{**}G \to G$, $(q_1, g, q_2) \mapsto g$. The verifications of the Cat^1-group axioms and the universal property of induced Cat^1-groups can be found in [1].

Proposition 2.2 The category of cat^1-groups is cocomplete.

We recall the definition of pushouts in a general category. Suppose we are given a commutative diagram of morphisms in a category **C**. Then



Recall [12] that (v_1, v_2) is pushout of (i_1, i_2), and also that the above square is a pushout square, if the following property holds: if $f_1 : X_1 \to H,\ f_2 : X_2 \to H$ are morphisms such that $f_1 i_1 = f_2 i_2$ then there is a unique map $f : X \to H$ such that $fv_1 = f_1,\ \ fv_2 = f_2$.

As usual, this property characterizes the pair (v_1, v_2) up to an automorphism of X. For this reason, it is common to coin an abuse of language and refer to X as the pushout of (i_1, i_2). In such case, we write

$$X = X_2 *_{X_0} X_1,$$

where $*_{X_0}$ denotes a free product with amalgamation over X_0.

Proposition 2.3 The functor $\iota^{**} : Cat^1Grp/U \to Cat^1Grp/R$ has a left adjoint $\iota_{**} : Cat^1Grp/R \to Cat^1Grp/U$.

Proof. The proof of proposition can be found in [2]. □

We now include some basic properties of commutators which we shall need to obtain some relations between the Peiffer subgroup $P = [\ker t, \ker h]$ and $P_{**} = [\ker t_{**}, \ker h_{**}]$.

Proposition 2.4 The following identities are easily verified.

$$\begin{array}{lllll}
\text{(ia)} & [ab, c] & = & [a, c]^b\, [b, c] \\
\text{(ib)} & [a, bc] & = & [a, c][a, b]^c \\
\text{(iia)} & [a_1 a_2 \dots a_n, c] & = & [a_1, c]^{a_2 \dots a_n} [a_2, c]^{a_3 \dots a_n} \dots [a_{n-1}, c]^{a_n} [a_n, c] \\
\text{(iib)} & [a, c_1 c_2 \dots c_n] & = & [a, c_n][a, c_{n-1}]^{c_n} \dots [a, c_2]^{c_3 \dots c_n} [a, c_1]^{c_2 \dots c_n}
\end{array}$$

Proposition 2.5 The Peiffer subgroup $P = [\ker t, \ker h]$ is normal in G and R-invariant.

Proof. If $a \in \ker t, c \in \ker h$ and $g \in G$ then $[a, c]^g = [a^g, c^g] \in P$. Since $r \in R$ acts on G by conjugation with er, it follows that $[a, c]^r \in P$. □

Proposition 2.6 Let X_t, X_h be generating sets for $\ker t, \ker h$, closed under conjugation in G. The Peiffer subgroup $[\ker t, \ker h]$ of G has generating set

$$\{[x, y] \mid x \in X_t, y \in X_h\}.$$

Proof. An element of $[\ker t, \ker h]$ has the form

$$z = \prod_i [a_i, c_i],$$

where $a_i = x_{i1}x_{i2}\dots x_{ir_i} \in \ker t, x_{ij} \in X_t$, and $c_i = y_{i1}y_{i2}\dots y_{is_i} \in \ker h, y_{ij} \in X_h$, so

$$z = \prod_i [x_{i1}x_{i2}\dots x_{ir_i}, y_{i1}y_{i2}\dots y_{is_i}].$$

¿From Proposition 2.4, z is a product of generating commutators. □

To any pre-cat[1]-group $\mathcal{P}$ there is a canonically associated a cat[1]-group $\mathcal{C}$, obtained by quotienting the source group by the Peiffer subgroup $[\ker t, \ker h]$. The corresponding functor is denoted

$$\mathbf{ass} : (\text{pre} - \text{cat}^1\text{−groups}) \to (\text{cat}^1 - \text{groups}) \tag{3}$$

and is clearly the identity when restricted to cat[1]-groups [7].

Our aim now is to find a convenient generating set for $[\ker t_{**}, \ker h_{**}]$. To this end we define, for an arbitrary pre-cat[1]-group $\mathcal{P} = (e; t, h, : G \to R)$, projections

$$\begin{array}{llllll} \pi_t : \mathrm{G} & \to & \ker t, & \mathrm{g} & \mapsto & (etg^{-1})g, \\ \pi_h : \mathrm{G} & \to & \ker h, & \mathrm{g} & \mapsto & (ehg^{-1})g. \end{array}$$

The maps π_t, π_h are respectively derivations for the conjugation crossed modules (inc : $\ker t \to G$) and (inc : $\ker h \to G$).

Since $\pi_t g = g$ when $g \in \ker t$ and $\pi_h g = g$ when $g \in \ker h$, both π_t and π_h are surjective. The following proposition gives values for π_t, π_h and their inverses in some special cases.

Proposition 2.7 The following identities are easily verified:

$$\begin{array}{llll} \text{(ia)} & \pi_t(ab) & = & (\pi_t a)^{tb}(\pi_t b) \\ \text{(ib)} & \pi_h(cd) & = & (\pi_h c)^{hd}(\pi_h d) \\ \text{(iia)} & (\pi_t a)^{-1} & = & (\pi_t a^{-1})^{ta} \\ \text{(iib)} & (\pi_h c)^{-1} & = & (\pi_h c^{-1})^{hc} \\ \text{(iiia)} & \pi_t(a_1 \dots a_n) & = & (\pi_t a_1)^{t(a_2 \dots a_n)}(\pi_t a_2)^{t(a_3 \dots a_n)} \dots (\pi_t a_{n-1})^{ta_n}(\pi_t a_n) \\ \text{(iiib)} & \pi_h(c_1 \dots c_n) & = & (\pi_h c_1)^{h(c_2 \dots c_n)}(\pi_h c_2)^{h(c_3 \dots c_n)} \dots (\pi_h c_{n-1})^{hc_n}(\pi_h c_n). \end{array}$$

Proof. We first verify (ia) :

$$\begin{aligned} \pi_t(ab) &= et(ab)^{-1}ab \\ &= (etb^{-1})(eta^{-1})ab \\ &= (etb^{-1})(\pi_t a)(etb)(etb^{-1})b \end{aligned}$$

$$= (\pi_t a)^{etb}(\pi_t b)$$
$$= (\pi_t a)^{tb}(\pi_t b)$$

by definition of the action of R on G. Then (iia) follows by setting $a = b^{-1}$ and (iiia) follows by induction. Identities (ib),(iib) and (iiib) can be proved in a similar way. □

A second set of identities, but with the order of the factors reversed, may be proved in a similar way.

Proposition 2.8

$$\begin{array}{lllll} \textit{(ia)} & \pi_t(ab) & = & (\pi_t b)(\pi_t a)^b \\ \textit{(ib)} & \pi_h(bc) & = & (\pi_h c)(\pi_h b)^c \\ \textit{(iia)} & (\pi_t a)^{-1} & = & (\pi_t a^{-1})^a \\ \textit{(iib)} & (\pi_h c)^{-1} & = & (\pi_h c^{-1})^c \\ \textit{(iiia)} & \pi_t(a_1 \dots a_n) & = & (\pi_t a_n)(\pi_t a_{n-1})^{a_n} \dots (\pi_t a_1)^{a_2 \dots a_n} \\ \textit{(iiib)} & \pi_h(c_1 \dots c_n) & = & (\pi_h c_n)(\pi_h c_{n-1})^{c_n} \dots (\pi_h c_1)^{c_2 \dots c_n}. \end{array}$$

We now obtain two pairs of identities expanding commutators containing terms $\pi_t(ab)$ or $\pi_h(bc)$.

Proposition 2.9

$$\begin{array}{llll} \textit{(i)} & [\pi_t(ab), \pi_h c] & = & [(\pi_t a)^{tb}, (\pi_h c)]^{\pi_t b}[\pi_t b, \pi_h c] \\ \textit{(ii)} & [\pi_t a, \pi_h(bc)] & = & [\pi_t a, \pi_h c][\pi_t a, (\pi_h b)^{(hc)}]^{\pi_h c}. \end{array}$$

Proof. Using the Proposition 2.7,

(i)

$$\begin{aligned} [\pi_t(ab), (\pi_h c)] &= [(\pi_t a)^{tb}(\pi_t b), (\pi_h c)] \\ &= (\pi_t b)^{-1}((\pi_t a)^{tb})^{-1}(\pi_h c)^{-1}(\pi_t a)^{tb}(\pi_t b)(\pi_h c) \\ &= (\pi_t b)^{-1}((\pi_t a)^{tb})^{-1}(\pi_h c)^{-1}(\pi_t a)^{tb}(\pi_h c)(\pi_t b)(\pi_t b)^{-1}(\pi_h c)^{-1}(\pi_t b)(\pi_h c) \\ &= [(\pi_t a)^{tb}, (\pi_h c)]^{(\pi_t b)}[(\pi_t b), (\pi_h c)] \\ &= [(\pi_t a)^{(tb)(\pi_t b)}, (\pi_h c)^{(\pi_t b)}][(\pi_t b), (\pi_h c)] \\ &= [(\pi_t a)^b, (\pi_h c)^{(\pi_t b)}][(\pi_t b), (\pi_h c)]. \end{aligned}$$

(ii)

$$\begin{aligned}[\pi_t(a), \pi_h(bc)] &= [(\pi_t a), (\pi_h b)^{hc}(\pi_h c)] \\ &= (\pi_t a)^{-1}(\pi_h c)^{-1}(\pi_h b)^{hc^{-1}}(\pi_t a)(\pi_h b)^{hc}(\pi_h c) \\ &= (\pi_t a)^{-1}(\pi_h c)^{-1}(\pi_t a)(\pi_h c)(\pi_h c)^{-1}(\pi_t a)(\pi_h b)^{hc^{-1}}(\pi_t a)(\pi_h b)^{hc}(\pi_h c) \\ &= [(\pi_t a), (\pi_h c)][(\pi_t a), (\pi_h b)^{(hc)}]^{(\pi_h c)}.\end{aligned}$$

□

Proposition 2.10

$$\begin{array}{lrcl} (i) & [\pi_t(ab), \pi_h c] & = & [\pi_t b, \pi_h c]^{(\pi_t a)^b}[(\pi_t a)^b, \pi_h c] \\ (ii) & [\pi_t a, \pi_h(bc)] & = & [\pi_t a, (\pi_h b)^c][\pi_t a, \pi_h c]^{(\pi_h b)^c}. \end{array}$$

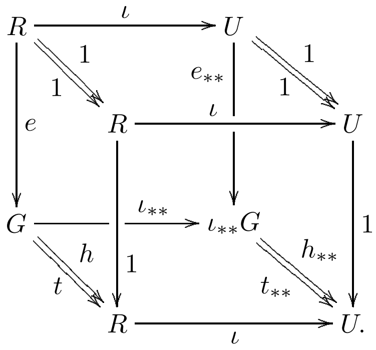
Proof. The proof is similar to the previous proof, but uses Proposition 2.8. □

Proposition 2.11 The maps π_t and π_h preserve the action of U.
Proof.

$$\begin{aligned}\pi_t(g^u) &= (et(g^u))^{-1}(g^u) \\ &= (et((eu)^{-1}g(eu)))^{-1}(g^{eu}) \\ &= ((eteu)^{-1}(etg)(eteu))^{-1}(eu)^{-1}g(eu) \\ &= (eu)^{-1}(etg)^{-1}g(eu) \\ &= ((etg)^{-1})g)^{eu} \\ &= (\pi_t g)^u\end{aligned}$$

The proof for π_h is similar. □

Let $\mathcal{C} = (e; t, h : G \to R)$ be a cat^1-group and let $\iota : R \to U$ be an inclusion. Denote by $\mathcal{U}, \mathcal{R}$ the identity cat^1-groups on U and R. Let $\iota_{**}\mathcal{C} = (e_{**}; t_{**}, h_{**} : \iota_{**}G \to U)$ be the pushout of the pre-cat^1-group morphisms $(\iota, \iota) : \mathcal{R} \to \mathcal{U}$ and $(e, 1) : \mathcal{R} \to \mathcal{C}$. The elements of $\iota_{**}G$ are words of the form $\kappa = g_1u_1g_2u_2 \ldots g_ku_k$, and the identity element is the empty word λ. We make no notational distinction between elements $g \in G,\ u \in U$ and there images under the inclusions $\iota_{**} : G \to \iota_{**}G$, $e_{**} : U \to \iota_{**}G$. Thus

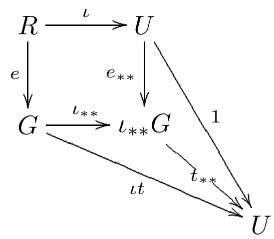


Proposition 2.12 The pushout $\iota_{**}\mathcal{C} = (e_{**}; t_{**}, h_{**} : \iota_{**}G \to U)$ is a pre-cat[1]-group, with tail t_{**} and head h_{**} given on words of length 1 by

$$h_{**}g = \iota hg,\ \ h_{**}u = u,\ \ t_{**}g = \iota tg,\ \ t_{**}u = u,$$

and extended componentwise to longer words, while $e_{**}u = u$.

Proof.



Since the above diagram is commutative and a pushout of groups then there is a unique homomorphism t_{**} such that $t_{**}e_{**} = 1$. Similarly, there is a unique h_{**} such that $h_{**}e_{**} = 1$. So the condition is satisfied. □

To turn this pre-cat[1]-group into a cat[1]-group we must find the Peiffer subgroup $P_{**} = [\ker t_{**}, \ker h_{**}]$. For this situation we denote the maps π_t and π_h by $\pi_{t_{**}}$ and $\pi_{h_{**}}$ so

$$\begin{array}{llllll} \pi_{t_{**}} : G *_R U & \to & G *_R U, & \kappa & \mapsto & (e_{**}t_{**}\kappa^{-1})\kappa \\ \pi_{h_{**}} : G *_R U & \to & G *_R U, & \kappa & \mapsto & (e_{**}h_{**}\kappa^{-1})\kappa. \end{array}$$

The following proposition gives values for $\pi_{t_{**}}$ and $\pi_{h_{**}}$ in some special cases.

Proposition 2.13 The following identities hold for any $g \in G, u \in U$:

$$\begin{aligned}
e_{**}t_{**}g &= etg \\
e_{**}h_{**}g &= ehg \\
\pi_{t_{**}}g &= \pi_t g \\
\pi_{t_{**}}u &= \lambda \\
\pi_{t_{**}}(gu) &= (\pi_t g)^u \\
\pi_{t_{**}}(ug) &= \pi_t g \\
\pi_{t_{**}}(g_1u_1g_2u_2) &= (\pi_t g_1)^{u_1(tg_2)u_2}(\pi_t g_2)^{u_2} \\
\pi_{t_{**}}(g_1u_1 \dots g_nu_n) &= (\pi_t g_1)^{u_1(tg_2)u_2 \dots (tg_n)u_n} \dots (\pi_t g_{n-1})^{u_{n-1}(tg_n)u_n}(\pi_t g_n)^{u_n}.
\end{aligned}$$

Proof. Using the Proposition 2.12 and $\iota_{**} : G \to \iota_{**}G$,

$$e_{**}t_{**}g = e_{**}\iota tg = \iota_{**}etg = etg.$$

Also using the definition of $\pi_{t_{**}}$,

$$\pi_{t_{**}}g = (e_{**}t_{**}g^{-1})g = (etg^{-1})g = \pi_t g,$$

$\pi_{t_{**}}u = (e_{**}t_{**}u^{-1})u = u^{-1}u = \lambda,$

$$\begin{aligned}
\pi_{t_{**}}(gu) &= (\pi_{t_{**}}g)^{(t_{**}u)}(\pi_{t_{**}}u) \\
&= (\pi_{t_{**}}g)^u\lambda \\
&= (\pi_t g)^u \\
\pi_{t_{**}}(ug) &= (\pi_{t_{**}}u)^{(t_{**}g)}(\pi_{t_{**}}g) \\
&= (\pi_{t_{**}}g) \\
&= \pi_t g.
\end{aligned}$$

Applying Proposition 2.7,

$$\begin{aligned}
\pi_{t_{**}}(g_1u_1g_2u_2) &= \pi_{t_{**}}(g_1u_1)^{(t_{**}g_2u_2)}(\pi_{t_{**}}g_2u_2) \\
&= [\pi_{t_{**}}(g_1)^{t_{**}u_1}\pi_{t_{**}}(u_1)]^{(t_{**}g_2u_2)}\pi_{t_{**}}(g_2)^{(t_{**}u_2)}\pi_{t_{**}}(u_2) \\
&= (\pi_t g_1)^{u_1(tg_2)u_2}(\pi_t g_2)^{u_2}.
\end{aligned}$$

The final identity follows from proposition 2.7 (iiia). □

Proposition 2.14 The following identities hold for any $g \in G, u \in U$:

$$\begin{aligned} \pi_{h_{**}} g &= \pi_h g, \\ \pi_{h_{**}} u &= \lambda, \\ \pi_{h_{**}}(gu) &= (\pi_h g)^u, \\ \pi_{h_{**}}(ug) &= \pi_h g, \\ \pi_{h_{**}}(g_1 u_1 g_2 u_2) &= (\pi_h g_1)^{u_1 (h g_2) u_2} (\pi_h g_2)^{u_2}, \\ \pi_{h_{**}}(g_1 u_1 \dots g_n u_n) &= (\pi_h g_1)^{u_1 (h g_2) u_2 \dots (h g_n) u_n} \dots (\pi_h g_n)^{u_n}. \end{aligned}$$

The following two pairs of identities, which expand commutators containing $\pi_{t_{**}}(ab)$ or $\pi_{h_{**}}(bc)$ follows immediately.

Corollary 2.15

$$\begin{aligned} [\pi_{t_{**}}(ab), (\pi_{h_{**}} c)] &= [(\pi_t a)^{tb}, (\pi_h c)]^{(\pi_t b)} [(\pi_t b), (\pi_h c)] \\ &= [(\pi_t b), (\pi_h c)]^{(\pi_t a)^b} [(\pi_t a)^b, (\pi_h c)] \end{aligned}$$

$$\begin{aligned} [\pi_{t_{**}}(a), \pi_{h_{**}}(bc)] &= [(\pi_t a), (\pi_h c)][(\pi_t a), (\pi_h b)^{(hc)}]^{(\pi_h c)} \\ &= [(\pi_t a), (\pi_h b)^c][(\pi_t a), (\pi_h c)]^{(\pi_h b)^c}. \end{aligned}$$

Let X_S be a generating set for $S = \ker t$, let $Y_S = X_S^R = \{g_1, \dots, g_n\}$ be the closure of X_S under the action of R, and let $T = \{c_1 = (), c_2, \dots, c_m\}$ be a transversal for the right cosets U/R.

Proposition 2.16 The kernels $\ker t_{**}$ and $\ker h_{**}$ have generating sets

$$Z_{t_{**}} = \{(1, g_i)^{c_j} \mid g_i \in Y_S, c_j \in T\},$$

$$Z_{h_{**}} = \{(\partial g_i^{-1}, g_i)^{c_j} \mid g_i \in Y_S, c_j \in T\},$$

where ∂ is an inclusion morphism.

Proof. In the cat^1-group $(\mathcal{C} = e; t, h : G \to R)$, where $G = R \ltimes S$, we have $t(r,s) = r,\ h(r,s) = r\partial s,\ e(r) = (r,1),\ \pi_t(r,s) = (1,s)$ and $\pi_h(r,s) = (\partial s^{-1}, s)$. Using Proposition 2.12, $t_{**}(r,s) = \iota t(r,s) = r,\ h_{**}(r,s) = \iota h(r,s) = r\partial s$ and $e_{**} r = (r, 1_S)$. We also have $t_{**}u = h_{**}u = e_{**}u = u, \pi_{t_{**}}(r,s) = r^{-1}(r,s) = (1_R, s),\ \pi_{t_{**}}u = \lambda,\ \pi_{h_{**}}(r,s) = \partial s^{-1}r^{-1}(r,s) = \partial s^{-1}(1_R, s)$ and $\pi_{h_{**}}u = \lambda$. Now $\pi_{t_{**}}$ is onto and, by Proposition 2.13,

$$\begin{aligned}\pi_{t_{**}}((r_1,s_1)u_1\dots(r_\ell,s_\ell)u_\ell) &= (1,s_1)^{u_1(r_2,1)u_2\dots(r_\ell,1)u_\ell}\dots(1,s_{\ell-1})^{u_{\ell-1}(r_\ell,1)u_\ell}(1,s_\ell)^{u_\ell}\\ &= (1,s_1)^{u_1r_2u_2}\dots(1,s_\ell)^{u_\ell}\\ &= (1,s_1)^{u_1'}\dots(1,s_\ell)^{u_\ell'},\end{aligned}$$

so every element of $\ker t_{**}$ has the form

$$(1,s_1)^{u_1}\dots(1,s_\ell)^{u_\ell},\quad s_i\in S, u_i\in U.$$

Since

$$(1,s_1s_2\dots s_\ell)^u = ((1,s_1)\dots(1,s_\ell))^u = (1,s_1)^u\dots(1,s_\ell)^u,$$

we need only take a generating set for S. Furthermore, since $u = rc$ for some $r\in R$ and coset representation $c\in U$,

$$(1,s)^u = (1,s)^{rc} = (1,s^r)^c.$$

So $\ker t_{**}$ has a generating set

$$\{(1,g_i)^{c_j} \mid g_i\in Y_S,\ c_j\in T\}$$

and similarly $\ker h_{**}$ has a generating set

$$\{(\partial g_i^{-1}, g_i)^{c_j} \mid g_i\in Y_S,\ c_j\in T\}.$$

□

Proposition 2.17 The Peiffer commutator subgroup $P_{**} = [\ker t_{**}, \ker h_{**}]$ has normal generating set

$$Z_{P_{**}} = \{[(1,g_i)^{c_j},\ (\partial g_k^{-1}, g_k)] \mid\ g_i, g_k\in Y_S, c_j\in T\}.$$

Proof. Since $\ker t_{**}$ is generated by $Z_{t_{**}}$ and $\ker h_{**}$ is generated by $Z_{h_{**}}$ it follows that P_{**} is normally generated by $\{[x,y]\mid x\in Z_{t_{**}},\ \ y\in Z_{h_{**}}\}$. Also,

$$\begin{aligned}[x,y] &= [(1,g_i)^{c_j}, (\partial g_k^{-1}, g_k)^{c_\ell}]\\ &= [(1,g_i)^{c_jc_\ell^{-1}}, (\partial g_k^{-1}, g_k)]^{c_\ell}\\ &= [(1,g_i^r)^{c'}, (\partial g_k^{-1}, g_k)]^{c_\ell},\end{aligned}$$

where $c_j c_\ell^{-1} = rc', \;\; r \in R, \; c' \in T.$ □

A smaller generating set may be obtained if we relax the requirement that the elements y are closed under the action of R.

Proposition 2.18 The Peiffer commutator subgroup P_{**} has a normal generating set

$$Z'_{P_{**}} = \{[(1, g_i)^{c_j}, \; (\partial g_k^{-1}, g_k)] \mid g_i \in Y_S, \; c_j \in T, \; g_k \in X_S\}.$$

Proof. Suppose $g_k = s_k^{r_k}$ where $s_k \in X_S, r_k \in R$. Then

$$\begin{aligned}[(1, g_i)^{c_j}, (\partial g_k^{-1}, g_k)^{c_\ell}] &= [(1, g_i)^{c_j}, ((\partial s_k^{-1})^{r_k}, s_k^{r_k})^{c_\ell}] \\ &= [(1, g_i)^{c_j r_k^{-1} c_\ell^{-1}}, (\partial s_k^{-1}, s_k)]^{r_k c_\ell} \\ &= [(1, g_i^{r'})^{c'}, (\partial s_k^{-1}, s_k)]^{r_k c_\ell}.\end{aligned}$$

□

3. Algorithm for Induced Cat^1-groups

The induced cat^1-group $\iota_{**}\mathcal{C}$ may be obtained by using **XModCat1** to construct $\mathcal{X}$, then **InducedXMod** to construct $\iota_*\mathcal{X}$ and then **Cat1XMod**. An alternative procedure is to calculate the induced cat^1-group $\iota_{**}G = (G *_R U)/P_{**}$ directly. This has been implemented for the case when $\mathcal{C} = (e; t, h : G \to R)$ and $\iota : R \to U$ is an inclusion.

3.1. Record Structure for InducedCat1

The record structure for an induced cat^1-group contains, in addition to the usual fields for a cat^1-group,

IC.cat1,	the cat^1-group $\mathcal{C}$,
IC.name,	written as IC(name of $\mathcal{C}$),
IC.morphism,	the morphism $\langle \iota_{**}, \iota \rangle : \mathcal{C} \to \iota_{**}\mathcal{C}$,
IC.isInducedCat1,	a boolean flag normally true.

3.2. Algorithm for InducedCat1

The function **InducedCat1** is called as:

```
gap> InducedCat1( G, R, U );
gap> InducedCat1( C, iota );
```

The function requires as data a conjugation cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ and an inclusion morphism $\iota : R \to U$. The data may be specified using either of the two forms shown, where the first form requires $G \geq R \geq U$. As output, the function returns $\iota_{**}\mathcal{C} = (e_{**}; t_{**}, h_{**} : \iota_{**}G \to U)$ together with a morphism $\langle \iota_{**}, \iota \rangle : \mathcal{C} \to \iota_{**}\mathcal{C}$.

Step 1	Check the argument: if the argument is a collection of groups then call **ConjugationCat1(G, R);** to construct $\mathcal{C}$, and call **InclusionMorphism(R, U);** to construct ι. Otherwise, **G:= C.source; R := C.range;** and **iota** is the second argument.
Step 2	Obtain finitely presented groups G', U' isomorphic to G and U.
Step 3	Construct a free F whose rank is the total length of the generating sets of G and U.
Step 4	Map the relators for G', U' into words in F and, for each generator r of R, add the relation $(r \in G') = (r \in U')$.
Step 5	Obtain coset representatives for U/R.
Step 6	Construct generators for the Peiffer subgroup P_{**}.
Step 7	Construct the finitely presented group $IG' = F/\text{rels}$ where **rels** is the set of relators constructed in steps 4 and 6.
Step 8	Obtain a faithful permutation representation IG of IG'.
Step 9	Construct homomorphisms **tstar**, **hstar** and **estar**.
Step 10	Call **Cat1(IG, tstar, hstar, estar);** to construct $\iota_{**}\mathcal{C}$.
Step 11	Use **Cat1Morphism** to construct $\langle \iota{**}, \iota \rangle$.
Step 12	Add the fields described in section 3.1

3.3. Comparative timing

The following table gives timing in miliseconds for the calculation of some induced crossed modules and induced cat^1-groups. Computations were performed on a Digital Alpha 64-bit workstation using 20M memory.

Conjugation cat¹-group crossed module	time $\mathcal{C}$ $\mathcal{X}$	$\mathcal{IC}$ $\mathcal{IX}$	$\mid I \mid$	iota
k4 → c2 c2 → c2	124 59	2605 1207	36 6	c2 → s3
a4a4 → a4 a4 → a4	9429 1705	107322 7328	864 36	a4 → s4
c4c4 → c4 c4 → c4	733 152	4395 2273	128 16	c4 → d8
c4c4 → c4 c4 → c4	248 150	5135 1777	128 16	c4 → c4c2
c3c3 → c3 c3 → c3	683 116	9789 3719	288 24	c3 → a4
d8d8 → d8 d8 → d8	4131 756	89649 21317	512 32	d8 → d8c2
q8q8 → q8 q8 → q8	4604 1073	229838 19663	512 32	q8 → q8c2
c4c2c4c2→c4c2 c4c2 → c4c2	723 532	6802340 10183	1024 64	c4c2 → d8y4
$c3^2c3^2 \to c3^2$ $c3^2 \to c3^2$	726 551	6227880 10813	1458 81	$c3^2 \to c3^2 \ltimes c2$

Table 1. Sample timings for induced crossed modules and cat^1-groups

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