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Fuzzy Maximal Ideals of Gamma Near-Rings*

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Abstract

Fuzzy maximal ideals and complete normal fuzzy ideals in Γ-near-rings are considered, and related properties are investigated.

Key words and phrases: (normal) fuzzy ideal, fuzzy maximal ideal, complete normal fuzzy ideal.

1. Introduction

Γ-near-rings were defined by Satyanarayana [16], and the ideal theory in Γ-near-rings was studied by Satyanarayana [16] and Booth [1]. Fuzzy ideals of rings were introduced by Liu [11], and it has been studied by several authors [2, 8, 9, 17]. The notion of fuzzy ideals and its properties were applied to various areas: semigroups [10, 12, 4], BCK-algebras [7, 14], and semirings [5]. In [6], Jun et al. considered the fuzzification of left (resp. right) ideals of Γ-near-rings, and investigated the related properties. Jun et al. [3] also introduced the notion of fuzzy characteristic left (resp. right) ideals and normal fuzzy left (resp. right) ideals of Γ-near-rings, and studied some of their properties. As a continuation of the papers [6] and [3], we state fuzzy maximal ideals and complete normal fuzzy ideals in Γ-near-rings, and investigate its properties.

2. Preliminaries

We first recall some basic concepts for the sake of completeness. Recall from [13, p. 3] that a non-empty set R with two binary operations "+" (addition) and "·" (multiplication) is called a *near-ring* if it satisfies the following axioms:

(i) $(R, +)$ is a group,

(ii) $(R, \cdot)$ is a semigroup,

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(iii) $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$.

Precisely speaking, it is a right near-ring because it satisfies the right distributive law. We will use the word "near-ring" to mean "right near-ring". We denote xy instead of $x \cdot y$.

A Γ-*near-ring* ([16]) is a triple $(M, +, \Gamma)$ where

(i) $(M, +)$ is a group,

(ii) Γ is a nonempty set of binary operators on M such that for each $\alpha \in \Gamma$, $(M, +, \alpha)$ is a near-ring,

(iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

A subset A of a Γ-near-ring M is called a *left* (resp. *right*) *ideal* of M if

(i) $(A, +)$ is a normal divisor of $(M, +)$,

(ii) $u\alpha(x + v) - u\alpha v \in A$ $(x\alpha u \in A)$ for all $x \in A$, $\alpha \in \Gamma$ and $u, v \in M$.

We now review some fuzzy logic concepts. A fuzzy set in a set M is a function $\mu : M \to [0, 1]$. We shall use the notation $U(\mu; t)$, called a *level subset* of μ, for $\{x \in M \mid \mu(x) \geq t\}$ where $t \in [0, 1]$.

3. Fuzzy maximal ideals of Γ-near-rings

In what follows let M denote a Γ-near-ring unless otherwise specified.

Definition 3.1 (Jun et al. [6]). A fuzzy set μ in M is called a *fuzzy left* (resp. *right*) *ideal* of M if

(i) μ is a fuzzy normal divisor with respect to the addition,

(ii) $\mu(u\alpha(x + v) - u\alpha v) \geq \mu(x)$ (resp. $\mu(x\alpha u) \geq \mu(x)$) for all $x, u, v \in M$ and $\alpha \in \Gamma$.

The condition (i) of Definition 3.1 means that μ satisfies:

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(y + x - y) \geq \mu(x)$,

for all $x, y \in M$.

Note that if μ is a fuzzy left (resp. right) ideal of M, then $\mu(0) \geq \mu(x)$ for all $x \in M$, where 0 is the zero element of M. Note also that if μ is a fuzzy left (resp. right) ideal of M, then the set

$$M_\mu := \{x \in M \mid \mu(x) = \mu(0)\}$$

is a left (resp. right) ideal of M (see [6]).

From now on, a (fuzzy) ideal shall mean a (fuzzy) left ideal. For a fuzzy ideal μ of M, we note that $\mu(0)$ is the largest value of μ. It is often convenient to have $\mu(0) = 1$.

Definition 3.2 (Jun et al. [3, Definition 3.16]). A fuzzy ideal μ of M is said to be *normal* if $\mu(0) = 1$.

Lemma 3.3 (Jun et al. [3]). *For an ideal A of M, if we define a fuzzy set in M by*

$$\mu_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in M$, then μ_A is a normal fuzzy ideal of M and $M_{\mu_A} = A$.

Theorem 3.4. *Let A and B be ideals of M. Then $A \subseteq B$ if and only if $\mu_A \subseteq \mu_B$.*

Proof. Straightforward. □

Proposition 3.5. *If μ and ν are normal fuzzy ideals of M, then $\mu \cap \nu$. is an ideal*
Proof. Straightforward. □

Lemma 3.6 (Jun et al. [3, Theorem 3.17]). *Let μ be a fuzzy ideal of M and let μ^* be a fuzzy set in M defined by $\mu^*(x) = \mu(x) + 1 - \mu(0)$ for all $x \in M$. Then μ^* is a normal fuzzy ideal of M containing μ.*

Lemma 3.7 (Jun et al. [3, Corollary 3.18]). *If μ is a fuzzy ideal of M satisfying $\mu^*(x) = 0$ for some $x \in M$, then $\mu(x) = 0$.*

Lemma 3.8 (Jun et al. [3, Theorem 3.22]). *Any fuzzy ideal μ of M is normal if and only if $\mu^* = \mu$.*

Using a given fuzzy ideal μ of M, we will construct a new fuzzy ideal. Let $t > 0$ be a real number, and define a mapping $\mu^t : M \to [0, 1]$ by $\mu^t(x) = (\mu(x))^t$ for all $x \in M$, where $(\mu(x))^t = \sqrt[2t]{\mu(x)}$ when $0 < t < 1$.

Theorem 3.9. *Let $t > 0$ be a real number. If μ is a normal fuzzy ideal of M, then μ^t is also a normal fuzzy ideal of M and $M_{\mu^t} = M_\mu$.*

Proof. For any $x, y \in M$, we have

$$\begin{aligned} \mu^t(x - y) &= (\mu(x - y))^t \geq (\min\{\mu(x), \mu(y)\})^t \\ &= \min\{(\mu(x))^t, (\mu(y))^t\} = \min\{\mu^t(x), \mu^t(y)\} \end{aligned}$$

and $\mu^t(y + x - y) = (\mu(y + x - y))^t \geq (\mu(x))^t = \mu^t(x)$. Let $x, u, v \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} \mu^t(u\alpha(x + v) - u\alpha v) &= (\mu(u\alpha(x + v) - u\alpha v))^t \\ &\geq (\mu(x))^t = \mu^t(x). \end{aligned}$$

Note that $\mu^t(0) = (\mu(0))^t = 1^t = 1$. Hence μ^t is a normal fuzzy ideal of M. Now

$$\begin{aligned} M_{\mu^t} &= \{x \in M \mid \mu^t(x) = \mu^t(0)\} \\ &= \{x \in M \mid (\mu(x))^t = 1\} \\ &= \{x \in M \mid \mu(x) = 1\} \\ &= \{x \in M \mid \mu(x) = \mu(0)\} \\ &= M_\mu. \end{aligned}$$

This completes the proof. □

Let $\mathcal{I}(M)$ (resp. $\mathcal{N}(M)$) denote the set of all ideals (resp. normal fuzzy ideals) of M. We define functions $\phi : \mathcal{I}(M) \to \mathcal{N}(M)$ and $\psi : \mathcal{N}(M) \to \mathcal{I}(M)$ by $\phi(A) = \mu_A$ and $\psi(\mu) = M_\mu$, respectively, for all $A \in \mathcal{I}(M)$ and $\mu \in \mathcal{N}(M)$. Then $\psi\phi = 1_{\mathcal{I}(M)}$ and $\phi\psi(\mu) = \phi(M_\mu) = \mu_{M_\mu} \subseteq \mu$.

Theorem 3.10. *If $A, B \in \mathcal{I}(M)$, then $\mu_{A\cap B} = \mu_A \cap \mu_B$, that is, $\phi(A \cap B) = \phi(A) \cap \phi(B)$. If $\mu, \nu \in \mathcal{N}(M)$, then $M_{\mu\cap\nu} = M_\mu \cap M_\nu$, that is, $\psi(\mu \cap \nu) = \psi(\mu) \cap \psi(\nu)$.*

Proof. Let $x \in M$. If $x \in A \cap B$, then $\mu_{A\cap B}(x) = 1$. From $x \in A$ and $x \in B$ it follows that $\mu_A(x) = 1 = \mu_B(x)$. Hence

$$\mu_{A\cap B}(x) = 1 = \min\{\mu_A(x), \mu_B(x)\} = (\mu_A \cap \mu_B)(x).$$

If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$. Thus

$$\mu_{A\cap B}(x) = 0 = \min\{\mu_A(x), \mu_B(x)\} = (\mu_A \cap \mu_B)(x).$$

Therefore $\mu_{A\cap B} = \mu_A \cap \mu_B$, and so $\phi(A \cap B) = \phi(A) \cap \phi(B)$ for all $A, B \in \mathcal{I}(M)$. Now let $\mu, \nu \in \mathcal{N}(M)$. Then

$$\begin{aligned} M_{\mu\cap\nu} &= \{x \in M \mid (\mu \cap \nu)(x) = (\mu \cap \nu)(0)\} \\ &= \{x \in M \mid \min\{\mu(x), \mu(y)\} = 1\} \\ &= \{x \in M \mid \mu(x) = 1 \text{ and } \nu(x) = 1\} \\ &= \{x \in M \mid \mu(x) = 1\} \cap \{x \in M \mid \nu(x) = 1\} \\ &= \{x \in M \mid \mu(x) = \mu(0)\} \cap \{x \in M \mid \nu(x) = \nu(0)\} \\ &= M_\mu \cap M_\nu, \end{aligned}$$

that is, $\psi(\mu \cap \nu) = M_{\mu\cap\nu} = M_\mu \cap M_\nu = \psi(\mu) \cap \psi(\nu)$. This completes the proof. □

Definition 3.11. A fuzzy ideal μ of M is said to be *fuzzy maximal* if it satisfies:

(i) μ is non-constant,

(ii) μ^* is a maximal element of $(\mathcal{N}(M), \subseteq)$.

Lemma 3.12 (Jun et al. [3, Theorem 3.28])**.** *Let μ be a non-constant normal fuzzy ideal of M, which is maximal in the poset of normal fuzzy ideals under set inclusion. Then μ takes only the values* 0 *and* 1.

Theorem 3.13. *If μ is a fuzzy maximal ideal of M, then*

(i) *μ is normal,*

(ii) *μ^* takes only the values* 0 *and* 1,

(iii) *$\mu_{M_\mu} = \mu$,*

(iv) *M_μ is a maximal ideal of M.*

Proof. Let μ be a fuzzy maximal ideal of M. Then μ^* is a non-constant maximal element of the poset $(\mathcal{N}(M), \subseteq)$. It follows from Lemma 3.12 that μ^* takes only the values 0 and 1. Note that $\mu^*(x) = 1$ if and only if $\mu(x) = \mu(0)$, and $\mu^*(x) = 0$ if and only if $\mu(x) = \mu(0) - 1$. By Lemma 3.7, we have $\mu(x) = 0$, that is, $\mu(0) = 1$. Hence μ is normal. This proves (i) and (ii).

(iii) Clearly, $\mu_{M_\mu} \subseteq \mu$ and μ_{M_μ} takes only the values 0 and 1. Let $x \in M$. If $\mu(x) = 0$, then obviously $\mu \subseteq \mu_{M_\mu}$. If $\mu(x) = 1$, then $x \in M_\mu$ and so $\mu_{M_\mu}(x) = 1$. This shows that $\mu \subseteq \mu_{M_\mu}$.

(iv) M_μ is a propser ideal of M because μ is non-constant. Let A be an ideal of M such that $M_\mu \subseteq A$. Using (iii) and Theorem 3.4, we have $\mu = \mu_{M_\mu} \subseteq \mu_A$. Since $\mu, \mu_A \in \mathcal{N}N(M)$ and $\mu = \mu^*$ is a maximal element of $\mathcal{N}(M)$, it follows that either $\mu = \mu_A$ or $\mu_A = \mathbf{1}$ where $\mathbf{1} : M \to [0, 1]$ is a fuzzy set defined by $\mathbf{1}(x) = 1$ for all $x \in M$. The later case implies that $A = M$. If $\mu = \mu_A$, then $M_\mu = M_{\mu_A} = A$ by Lemma 3.3. This proves that M_μ is a maximal ideal of M. This completes the proof. □

Definition 3.14. A normal fuzzy ideal μ of M is said to be *complete* if there exists $c \in M$ such that $\mu(c) = 0$.

Note that μ_A is a complete normal fuzzy ideal of M for every ideal A of M.

Denote by $\mathcal{C}(M)$ the set of all complete normal fuzzy ideals of M. Note that $\mathcal{C}(M) \subseteq \mathcal{N}(M)$ and the restriction of the partial ordering "$\subseteq$" of $\mathcal{N}(M)$ gives a partial ordering of $\mathcal{C}(M)$.

Theorem 3.15. *Every non-constant maximal element of $(\mathcal{N}(M), \subseteq)$ is also a maximal element of $(\mathcal{C}(M), \subseteq)$.*

Proof. Let μ be a non-constant maximal element of $(\mathcal{N}(M), \subseteq)$. By Lemma 3.12, μ takes only the values 0 and 1, and in fact $\mu(0) = 1$ and $\mu(c) = 0$ for some $c (\neq 0) \in M$.

Hence μ is complete. Assume that there exists $\nu \in \mathcal{C}(M)$ such that $\mu \subseteq \nu$. It follows that $\mu \subseteq \nu$ in $\mathcal{N}(M)$. Since μ is maximal in $(\mathcal{N}(M), \subseteq)$ and since ν is non-constant, therefore $\mu = \nu$. Thus μ is a maximal element of $(\mathcal{C}(M), \subseteq)$. □

Theorem 3.16. *Every fuzzy maximal ideal of M is complete normal.*

Proof. Let μ be a fuzzy maximal ideal of M. By Theorem 3.13 and Lemma 3.8, μ is normal and $\mu = \mu^*$ takes only the values 0 and 1. Since μ is non-constant and $\mu(0) = 1$, it is clear that there exists $c(\neq 0) \in M$ such that $\mu(c) = 0$. Hence μ is complete. This completes the proof. □

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