Turk J Math
25 (2001) , 283 – 298.

Inequalities for the Vibrating Clamped Plate Problem

K.P. McHale, U. Ufuktepe

Abstract

We study the eigenvalues of the vibrating clamped plate problem. We have made improvements on the bounds of the ratios of the eigenvalues of the biharmonic operator (clamped plate) using the methods of Payne, Polya, and Weinberger. The difference in our proof lies mainly with the trial functions and the orthogonality arguments. While Payne, Polya, and Weinberger and Hile and Yeh project away components along $u_1, u_2, ..., u_k$ to meet the orthogonality conditions,we use a translation/rotation argument to meet these conditions.

1. Introduction

In the present paper, we consider the biharmonic eigenvalue problem

$$\left.\begin{array}{ll} \Delta^2 u - \mu u = 0 & \text{in } D \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \end{array}\right\} \tag{1.1}$$

with "clamped" boundary conditions on a bounded domain $D \subset \mathbb{R}^n$ $(n \geq 2)$ where Δ denotes the Laplace operator, n denotes the outward normal to ∂D, and $0 < \mu_1 < \mu_2 < ...$ denote the successive eigenvalues, with multiplicity, and u_i denotes the corresponding eigenfunction.

Payne, Polya, and Weinberger [3] showed that for domains in the plane,

$$\mu_{k+1} \leq \mu_k + \frac{8}{k} \sum_{i=1}^{k} \mu_i \leq 9\mu_k. \tag{1.2}$$

A related result for a domain $D \subset \mathbb{R}^n$ $(n \geq 2)$, was obtained by Hile and Yeh [1] :

$$\sum_{i=1}^{k} \frac{\sqrt{\mu_i}}{\mu_{k+1} - \mu_i} \geq \frac{n^2 k^{3/2}}{8(n+2)} \left(\sum_{i=1}^{k} \sqrt{\mu_i} \right)^{-1/2}. \tag{1.3}$$

From this they derived the weaker, but explicit, bound

$$\mu_{k+1} \leq \mu_k + \frac{8(n+2)}{n^2 k^{3/2}} \left(\sum_{i=1}^{k} \mu_i \right)^{1/2} \left(\sum_{i=1}^{k} \sqrt{\mu_i} \right). \tag{1.4}$$

and also obtained the still weaker bound

$$\mu_{k+1} \leq \mu_k + \frac{8(n+2)}{n^2 k} \left(\sum_{i=1}^{k} \mu_i \right) \leq \left(1 + \frac{4}{n} \right)^2 \mu_k, \tag{1.5}$$

which is the natural extension to dimensions $n \geq 2$ of the bound (1.2) for domains in the plane. In particular, for $k = 1$, we have

$$\mu_2 \leq \left(1 + \frac{4}{n} \right)^2 \mu_1. \tag{1.6}$$

Hook [2] showed, using abstract operator theory,

$$\sum_{i=1}^{k} \frac{\sqrt{\mu_i}}{\mu_{k+1} - \mu_i} \geq \frac{n^2 k^2}{8(n+2)} \left(\sum_{i=1}^{k} \sqrt{\mu_i} \right)^{-1}. \tag{1.7}$$

Considering low eigenvalues in $\mathbb{R}^2$, we see from (1.2) that

$$\mu_2 \leq 9\mu_1 \text{ and } \quad \mu_3 \leq 9\mu_2 \leq 81\mu_1. \tag{1.8}$$

Even better, (1.2) yields the following bound in $\mathbb{R}^2$

$$\mu_3 \leq 4\mu_1 + 5\mu_2 \leq 49\mu_1. \tag{1.9}$$

Improved estimates for some of the lower eigenvalues of (1.1) where also found by Hile and Yeh [1]. They showed that for any $\sigma > 0$

$$\mu_{k+1} \leq (1+\sigma)\mu_k + q(\sigma) \frac{M(n)}{k} \sum_{i=1}^{k} \mu_i \tag{1.10}$$

where

$$q(\sigma) = \left(\frac{(1+\sigma)^3}{\sigma}\right)^{1/2},$$

$$M(n) = \frac{32}{3}\sqrt{\frac{2}{3}}n^{-1}(n+2)^{-1/2},$$

and furthermore, they proved that (1.10) is better than (1.5) for certain low values of k (depending on the dimension n). In particular, for $n = 2$, $k = 1$, $\sigma = .4$ they found

$$\mu_2 \le 7.103\mu_1, \tag{1.11}$$

and for $n = 2$, $k = 2$, $\sigma = .34$ they obtained $\mu_3 \le 2.897\mu_1 + 4.237\mu_2$,

2. Trial Function Method

An ever-present tool in our method will be the Rayleigh-Ritz inequality given by

$$\mu_{i+1} \le \frac{\int_D \Psi_i \Delta^2 \Psi_i}{\int_D \Psi_i^2}, \qquad i = 1, 2, ..., n$$

which is satisfied by any sufficiently smooth function Ψ_i, such that

$$\Psi_i = \frac{\partial \Psi_i}{\partial n} = 0 \text{ on } D$$

and

$$\int_D \Psi_i u_j = 0, \ j \le i, \ i = 1, ..., n$$

For the vibrating clamped plate problem, Payne, Pólya, and Weinberger (PPW)and Hile and Yeh (HY) use trial functions $\Psi_i = xu_i - \sum_{j=1}^{k} a_{ij}u_j$.With the appropriate choice of a_{ij} , the orthogonality conditions necessary to use the Rayleigh-Ritz quotient are met. They both use rotations later in their proofs to simplify the Rayleigh-Ritz inequality. The method used here is to rotate coordinates so that the trial functions used in the PPW/HY method are simpler to use. For Theorem 2.1, we would like to use just $\Psi_i = x_i u_1$ as trial functions for μ_{i+1} for $i = 1, ..., n$. However, given an arbitrary choice of Cartesian coordinates x_i, we have no guarantee that the appropriate orthogonalities (i.e., $< \Psi_i, u_j >= 0$ for all $j \leq i$, where $< \cdot, \cdot >$ denotes the appropriate inner product) will hold. To remedy this situation, we argue that we can always find a suitable rotation of axes to a new system of Cartesian coordinates $x_i^{'}$ so that the desired orthogonalities are obtained (i.e., $< \tilde{\Psi}_i, u_j >= 0$ for all $j \leq i$ and $i = 1, ..., n$ where $\tilde{\Psi}_i = x_i^{'} u_1$). Thus the necessary orthogonality conditions will hold for the trial functions $\tilde{\Psi}_i = x_i^{'} u_1$, where the new Cartesian variables $x_i^{'}$ are obtained via a rotation from our original variables. In other words, there exists a real orthogonal matrix S such that $x_i^{'} = \sum_{j=1}^{n} S_{ij} x_j$, for $i = 1, ..., n$, and the following theorem holds.

Theorem 1 *There exists a set of Cartesian coordinates $x_i^{'}$ such that the functions $\tilde{\Psi}_i = x_i^{'} u_1$ are suitable trial functions for μ_{i+1} in the corresponding Rayleigh quotient. That is, we have*

$$\tilde{\Psi}_i = \frac{\partial \tilde{\Psi}_i}{\partial n} = 0 \quad \text{on } \partial D \tag{2.12}$$

$$< \tilde{\Psi}_i, u_j >= 0, \ \text{for all } 1 \leq j \leq i \leq n. \tag{2.13}$$

Proof. We start from the arbitrary system of Cartesian coordinates x_i and let $\Psi_i = x_i u_1$. We can assume that these obey $< x_i u_1, u_1 >= 0$, for, if not, we can simply translate each x_i by $a_i = \frac{< x_i u_1, u_1 >}{< u_1, u_1 >}$. Assuming this has been done, we find that $\tilde{\Psi}_i = \tilde{x}_i u_1$ satisfies (2.12) for all i and satisfies (2.13) for $j = 1$ and all i, $1 \leq i \leq n$. To prove (2.13) for $j = 2, 3, ..., i$, let C be an nxn matrix such that $C = [c_{ij}]_{1 \leq i \leq j \leq n}$ where $c_{ij} =< \Psi_i, u_{j+1} >=< x_i u_1, u_{j+1} >$.

Then

$$C = \begin{bmatrix} < \Psi_i, u_2 > & < \Psi_i, u_3 > & \cdots & < \Psi_i, u_{n+1} > \\ < \Psi_i, u_2 > & < \Psi_i, u_3 > & \cdots & < \Psi_i, u_{n+1} > \\ \vdots & \vdots & & \vdots \\ < \Psi_i, u_2 > & < \Psi_i, u_3 > & \cdots & < \Psi_i, u_{n+1} > \end{bmatrix} = [\vec{c}_1, \vec{c}_2, ..., \vec{c}_n]$$

where the $\vec{c}_j$'s are the columns of C. Now using the Gram-Schmidt procedure we can orthogonalize the $\vec{c}_j$'s in order, followed by the standard basis vectors $\vec{e}_i$ as needed, to get n independent, orthonormal column vectors $\vec{r}_j$. Let R be the matrix with columns $\vec{r}_j$, $1 \leq j \leq n$. Then we have $C = RT$ where R is an nxn real orthogonal matrix and T is an nxn upper triangular matrix. Therefore, $R^T C = T$ and each entry in the matrix T, denoted by T_{ij}, can be represented as follows:

$$T_{ij} = \sum_{k=1}^{n} (R^T)_{ik} c_{kj} = \sum_{k=1}^{n} (R^T)_{ik} < x_k u_1, u_{j+1} >$$
$$= \left\langle \left(\sum_{k=1}^{n} (R^T)_{ik} x_k u_1 \right), u_{j+1} \right\rangle =< x_k^{'} u_1, u_{j+1} > .$$

Thus we identify S from our discussion leading up to this theorem as R^T. Since T is an upper triangular matrix, we have $< x_i^{'} u_1, u_{j+1} >= 0$ for $1 \leq j \leq i$, $i = 2, ..., n$. So $< x_i^{'} u_1, u_j >= 0$ for $2 \leq j \leq i$, and thus $\tilde{\Psi}_i = x_i^{'} u_1 \perp u_2, u_3, ..., u_i$ for $i = 2, 3, ..., n$. We note also that since $< x_k u_1, u_1 >= 0$ for $k = 1, ..., n$, $< x_i^{'} u_1, u_1 >= \sum_{k=1}^{n} (R^T)_{ik} < x_k u_1, u_1 >= 0$ for each $i = 1, ..., n$. We therefore have $< \tilde{\Psi}_i, u_j >=< x_i^{'} u_1, u_j >= 0$ for $1 \leq j \leq i$ and $i = 1, ..., n$, which shows that (2.13) is satisfied.

3. Useful Calculations

In this section, some of the more useful formulas used later in this paper are given. Calculating the numerator in the Rayleigh-Ritz inequality, we have the following lemmas.

Lemma 1 *Given the trial functions $\Psi_i = x_i u_1$, we have*

$$\int_D \Psi_i \Delta^2 \Psi_i = \mu_1 \int_D x_i^2 u_1^2 + 4 \int_D x_i u_1 \Delta u_{x_i}$$

Proof.

$$\begin{aligned}
\int_D \Psi_i \Delta^2 \Psi_i &= \int_D \Psi_i (\Delta^2 x_i \Psi_i) \\
&= \int_D \Psi_i (\Delta(\Delta x_i u)) = \int_D \Psi_i \Delta(x_i \Delta u + 2u_{x_i}) \\
&= \int_D \Psi_i (x_i \Delta^2 u + 2\Delta u_{x_i} + 2\Delta u_{x_i}) \\
&= \int_D x_i u (x_i \mu_1 u + 4\Delta u_{x_i}) \\
&= \mu_1 \int_D x_i^2 u^2 + 4 \int_D x_i u_1 \Delta u_{x_i}
\end{aligned}$$

□

Lemma 2

$$\int_D |\nabla u_i|^2 \leq \sqrt{\mu_i}$$

Proof.

$$\begin{aligned}
\int_D |\nabla u_i|^2 &= \int_D u_i(-\Delta u_i) \\
&\leq \int_D |u_i \Delta u_i| \\
&\leq \left(\int_D u_i^2 \right)^{1/2} \left(\int_D (\Delta u_i)^2 \right)^{1/2} \\
&\leq \left(\int_D u_i \Delta^2 u_i \right)^{1/2}
\end{aligned}$$

$$= \left(\mu_1 \int_D u_1^2 \right)^{1/2}$$
$$= \sqrt{\mu_i}$$

□

Here we made use of Cauchy-Schwarz inequality, the fact that our eigenfunctions u_i are real-valued and enjoy the orthonormality property $\int_D u_1^2 = 1$. Finally, we need the following lemma. We define

$$J_l = \sum_{i=1}^{k} \int_D \Psi_{il} \Delta u_{ix_l} \text{ and } J = \frac{n+2}{2n} \sum_{i=1}^{k} \int_D |\nabla u_i|^2 . \tag{3.14}$$

Lemma 3 *The inequalities J_l and J satisfy:*

$$(i)\ \frac{1}{n} \sum_{l=1}^{n} J_l = J,$$

$$(ii)\ J \le \frac{n+2}{2n} \sum_{i=1}^{k} \sqrt{\mu_i},$$

$$(iii)\ \frac{nk^2(n+2)}{8} \le J \sum_{l=1}^{n} \sum_{i=1}^{k} \Psi_{il}^2 .$$

Proof. of (i). We have

$$\sum_{l=1}^{n} J_l = \sum_{l=1}^{n} \sum_{i=1}^{k} \int_D \Psi_{il} \Delta u_{ix_l}$$
$$= \sum_{l=1}^{n} \left(\sum_{i=1}^{k} \int_D x_l u_i \Delta u_{ix_l} - \sum_{i,j=1}^{k} a_{ij} \int_D u_j \Delta u_{ix_l} \right) . \tag{3.15}$$

The second term on the left-hand side of (3.15) vanishes because of the symmetry of a_{ij} and the antisymmetry of $\int_D u_j \Delta u_{ix_l}$. Next we observe that

$$\int_D x_l u_i \Delta u_{ix_l} = \frac{1}{2} \int_D |\nabla u_i|^2 + \int_D u_{ix_l}^2. \tag{3.16}$$

Summing (3.16) over i and l, and substituting the result into (3.15) yields

$$\begin{aligned}
\sum_{l=1}^{n} J_l &= \sum_{l=1}^{n} \left(\sum_{i=1}^{k} \int_D x_l u_i \Delta u_{ix_l} \right) \\
&= \sum_{i=1}^{k} \frac{n}{2} \int_D |\nabla u_i|^2 + \sum_{i=1}^{k} \int_D |\nabla u_i|^2 \\
&= \sum_{i=1}^{k} \frac{n+2}{2} \int_D |\nabla u_i|^2 \\
&= nJ.
\end{aligned}$$

This completes the proof of (i). To prove (ii), we employ Lemma 3.1, making use of the Cauchy-Schwarz inequality and the fact that $\int_D (\Delta u_i)^2 = \mu_i$. Thus:

$$J = \frac{n+2}{2n} \sum_{i=1}^{k} \int_D |\nabla u_i|^2 \leq \frac{n+2}{2n} \sum_{i=1}^{k} \sqrt{\mu_i}.$$

To prove (iii), we consider the following:

$$\sum_{i=1}^{k} \int_D \Psi_{il} u_{ix_l} = \sum_{i=1}^{k} \int_D x_l u_i u_{ix_l} = \sum_{i,j=1}^{k} a_{ij} \int_D u_j u_{ix_l}. \tag{3.17}$$

Using integration by parts, we have

$$\int_D x_l u_i u_{ix_l} = -\frac{1}{2}. \tag{3.18}$$

Thus, again using the symmetry of a_{ij}, the antisymmetry of $\int\limits_D u_i u_{ix_l}$, and substituting (3.18) into (3.17), we have

$$\sum_{i=1}^{k} \int\limits_D \Psi_{il} u_{ix_l} = -\frac{k}{2}. \tag{3.19}$$

Applying the Cauchy-Schwarz inequalities both for integrals and sums to (3.19), yields

$$\frac{k}{2} \leq \left(\sum_{i=1}^{k} \int\limits_D \Psi_{il}^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{k} \int\limits_D u_{ix_l}^2 \right)^{\frac{1}{2}}.$$

Summing over l and using the Cauchy-Schwarz inequality again we have

$$\frac{nk}{2} \leq \left(\sum_{l=1}^{n} \sum_{i=1}^{k} \int\limits_D \Psi_{il}^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{k} \int\limits_D |\nabla u_i|^2 \right)^{\frac{1}{2}}$$

or, upon squaring and using the definition of J,

$$\left(\frac{n+2}{2n} \right) \frac{n^2 k^2}{4} \leq J \sum_{l=1}^{n} \sum_{i=1}^{k} \int\limits_D \Psi_{il}^2.$$

The proof of Lemma 3 is now complete. □

4. Bounds on $\frac{\mu_2 + \ldots + \mu_{n+1}}{\mu_1}$

Theorem 2 *Let D be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, with boundary ∂D. Let $\mu_1, \mu_2, \ldots$ denote the successive eigenvalues of* (1.1) *with $u = u_1$the eigenfunction corresponding to μ_1. Then*

$$\frac{\mu_1 + \mu_2 + \ldots + \mu_{n+1}}{\mu_2} \leq n \left(1 + \frac{24}{n} \right), \tag{4.20}$$

and

$$\sum_{i=1}^{n} \sqrt{\mu_{i+1}} \le (n+4)\sqrt{\mu_1}. \tag{4.21}$$

Proof. of (4.20).

Let $\Psi_i = x_i u$, where $u = u_1$. The conditions for the Rayleigh-Ritz inequality are met by virtue of the boundary conditions and an appropriate translation and rotation of the coordinate axes, which is detailed in the Theorem 2.1. By Lemma 3.1, we have

$$\mu_{i+1} \le \frac{\mu_1 \int\limits_D x_i^2 u^2 + 4 \int\limits_D x_i u \Delta u_{x_i}}{\int\limits_D x_i^2 u^2}. \tag{4.22}$$

By normalization and integration by parts, we have

$$1 = \int\limits_D u^2 = -\int\limits_D x_i 2 u u_{x_i}.$$

Squaring both sides and using the Cauchy-Schwarz inequality we have

$$1 = \left(2 \int\limits_D x_i u u_{x_i}\right)^2 \le \left(4 \int\limits_D u_{x_i}^2\right)\left(\int\limits_D x_i^2 u^2\right).$$

Thus

$$\frac{1}{\int\limits_D x_i^2 u^2} \le 4 \int\limits_D u_{x_i}^2. \tag{4.23}$$

Substituting (4.23) into (4.22) yields

$$\mu_{i+1} - \mu_1 \le \left(4 \int\limits_D x_i u \Delta u_{x_i}\right)\left(4 \int\limits_D u_{x_i}^2\right). \tag{4.24}$$

From the divergence theorem and integration by parts, we have

$$2 \int\limits_D x_i u \Delta u_{x_i} = \int\limits_D |\nabla u|^2 + 2 \int\limits_D u_{x_i}^2$$

Note that we avoid using Hile and Yeh's rotation convention here. Instead, we use the rotation convention outlined in Theorem 2.1 to meet the orthogonality conditions and simplify the Rayleigh-Ritz inequality further before summing over i. Thus (4.24) becomes

$$\begin{aligned}\mu_{i+1}-\mu_1 &\le 2\left(\int_D |\nabla u|^2 + 2\int_D u_{x_i}^2\right)\left(4\int_D u_{x_i}^2\right)\\ &= 8\int_D u_{x_i}^2 \int_D |\nabla u|^2 + 16\left(\int_D u_{x_i}^2\right)^2 \qquad (4.25)\end{aligned}$$

Summing on i from 1 to n, and utilizing Lemma 3.2, we have

$$\begin{aligned}\mu_2+\mu_3+...+\mu_{n+1}-n\mu_1 &\le 24\left(\int_D |\nabla u|^2\right)^2\\ &\le 24\mu_1,\end{aligned}$$

achieving the desired result,

$$\frac{\mu_2+\mu_3+...+\mu_{n+1}}{\mu_1} \le n+24.$$

Proof of (4.21). Let $a_i = \int_D u_{x_i}^2$ and $I = \int_D |\nabla u|^2$. Then from (4.25) we have

$$\begin{aligned}\mu_{i+1}-\mu_1 &\le 8a_i(2a_i+I)\\ &= 16a_i^2 + 8a_i I\\ &\le 16a_i^2 + 8a_i\sqrt{\mu_1}\\ &= (4a_i+\sqrt{\mu_1})^2 - \mu_1. \qquad (4.26)\end{aligned}$$

Canceling the μ_1's and taking the square root of both sides, we have

$$\sqrt{\mu_{i+1}} \le 4a_i + \sqrt{\mu_1}.$$

Summing over i, we have (using $\sum_{i=1}^{n} a_i = I \le \sqrt{\mu_1}$),

$$\sum_{i=1}^{n} \sqrt{\mu_{i+1}} \le (n+4)\sqrt{\mu_1}, \qquad (4.27)$$

5. Improved ratio results

Improvements to Hile and Yeh's work are shown in the next theorem. In addition, we offer an alternate proof of (5.28) originally found by Hook [2] using an operator method.

Theorem 3 *Let D be bounded domain in $\mathbb{R}^n$, $n \geq 2$, with boundary ∂D. Let the eigenvalues of (1.1) be designated by $0 \leq \mu_1 \leq \mu_2 \leq \ldots$. Then we have for $n \geq 2$ and $k \geq 1$, the implicit bound*

$$\sum_{i=1}^{k} \frac{\sqrt{\mu_i}}{\mu_{k+1}-\mu_i} \geq \frac{n^2k^2}{8(n+2)} \left(\sum_{i=1}^{k} \sqrt{\mu_i} \right)^{-1} \tag{5.28}$$

and the explicit bound

$$\mu_{k+1}-\mu_k \leq \frac{8(n+2)}{n^2k^2} \left(\sum_{i=1}^{k} \sqrt{\mu_i} \right)^{2} \tag{5.29}$$

Proof. Following Payne, Polya, and Weinberger, we consider the trial functions

$$\Psi_{il} = x_l u_i - \sum_{j=1}^{k} a_{ij} u_j, \quad i = 1, 2, ..., k, \quad l = 1, 2, ..., n$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$, and the constants a_{ij} are defined by

$$a_{ij} = \int_D x_l u_i u_j$$

(we suppress the dependence on l) so that

$$\begin{aligned} \int_D \Psi_{il} u_k &= \int_D x_l u_i u_k - \sum_{j=1}^{k} a_{ij} \int_D u_j u_k \\ &= \int_D x_l u_i u_j - a_{ik} = 0. \end{aligned}$$

In addition,

$$\Psi_{il} = \frac{\partial \Psi_{il}}{\partial n} = 0 \quad \text{on } \partial D.$$

Therefore, the Rayleigh-Ritz inequality holds

$$\mu_{k+1} \leq \frac{\int\limits_D \Psi_{il} \Delta^2 \Psi_{il}}{\int\limits_D \Psi_{il}^2}, \quad i = 1, 2, ..., k. \tag{5.30}$$

Using Lemma 3.1 and summing over i, (5.30) becomes

$$\mu_{k+1} \sum_{i=1}^{k} \int\limits_D \Psi_{il}^2 \leq \sum_{i=1}^{k} \mu_i \int\limits_D \Psi_{il}^2 + 4 \sum_{i=1}^{k} \int\limits_D \Psi_{il} \Delta u_{ix_l}. \tag{5.31}$$

Introducing a new parameter $\alpha, \alpha > \mu_k$ we have from (5.31) and (3.14),

$$(\mu_{k+1} - \alpha) \sum_{i=1}^{k} \int\limits_D \Psi_{il}^2 \leq \sum_{i=1}^{k} (\mu_i - \alpha) \int\limits_D \Psi_{il}^2 + 4J_l. \tag{5.32}$$

Using the simple inequality $ab \leq |ab| \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2$ for any $\delta > 0$, the new parameter α,and (3.19), we obtain $\forall\ \delta > 0$,

$$\frac{k}{2} \leq \frac{\delta}{2} \sum_{i=1}^{k} (\alpha - \mu_i) \int\limits_D \Psi_{il}^2 + \frac{1}{2\delta} \sum_{i=1}^{k} (\alpha - \mu_i)^{-1} \int\limits_D u_{ix_l}^2. \tag{5.33}$$

Denote $S = \sum\limits_{l=1}^{n} \sum\limits_{i=1}^{k} \int\limits_D \Psi_{il}^2$, $T = \sum\limits_{l=1}^{n} \sum\limits_{i=1}^{k} (\alpha - \mu_i) \int\limits_D \Psi_{il}^2$; then (5.32) when summed on l from 1 to n gives (using (i))

$$(\mu_{k+1} - \alpha) S + T \leq 4nJ \tag{5.34}$$

and similarly (5.33) gives

$$\begin{aligned} nk &\leq \delta T + \delta^{-1} \sum_{i=1}^{k} (\alpha - \mu_i)^{-1} \int\limits_D |\nabla u_i|^2 \\ &\leq \delta T + \delta^{-1} \sum_{i=1}^{k} \frac{\sqrt{\mu_i}}{\alpha - \mu_i}, \end{aligned} \tag{5.35}$$

since

$$\int_D |\nabla u_i|^2 \leq \sqrt{\mu_i}.$$

To minimize the right-hand side of (5.35) we need to find δ such that

$$T - \frac{1}{\delta^2}\sum_{i=1}^{k}\frac{\sqrt{\mu_i}}{\alpha-\mu_i} = 0, \tag{5.36}$$

and thus

$$\delta = T^{-\frac{1}{2}}\left(\sum_{i=1}^{k}\frac{\sqrt{\mu_i}}{\alpha-\mu_i}\right)^{1/2} \tag{5.37}$$

Putting this value of δ into (5.35), squaring both sides, and solving for T yields

$$\begin{aligned} nk &\leq 2T^{\frac{1}{2}}\left(\sum_{i=1}^{k}\frac{\sqrt{\mu_i}}{\alpha-\mu_i}\right)^{1/2}, \\ \frac{n^2k^2}{4} &\leq T\sum_{i=1}^{k}\frac{\sqrt{\mu_i}}{\alpha-\mu_i}, \qquad (5.38) \\ T &\geq \frac{n^2k^2}{4}\left(\sum_{i=1}^{k}\frac{\sqrt{\mu_i}}{\alpha-\mu_i}\right)^{-1}, \end{aligned}$$

From Lemma 3.3 (ii) , we have

$$J \leq \frac{n+2}{2n}\sum_{i=1}^{k}\sqrt{\mu_i}. \tag{5.39}$$

Substitution of (5.38) and (5.39) into (5.34) yields

$$(\mu_{k+1}-\alpha)\,S \leq 2(n+2)\sum_{i=1}^{k}\sqrt{\mu_i} - \frac{n^2k^2}{4}\left(\sum_{i=1}^{k}\frac{\sqrt{\mu_i}}{\alpha-\mu_i}\right)^{-1}. \tag{5.40}$$

Recall that $\alpha > \mu_i$, $i = 1, ..., k$. Choose α such that the right-hand side of (5.40) is 0. Then $\alpha > \mu_{k+1}$ (since $S > 0$) and it follows that

$$\sum_{i=1}^{k}\frac{\sqrt{\mu_i}}{\mu_{k+1}-\mu_i} \geq \sum_{i=1}^{k}\frac{\sqrt{\mu_i}}{\alpha-\mu_i} = \frac{n^2k^2}{8(n+2)}\left(\sum_{i=1}^{k}\sqrt{\mu_i}\right)^{-1}. \tag{5.41}$$

Thus we have recovered Hook's bound using the trial function method. To prove (5.29) we transpose (5.41)

$$\frac{n^2k^2}{8(n+2)} \leq \left(\sum_{i=1}^{k} \sqrt{\mu_i}\right)\left(\sum_{i=1}^{k} \frac{\sqrt{\mu_i}}{\mu_{k+1}-\mu_i}\right)$$

$$\leq \left(\frac{1}{\mu_{k+1}-\mu_k}\right)\left(\sum_{i=1}^{k} \sqrt{\mu_i}\right)^2 \tag{5.42}$$

or

$$\mu_{k+1}-\mu_k \leq \frac{8(n+2)}{n^2k^2}\left(\sum_{i=1}^{k} \sqrt{\mu_i}\right)^2 \tag{5.43}$$

Additionally, let $a_i = \frac{\sqrt{\mu_i}}{\mu_{k+1}-\mu_i}$, $b_i = \sqrt{\mu_i}$. Then both of these terms are increasing with i and now Chebyshev's inequality, which states that $\sum_{i=1}^{k} a_i \sum_{i=1}^{k} b_i \leq k \sum_{i=1}^{k} a_i b_i$ when a_i and b_i are similarly ordered, yields the following when applied to (5.42)

$$\sum_{i=1}^{k} \frac{\sqrt{\mu_i}}{\mu_{k+1}-\mu_i} \geq \frac{n^2k}{8(n+2)}.$$

To recover (1.3) we note that

$$\sum_{i=1}^{k} \sqrt{\mu_i} \leq \sqrt{k}\left(\sum_{i=1}^{k} \mu_i\right)^{1/2}$$

which implies

$$k^{-1/2}\left(\sum_{i=1}^{k} \mu_i\right)^{-1/2} \leq \left(\sum_{i=1}^{k} \sqrt{\mu_i}\right)^{-1} \tag{5.44}$$

Therefore, from (5.28) we have

$$\sum_{i=1}^{k} \frac{\sqrt{\mu_i}}{\mu_{k+1}-\mu_i} \geq \frac{n^2k^2}{8(n+2)}\left(\sum_{i=1}^{k} \sqrt{\mu_i}\right)^{-1}$$

$$\geq \frac{n^2k^{\frac{3}{2}}}{8(n+2)}\left(\sum_{i=1}^{k} \sqrt{\mu_i}\right)^{-1/2}. \tag{5.45}$$

To recover (1.5) we apply (5.44) to (5.45), obtaining

$$\mu_{k+1} - \mu_k \leq \frac{8(n+2)}{n^2k^2}\left(\sum_{i=1}^{k}\sqrt{\mu_i}\right)^2$$
$$\leq \frac{8(n+2)}{n^2k}\left(\sum_{i=1}^{k}\mu_i\right).$$

References

[1] G.N.Hile and R.Z.Yeh, *"Inequalities for eigenvalues of the biharmonic operator "*, Pac.J.Math **112** (1984), 115-133.

[2] S.M. Hook, *"Domain-independent upper bounds for eigenvalues of elliptic operators"*,Tran. AMer.Math.Soc. **318** (1990), 615-642.

[3] L.E.Payne, G.Polya, and H.F.Weinberger, *"On the ratio of consecutive eigenvalues"*, J.Math. and Phys. **35** (1956), 289-298 [reprinted in G.Polya: Collected Papers, Vol. III:Analysis, J.Hersch and G.C.Rota, editors, MIT Press, Cambridge, Massachusetts, 1984, pp. 420-429, with comments by J.Hersch on pp. 521-522].

K.P. MCHALE,
Department of Computer Science
and Mathematics, 1001 Rogers Street
Columbia, MO 65201, USA
(573) 875 - 7609
U. UFUKTEPE
İzmir Institute of Technology,
Faculty of Sciences,
Department of Mathematics,
Gülbahçe, Urla, İzmir-TURKEY

Received 27.03.2000