

## On the Expansions in Eigenfunctions of Hill's Operator

*F. Aras and G. Sh. Guseinov*

### Abstract

In this paper we show how one can deduce the Titchmarsh expansion formula in eigenfunctions of Hill's operator from the Gel'fand expansion formula.

**Key Words:** Hill's operator, spectrum, eigenvalues, eigenfunctions.

### 1. Introduction

Consider the second-order differential equation

$$-[p(x)y']' + q(x)y = \lambda\rho(x)y \quad (-\infty < x < \infty), \quad (1)$$

where  $\lambda$  is a complex parameter (spectral parameter), the coefficients  $p(x)$ ,  $q(x)$ , and  $\rho(x)$  are real-valued measurable functions defined on  $(-\infty, \infty)$  and periodic with the period  $\omega > 0$ :

$$p(x + \omega) = p(x), \quad q(x + \omega) = q(x), \quad \rho(x + \omega) = \rho(x).$$

In addition, we assume that  $p(x) > 0$  and  $\rho(x) > 0$  almost everywhere, and

$$\int_0^\omega \frac{dx}{p(x)} < \infty, \quad \int_0^\omega |q(x)|dx < \infty, \quad \int_0^\omega \rho(x)dx < \infty. \quad (2)$$

Notice that we do not assume the differentiability and even the continuity of  $p(x)$ . A function  $y = y(x)$  is called a solution of the equation (1) if its first derivative  $y'(x)$  exists,  $p(x)y'(x)$  is absolutely continuous and (1) is satisfied almost everywhere on  $(-\infty, \infty)$ . Let us set

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$$y^{[1]}(x) = p(x)y'(x).$$

This is the so-called quasi-derivative of  $y(x)$ . For any solution  $y(x)$  and any point  $a \in (-\infty, \infty)$  the value  $y(a)$  is finite, whereas the value  $y'(a)$  may be infinite. However, the value

$$y^{[1]}(a) = \lim_{x \rightarrow a} p(x)y'(x)$$

certainly will be finite. Under condition (2) the existence and uniqueness theorem for solution  $y(x)$  of the equation (1) satisfying the initial conditions

$$y(a) = c_0, y^{[1]}(a) = c_1$$

is valid (See, for example, [12, Kapitel 5].)

For the results relating to eigenvalue and eigenfunction theory of periodic differential equations we refer to [1, 2, 11, 14].

Denote by

$$\mu_0^+ < \mu_2^- \leq \mu_2^+ < \mu_4^- \leq \mu_4^+ < \dots < \mu_{2j}^- \leq \mu_{2j}^+ < \dots$$

the eigenvalues of the periodic boundary value problem (BVP) generated on the segment  $0 \leq x \leq \omega$  by Equation (1) and the boundary conditions

$$y(0) = y(\omega), y^{[1]}(0) = y^{[1]}(\omega), \tag{3}$$

and by

$$\mu_1^- \leq \mu_1^+ < \mu_3^- \leq \mu_3^+ < \dots < \mu_{2j+1}^- \leq \mu_{2j+1}^+ < \dots$$

the eigenvalues of the semi-periodic (or anti-periodic) BVP generated on the segment  $0 \leq x \leq \omega$  by the equation (1) and the boundary conditions

$$y(0) = -y(\omega), y^{[1]}(0) = -y^{[1]}(\omega). \tag{4}$$

In the above inequalities the equality holds in the case of double eigenvalue. These values occur in the order

$$\mu_0^+ < \mu_1^- \leq \mu_1^+ < \mu_2^- \leq \mu_2^+ < \mu_3^- \leq \mu_3^+ < \mu_4^- \leq \mu_4^+ < \dots .$$

Furthermore, denote by  $H$  the minimal closed operator generated in the Hilbert space

$$L^2_\rho(-\infty, \infty) = \{f : (-\infty, \infty) \rightarrow \mathbf{C} \mid \int_{-\infty}^{\infty} \rho(x)|f(x)|^2 dx < \infty\}$$

by the differential expression

$$\frac{1}{\rho(x)} \left\{ -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right\}.$$

The operator  $H$ , which is called the Hill operator is selfadjoint, its spectrum is continuous and consists of the sequence of segments

$$[\mu_0^+, \mu_1^-], [\mu_1^+, \mu_2^-], \dots, [\mu_{j-1}^+, \mu_j^-], \dots ,$$

which are separated from each other by the gaps

$$(-\infty, \mu_0^+), (\mu_1^-, \mu_1^+), \dots, (\mu_j^-, \mu_j^+), \dots .$$

An important role in the spectral analysis of the operator  $H$  is played by the so-called  $t$ -periodic BVP defined on the segment  $0 \leq x \leq \omega$  by Equation (1) and the boundary conditions

$$y(\omega) = e^{it}y(0) , y^{[1]}(\omega) = e^{it}y^{[1]}(0), \tag{5}$$

where  $t$  is an arbitrary fixed number that belongs to  $[-\pi, \pi]$ .

We denote by  $\theta(x, \lambda)$  and  $\varphi(x, \lambda)$  the solutions of Equation (1) satisfying the conditions

$$\theta(0, \lambda) = \varphi^{[1]}(0, \lambda) = 1 , \theta^{[1]}(0, \lambda) = \varphi(0, \lambda) = 0. \tag{6}$$

These solutions and their first quasi-derivatives with respect to  $x$  are entire functions of the variable  $\lambda \in \mathbf{C}$  and are real-valued for real values of  $\lambda$  . Let us set

$$F(\lambda) = \theta(\omega, \lambda) + \varphi^{[1]}(\omega, \lambda). \tag{7}$$

Then the eigenvalues of the  $t$  - periodic BVP (1), (5) coincide with the roots  $\lambda$  of the

equation

$$F(\lambda) = 2 \cos t. \tag{8}$$

The BVP (1), (5) is self adjoint and possesses a countable set of real eigenvalues with the accumulation point at  $+\infty$ , and the corresponding eigenfunctions form an orthogonal basis of  $L^2_\rho[0, \omega]$ . Denote by  $\lambda_j(t), j = 1, 2, \dots$ , the eigenvalues of the problem (1), (5) numerated with regard of their multiplicity and in increasing order, so that  $\lambda_j(t)$  for any  $j$  be a continuous function of the variable  $t$ . Let  $\psi_j(x, t), j = 1, 2, \dots$ , be the corresponding orthonormal eigenfunctions.

Extending each of the functions  $\psi_j(x, t)$  to the whole axis  $-\infty < x < \infty$  as a solution of Equation (1) with  $\lambda = \lambda_j(t)$ , we arrive at the functional equation

$$\psi_j(x + \omega, t) = e^{it}\psi_j(x, t) \quad (-\infty < x < \infty).$$

With this, we obtain a set of solutions of the equation (1)

$$\psi_j(x, t), j = 1, 2, \dots, t \in [-\pi, \pi],$$

bounded with respect to variable  $x \in (-\infty, \infty)$ . These functions are called generalized or Bloch eigenfunctions of the Hill operator  $H$ .

In the work of Gel'fand [3] (see also [13]) the following theorem is proved.

**Theorem 1** *For an arbitrary function  $f \in L^2_\rho(-\infty, \infty)$  there exist limit functions*

$$\alpha_j(t) = \lim_{r \rightarrow \infty} \int_{-r}^r \rho(x) f(x) \overline{\psi_j(x, t)} dx, j = 1, 2, \dots, \tag{9}$$

*and the Parseval identity*

$$\int_{-\infty}^{\infty} \rho(x) |f(x)|^2 dx = \frac{1}{2\pi} \sum_{j=1}^{\infty} \int_{-\pi}^{\pi} |\alpha_j(t)|^2 dt \tag{10}$$

*and the expansion formula*

$$f(x) = \frac{1}{2\pi} \sum_{j=1}^{\infty} \int_{-\pi}^{\pi} \alpha_j(t) \psi_j(x, t) dt \quad (11)$$

hold. The limit in (9) is taken with respect to the metric of  $L^2[-\pi, \pi]$ . The bar over a function here and below indicates complex conjugation; and the series (11) converges with respect to the metric of  $L^2_\rho(-\infty, \infty)$ .

In the monograph of Titchmarsh [14, Chap. XXI] the Parseval identity and the expansion formula are obtained in other terms: that is the following theorem is proved.

**Theorem 2** For an arbitrary real-valued function  $f \in L^2_\rho(-\infty, \infty)$  with compact support, the Parseval identity

$$\begin{aligned} \int_{-\infty}^{\infty} \rho(x) f^2(x) dx &= \frac{1}{\pi} \left( \sum_{j=1}^{\infty} \int_{\mu_{2j-2}^+}^{\mu_{2j-1}^-} - \sum_{j=1}^{\infty} \int_{\mu_{2j-1}^+}^{\mu_{2j}^-} \right) \{ \varphi(\omega, \lambda) g^2(\lambda) \\ &\quad - \theta^{[1]}(\omega, \lambda) h^2(\lambda) + [\varphi^{[1]}(\omega, \lambda) - \theta(\omega, \lambda)] g(\lambda) h(\lambda) \} \eta(\lambda) d\lambda \end{aligned} \quad (12)$$

and the expansion formula

$$\begin{aligned} f(x) &= \frac{1}{\pi} \left( \sum_{j=1}^{\infty} \int_{\mu_{2j-2}^+}^{\mu_{2j-1}^-} - \sum_{j=1}^{\infty} \int_{\mu_{2j-1}^+}^{\mu_{2j}^-} \right) \{ \varphi(\omega, \lambda) g(\lambda) \theta(x, \lambda) - \theta^{[1]}(x, \lambda) h(\lambda) \varphi(x, \lambda) + \\ &\quad \frac{1}{2} [\varphi^{[1]}(\omega, \lambda) - \theta(\omega, \lambda)] [h(\lambda) \theta(x, \lambda) + g(\lambda) \varphi(x, \lambda)] \} \eta(\lambda) d\lambda \end{aligned} \quad (13)$$

hold, where

$$g(\lambda) = \int_{-\infty}^{\infty} \rho(x) f(x) \theta(x, \lambda) dx, \quad h(\lambda) = \int_{-\infty}^{\infty} \rho(x) f(x) \varphi(x, \lambda) dx \quad (14)$$

and

$$\eta(\lambda) = \{ 4 - [\theta(\omega, \lambda) + \varphi^{[1]}(\omega, \lambda)]^2 \}^{-1/2} \quad (15)$$

is a positive on the interval  $(\mu_0^+, \mu_1^-)$  branch of the root.

Note that Theorem 2, in contrast to Theorem 1, tells us much about the structure of the spectral matrix of the operator  $H$ . The proof of this theorem as given in [14,

Chapter.XXI] is (as it seems to us) rather heavy and is constructed by means of a resolvent method via using a contour integral of the Green function.

At the same time, the proof of Theorem 1 is more elementary (for the detailed proof of this theorem see [8]).

In this paper we show that Theorem 2 may be deduced from Theorem 1 using some simple tools. Such a way unlike the resolvent method can be applied also to the multiparameter differential and difference equations with periodic coefficients (see [4, 7, 8 ])to which the resolvent method has not yet been developed. This way was exploited by the one of the authors in [5, 6, 9].

## 2. Deduction Theorem 2 from Theorem 1

First, we study some needed properties of the eigenvalue  $\lambda_j(t)$  as the function of variable  $t$ .

Since the eigenvalues of the  $t$ -periodic BVP (1), (5) coincide with the roots of Equation (8), for all  $j = 1, 2, \dots$  we have

$$F(\lambda_j(t)) = 2 \cos t. \tag{16}$$

By (7) the function  $F(\lambda)$  is holomorphic in the complex plane  $\mathbf{C}$ .

It is easily verified (see [1, p. 216]) that

$$\begin{aligned} \frac{dF(\lambda)}{d\lambda} = & \int_0^\omega \{ \theta^{[1]}(\omega, \lambda) \varphi^2(x, \lambda) - \varphi(\omega, \lambda) \theta^2(x, \lambda) \\ & + [\theta(\omega, \lambda) - \varphi^{[1]}(\omega, \lambda)] \theta(x, \lambda) \varphi(x, \lambda) \} \rho(x) dx . \end{aligned} \tag{17}$$

Denote  $G = \{ \lambda \in (-\infty, \infty) : |F(\lambda)| < 2 \}$ . From the continuity of the function  $F(\lambda)$ , it follows that  $G$  is an open subset of  $(-\infty, \infty)$ .

**Lemma 1** *If  $\lambda \in G$ , then  $\frac{dF(\lambda)}{d\lambda} \neq 0$*

*Proof.* First of all, we observe that if  $\lambda \in (-\infty, \infty)$  and  $|F(\lambda)| < 2$ , then  $\varphi(\omega, \lambda)$  and  $\theta^{[1]}(\omega, \lambda)$  are nonzero and have opposite signs. Indeed, by the definition (7) of the function  $F(\lambda)$ , and by the identity

$$\theta(\omega, \lambda)\varphi^{[1]}(\omega, \lambda) - \theta^{[1]}(\omega, \lambda)\varphi(\omega, \lambda) = 1 \tag{18}$$

(which follows from the constancy of the Wronskian), the inequality  $F^2(\lambda) < 4$  is equivalent to

$$[\theta(\omega, \lambda) - \varphi^{[1]}(\omega, \lambda)]^2 < -4\theta^{[1]}(\omega, \lambda)\varphi(\omega, \lambda),$$

which implies the desired assertion.

Since for  $\lambda \in G$  we have  $\varphi(\omega, \lambda) \neq 0$ , formula (17) can be rewritten in the form

$$\begin{aligned} \frac{dF(\lambda)}{d\lambda} = & -\varphi(\omega, \lambda) \int_0^\omega \left\{ \left[ \theta(x, \lambda) - \frac{\theta(\omega, \lambda) - \varphi^{[1]}(\omega, \lambda)}{2\varphi(\omega, \lambda)} \varphi(x, \lambda) \right]^2 + \right. \\ & \left. \frac{4 - F^2(\lambda)}{4\varphi^2(\omega, \lambda)} \varphi^2(x, \lambda) \right\} \rho(x) dx. \end{aligned}$$

This implies the lemma.

**Lemma 2** *If  $t \in (0, \pi)$ , then  $\lambda_j(t) \neq \lambda_\ell(t)$  for  $j \neq \ell$ . In other words, the eigenvalues of the  $t$ -periodic BVP (1), (5) are simple for  $t \in (0, \pi)$ .*

*Proof.* Suppose the contrary:  $\lambda_j(t) = \lambda_\ell(t)$ ,  $j \neq \ell$ , for some  $t \in (0, \pi)$ . This means that  $\lambda_j(t)$  is a multiple eigenvalue of the  $t$ -periodic BVP (1), (5). Then for  $\lambda = \lambda_j(t)$  the BVP (1), (5) will have two linearly independent solutions  $\psi(x)$  and  $\tilde{\psi}(x)$ . Hence for this  $\lambda$  any solution  $y(x)$  of Equation (1), being a linear combination of the solutions  $\psi(x)$  and  $\tilde{\psi}(x)$ , will satisfy boundary conditions (5). In particular,  $\theta(x, \lambda)$  and  $\varphi(x, \lambda)$  will satisfy these conditions. Hence

$$F(\lambda) = \theta(\omega, \lambda) + \varphi^{[1]}(\omega, \lambda) = e^{it}\theta(0, \lambda) + e^{it}\varphi^{[1]}(0, \lambda) = 2e^{it}.$$

On the other hand, the eigenvalue  $\lambda$  of the problem (1), (5) should satisfy Equation (8). Thus, we arrive at  $e^{it} = \cos t$ . Obviously, the latter is possible only for  $t = m\pi$  ( $m = 0, \pm 1, \pm 2, \dots$ ). But by the condition of the lemma,  $0 < t < \pi$ : the contradiction proves the lemma.

**Lemma 3** *For any  $j = 1, 2, \dots$  the function  $\lambda_j(t)$  is real-analytic ( $\mathbf{R}$ -analytic) in the interval  $(0, \pi)$  and continuous in its closure  $[0, \pi]$ .*

**Proof.** Take an arbitrary point  $t^0 \in (0, \pi)$  and set  $\lambda^0 = \lambda_j(t^0)$ , where  $j$  is fixed. Since the inclusion  $t^0 \in (0, \pi)$  implies  $|\cos t^0| < 1$ , it follows from (16) that  $|F(\lambda^0)| < 2$ . Hence  $\lambda^0 \in G$ .

The function  $\Phi(\lambda, t) = F(\lambda) - 2 \cos t$  of two arguments  $\lambda$  and  $t$  is  $\mathbf{R}$ -analytic in  $\mathbf{R}^2$ ; and  $\Phi(\lambda^0, t^0) = 0$ . Moreover, by Lemma 1

$$\frac{\partial \Phi(\lambda, t)}{\partial \lambda} = \frac{dF(\lambda)}{d\lambda} \neq 0,$$

for  $\lambda \in G$ . By the implicit function theorem applied to analytic equations, in a neighbourhood of the point  $(\lambda^0, t^0) \in G \times (0, \pi)$  the equation  $\Phi(\lambda, t) = 0$  determines  $\lambda$  as a single-valued function of  $t : \lambda = \varphi(t)$ , and, moreover the function  $\varphi$  is  $\mathbf{R}$ -analytic.

These arguments together with (16) imply that the function  $\lambda_j(t)$  is  $\mathbf{R}$ -analytic in  $(0, \pi)$ .

From the variational principles for the eigenvalues applied to the BVP (1), (5) it follows that the function  $\lambda_j(t)$  depends continuously on  $t \in \mathbf{R}$ . The lemma is proved.  $\square$

Since by Lemma 3 the function  $\lambda_j(t)$  is differentiable for  $0 < t < \pi$ , differentiating the identity (16) with respect to  $t$ , we get

$$\frac{dF(\lambda)}{d\lambda} \Big|_{\lambda=\lambda_j(t)} \cdot \frac{d\lambda_j(t)}{dt} = -2 \sin t \tag{19}$$

for  $t \in (0, \pi)$ .

It follows from (19) that the function  $\frac{d\lambda_j(t)}{dt}$  is nonzero for  $t \in (0, \pi)$ . Hence, by continuity, it is of constant sign (but depending on  $j$ ).

Let  $S_j$  be the image of the interval  $[0, \pi]$  under the mapping  $\lambda_j(\cdot) : [0, \pi] \longrightarrow \mathbf{R}$ ,

$$S_j = \{\lambda_j(t) : t \in [0, \pi]\}.$$

Since from (19) we have

$$\frac{d\lambda_j(t)}{dt} \neq 0 \quad \text{for } t \in (0, \pi),$$



it follows by the inverse mapping theorem that

$$\lambda_j(\cdot) : (0, \pi) \longrightarrow \text{int } S_j$$

is a one-to-one bicontinuous mapping and, moreover, the inverse of it is also  $\mathbf{R}$ -analytic. The latter implies that the boundary points of  $[0, \pi]$  correspond to the boundary points of  $S_j$  and conversely. Hence  $S_j$  is a bounded and closed interval of  $\mathbf{R}$ .

**Lemma 4** *The intervals  $S_j$  ( $j = 1, 2, \dots$ ) do not overlap, i.e., no two of them have common interior points.*

**Proof.** Suppose the contrary. Then for some  $t, \tau \in (0, \pi)$  and  $j \neq \ell$  the equality  $\lambda_j(t) = \lambda_\ell(\tau) \stackrel{\text{def}}{=} \mu$  will be valid. By Lemma 2 we have  $t \neq \tau$ . The functions  $\psi_j(x, t) = y(x)$  and  $\psi_\ell(x, \tau) = z(x)$  are solutions of Equation (1) for  $\lambda = \mu$  satisfying the boundary conditions

$$y(\omega) = e^{it}y(0) , y^{[1]}(\omega) = e^{it}y^{[1]}(0) \tag{20}$$

$$z(\omega) = e^{i\tau}z(0) , z^{[1]}(\omega) = e^{i\tau}z^{[1]}(0).$$

One can easily see that the solutions  $y(x)$  and  $z(x)$  are linearly independent. Indeed, if not, then from the boundary conditions (20) we would obtain  $t - \tau = 2m\pi$  for some integer  $m$ , which is impossible since  $t, \tau \in (0, \pi)$  and  $t \neq \tau$ .

From the linear independence of the solutions  $y(x)$  and  $z(x)$  it follows that the Wronskian

$$W_x(y, z) = y(x)z^{[1]}(x) - y^{[1]}(x)z(x)$$

is not zero. Since the Wronskian does not depend on  $x$ , it has the same value for  $x = \omega$  and  $x = 0$ . Hence by (20) we obtain  $\exp\{i(t+\tau)\} = 1$ , which implies that  $t+\tau = 2m\pi$  for same integer  $m$ . The latter cannot be true since  $t, \tau \in (0, \pi)$ . The contradiction proves the lemma.  $\square$

**Lemma 5** For each  $j$  ( $j = 1, 2, \dots$ ) as the parameter  $t$  increases continuously from zero to  $\pi$ , the function  $\lambda_{2j-1}(t)$  increases continuously from the value  $\mu_{2j-2}^+$  to the value  $\mu_{2j-1}^-$  while  $\lambda_{2j}(t)$  decreases continuously from  $\mu_{2j}^-$  to  $\mu_{2j-1}^+$ . So, if  $t$  runs over the segment  $[0, \pi]$  from 0 to  $\pi$ , then  $\lambda_j(t)$  strictly monotonically runs over the segment  $[\mu_{j-1}^+, \mu_j^-]$ . In addition, for the odd  $j$  the motion of  $\lambda_j(t)$  starts from the left edge of the segment  $[\mu_{j-1}^+, \mu_j^-]$  while for the even  $j$  from the right edge of this segment.

**Proof.** For a fixed  $t \in (0, \pi)$  we consider the function  $\Phi(\lambda) = F(\lambda) - 2 \cos t$ . Since  $F(\mu_j^\pm) = (-1)^j \cdot 2$ , the values of  $\Phi(\lambda)$  at the edges of the segment  $[\mu_{j-1}^+, \mu_j^-]$  are

$$\begin{aligned} \Phi(\mu_{j-1}^+) &= (-1)^{j-1} \cdot 2 - 2 \cos t = -2[(-1)^j + \cos t] \\ \Phi(\mu_j^-) &= (-1)^j \cdot 2 - 2 \cos t = 2[(-1)^j - \cos t]. \end{aligned}$$

Obviously, these values have opposite sign. Hence, by the continuity, in the segment  $[\mu_{j-1}^+, \mu_j^-]$  there is at least one root of the function  $\Phi(\lambda)$ , i.e., at least one eigenvalue of the  $t$ -periodic BVP (1), (5). Since  $F(\lambda)$  is monotonic on  $[\mu_{j-1}^+, \mu_j^-]$ , there must be exactly one. Consequently, the numbers  $\lambda_j(t)$  ( $j = 1, 2, \dots$ ) lie one each in the segments  $[\mu_{j-1}^+, \mu_j^-]$  ( $j = 1, 2, \dots$ ):

$$\mu_{j-1}^+ < \lambda_j(t) < \mu_j^-, \quad t \in (0, \pi), \quad j = 1, 2, \dots \tag{21}$$

The cases  $t = 0$  and  $t = \pi$  of the  $t$ -periodic BVP (1), (5) are the periodic and semi-periodic problems (1), (3) and (1), (4) respectively. Therefore,  $\lambda_j(0)$  are the eigenvalues of the periodic BVP and  $\lambda_j(\pi)$  are the eigenvalues of the semi-periodic BVP. This yields, by (21), that

$$\begin{aligned} \lambda_{2j-1}(0) &= \mu_{2j-2}^+, \quad \lambda_{2j-1}(\pi) = \mu_{2j-1}^- \\ \lambda_{2j}(0) &= \mu_{2j}^-, \quad \lambda_{2j}(\pi) = \mu_{2j-1}^+ \end{aligned} \tag{22}$$

It is known (see [2, p.27] and [14, Chap.XXI]) that

$$\frac{dF(\lambda)}{d\lambda} < 0 \text{ for } \mu_{2j-2}^+ < \lambda < \mu_{2j-1}^-,$$

$$\frac{dF(\lambda)}{d\lambda} > 0 \text{ for } \mu_{2j-1}^+ < \lambda < \mu_{2j}^-.$$

Consequently, by (19) and (21), we get

$$\frac{d\lambda_{2j-1}(t)}{dt} > 0, \frac{d\lambda_{2j}(t)}{dt} < 0, \text{ for } 0 < t < \pi.$$

From the latter inequalities and (21), (22) follow the statements of the lemma.  $\square$

Ending the analysis of  $\lambda_j(t)$  we now proceed to describe the functions  $\psi_j(x, t)$  and  $\alpha_j(t)$  presented in (10) and (11).

Since  $\theta(x, \lambda)$  and  $\varphi(x, \lambda)$  form a fundamental system of solutions of Equation (1), taking the initial conditions (6) into account we have

$$\psi_j(x, t) = \psi_j(0, t)\theta(x, \lambda_j(t)) + \psi_j^{[1]}(0, t)\varphi(x, \lambda_j(t)). \quad (23)$$

Let  $f \in L^2_\rho(-\infty, \infty)$  be a real function with compact support. By (9) and (23) we obtain

$$\alpha_j(t) = \int_{-\infty}^{\infty} \rho(x)f(x)\overline{\psi_j(x, t)}dx = \overline{\psi_j(0, t)}g(\lambda_j(t)) + \overline{\psi_j^{[1]}(0, t)}h(\lambda_j(t)),$$

where  $g(\lambda)$  and  $h(\lambda)$  are defined by (14). Therefore,

$$\begin{aligned} |\alpha_j(t)|^2 &= |\psi_j(0, t)|^2g^2(\lambda_j(t)) + |\psi_j^{[1]}(0, t)|^2h^2(\lambda_j(t)) \\ &+ [\psi_j(0, t)\overline{\psi_j^{[1]}(0, t)} + \overline{\psi_j(0, t)}\psi_j^{[1]}(0, t)]g(\lambda_j(t))h(\lambda_j(t)), \end{aligned} \quad (24)$$

$$\begin{aligned} \alpha_j(t)\psi_j(x, t) &= |\psi_j(0, t)|^2g(\lambda_j(t))\theta(x, \lambda_j(t)) + |\psi_j^{[1]}(0, t)|^2h(\lambda_j(t))\varphi(x, \lambda_j(t)) \\ &+ \psi_j(0, t)\overline{\psi_j^{[1]}(0, t)}h(\lambda_j(t))\theta(x, \lambda_j(t)) + \overline{\psi_j(0, t)}\psi_j^{[1]}(0, t)g(\lambda_j(t))\varphi(x, \lambda_j(t)). \end{aligned} \quad (25)$$

Further, from (23) and the equality  $\psi_j(\omega, t) = e^{it}\psi_j(0, t)$  we get

$$\psi_j^{[1]}(0, t) = \frac{e^{it} - \theta(\omega, \lambda_j(t))}{\varphi(\omega, \lambda_j(t))}\psi_j(0, t). \quad (26)$$

Note that, by (16),  $\varphi(\omega, \lambda_j(t)) \neq 0$  for  $t \in (0, \pi)$  (cf. the proof of Lemma1). Therefore,

taking into account the relation

$$\theta(\omega, \lambda_j(t)) + \varphi^{[1]}(\omega, \lambda_j(t)) = 2 \cos t, \quad (27)$$

which follows from (16), we find

$$\begin{aligned} \psi_j(0, t) \overline{\psi_j^{[1]}(0, t)} + \overline{\psi_j(0, t)} \psi_j^{[1]}(0, t) &= 2 \frac{\cos t - \theta(\omega, \lambda_j(t))}{\varphi(\omega, \lambda_j(t))} |\psi_j(0, t)|^2 \\ &= \frac{\varphi^{[1]}(\omega, \lambda_j(t)) - \theta(\omega, \lambda_j(t))}{\varphi(\omega, \lambda_j(t))} |\psi_j(0, t)|^2, \end{aligned} \quad (28)$$

$$\begin{aligned} |\psi_j^{[1]}(0, t)|^2 &= \frac{1 - 2\theta(\omega, \lambda_j(t)) \cos t + \theta^2(\omega, \lambda_j(t))}{\varphi^2(\omega, \lambda_j(t))} |\psi_j(0, t)|^2 \\ &= \frac{1 - \theta(\omega, \lambda_j(t)) \varphi^{[1]}(\omega, \lambda_j(t))}{\varphi^2(\omega, \lambda_j(t))} |\psi_j(0, t)|^2 = -\frac{\theta^{[1]}(\omega, \lambda_j(t))}{\varphi(\omega, \lambda_j(t))} |\psi_j(0, t)|^2. \end{aligned} \quad (29)$$

In the latter equality we used the identity (18). Substitution of (28), (29) in (24), (25) yields

$$\begin{aligned} |\alpha_j(t)|^2 &= \{\varphi(\omega, \lambda_j(t)) g^2(\lambda_j(t)) - \theta^{[1]}(\omega, \lambda_j(t)) h^2(\lambda_j(t)) \\ &+ [\varphi^{[1]}(\omega, \lambda_j(t)) - \theta(\omega, \lambda_j(t))] g(\lambda_j(t)) h(\lambda_j(t))\} \frac{|\psi_j(0, t)|^2}{\varphi(\omega, \lambda_j(t))}, \end{aligned} \quad (30)$$

$$\begin{aligned} \alpha_j(t) \psi_j(x, t) &= \{\varphi(\omega, \lambda_j(t)) g(\lambda_j(t)) \theta(x, \lambda_j(t)) - \theta^{[1]}(\omega, \lambda_j(t)) h(\lambda_j(t)) \varphi(x, \lambda_j(t)) \\ &+ [e^{-it} - \theta(\omega, \lambda_j(t))] h(\lambda_j(t)) \theta(x, \lambda_j(t)) + [e^{it} - \theta(\omega, \lambda_j(t))] g(\lambda_j(t)) \varphi(x, \lambda_j(t))\} \frac{|\psi_j(0, t)|^2}{\varphi(\omega, \lambda_j(t))}. \end{aligned} \quad (31)$$

Let us now show that

$$\frac{|\psi_j(0, t)|^2}{\varphi(\omega, \lambda_j(t))} = \eta(\lambda_j(t)) \frac{d\lambda_j(t)}{dt} \quad (32)$$

for  $t \in (0, \pi)$ , where the function  $\eta(\lambda)$  is defined by (15). Indeed, using the formulas (23), (28), (29), and (17), we have

$$\begin{aligned}
 1 &= \int_0^\omega \rho(x) |\psi_j(x, t)|^2 dx = \int_0^\omega \rho(x) \{ |\psi_j(0, t)|^2 \theta^2(x, \lambda_j(t)) + |\psi_j^{[1]}(0, t)|^2 \varphi^2(x, \lambda_j(t)) \\
 &\quad + [\psi_j(0, t) \overline{\psi_j^{[1]}(0, t)} + \overline{\psi_j(0, t)} \psi_j^{[1]}(0, t)] \theta(x, \lambda_j(t)) \varphi(x, \lambda_j(t)) \} dx \\
 &= |\psi_j(0, t)|^2 \cdot \int_0^\omega \rho(x) \{ \theta^2(x, \lambda_j(t)) - \frac{\theta^{[1]}(\omega, \lambda_j(t))}{\varphi(\omega, \lambda_j(t))} \varphi^2(x, \lambda_j(t)) \\
 &\quad + \frac{\varphi^{[1]}(\omega, \lambda_j(t)) - \theta(\omega, \lambda_j(t))}{\varphi(\omega, \lambda_j(t))} \theta(x, \lambda_j(t)) \varphi(x, \lambda_j(t)) \} dx = - \frac{|\psi_j(0, t)|^2}{\varphi(\omega, \lambda_j(t))} \cdot \frac{dF(\lambda_j(t))}{d\lambda}.
 \end{aligned}$$

So, we have established the identity

$$- \frac{|\psi_j(0, t)|^2}{\varphi(\omega, \lambda_j(t))} \cdot \frac{dF(\lambda_j(t))}{d\lambda} = 1. \tag{33}$$

On the other hand, since by (27) and (15) we have for  $t \in (0, \pi)$ ,

$$2 \sin t = \{4 - (2 \cos t)^2\}^{1/2} = \{4 - [\theta(\omega, \lambda_j(t)) + \varphi^{[1]}(\omega, \lambda_j(t))]^2\}^{1/2} = \frac{1}{\eta(\lambda_j(t))},$$

we get from (19) that

$$\frac{dF(\lambda_j(t))}{d\lambda} \cdot \frac{d\lambda_j(t)}{dt} = - \frac{1}{\eta(\lambda_j(t))}. \tag{34}$$

Now, comparing (33) with (34) we obtain (32).

Substituting (32) in (30), (31) and taking into account that by (27)

$$\begin{aligned}
 Re[e^{-it} - \theta(\omega, \lambda_j(t))] &= Re[e^{it} - \theta(\omega, \lambda_j(t))] = \cos t - \theta(\omega, \lambda_j(t)) \\
 &= \frac{1}{2} [\theta(\omega, \lambda_j(t)) + \varphi^{[1]}(\omega, \lambda_j(t))] - \theta(\omega, \lambda_j(t)) = \frac{1}{2} [\varphi^{[1]}(\omega, \lambda_j(t)) - \theta(\omega, \lambda_j(t))],
 \end{aligned}$$

we get

$$\begin{aligned}
 |\alpha_j(t)|^2 &= \{ \varphi(\omega, \lambda_j(t)) g^2(\lambda_j(t)) - \theta^{[1]}(\omega, \lambda_j(t)) h^2(\lambda_j(t)) \\
 &\quad + [\varphi^{[1]}(\omega, \lambda_j(t)) - \theta(\omega, \lambda_j(t))] g(\lambda_j(t)) h(\lambda_j(t)) \} \eta(\lambda_j(t)) \frac{d\lambda_j(t)}{dt}, \tag{35}
 \end{aligned}$$

$$\begin{aligned} \operatorname{Re}[\alpha_j(t)\psi_j(x, t)] &= \{\varphi(\omega, \lambda_j(t))g(\lambda_j(t))\theta(x, \lambda_j(t)) - \theta^{[1]}(\omega, \lambda_j(t))h(\lambda_j(t))\varphi(x, \lambda_j(t)) \\ &+ \frac{1}{2}[\varphi^{[1]}(\omega, \lambda_j(t)) - \theta(\omega, \lambda_j(t))][h(\lambda_j(t))\theta(x, \lambda_j(t)) + g(\lambda_j(t))\varphi(x, \lambda_j(t))]\} \eta(\lambda_j(t)) \frac{d\lambda_j(t)}{dt}. \end{aligned} \tag{36}$$

Further, note that comparing the two cases of  $t$ -periodic BVP (1), (5) in which  $t = \tau$  and  $t = -\tau$  ( $0 < \tau < \pi$ ), we see that the eigenvalues are the same in the two cases but the eigenfunctions in one case are the complex conjugates of those in the other. Consequently, we may assume that  $\overline{\psi_j}(x, t) = \psi_j(x, -t)$ . Because of this, in the case of real function  $f(x)$  the formulas (10) and (11) can be written as

$$\int_{-\infty}^{\infty} \rho(x) f(x)^2 dx = \frac{1}{\pi} \sum_{j=1}^{\infty} \int_0^{\pi} |\alpha_j(t)|^2 dt, \tag{37}$$

$$f(x) = \frac{1}{\pi} \sum_{j=1}^{\infty} \int_0^{\pi} \operatorname{Re}[\alpha_j(t)\psi_j(x, t)] dt. \tag{38}$$

Now we represent the right-hand side of (37) as

$$\frac{1}{\pi} \sum_{j=1}^{\infty} \int_0^{\pi} |\alpha_{2j-1}(t)|^2 dt + \frac{1}{\pi} \sum_{j=1}^{\infty} \int_0^{\pi} |\alpha_{2j}(t)|^2 dt$$

and then replace the function  $|\alpha_j(t)|^2$  by its expression (35). Next make the change of variable  $\lambda_j(t) = \lambda$  for each  $j$ . Using Lemma 5 we get (12). Similarly from (38) and (36) we get (13).

### References

- [1] Coddington, E. A. and Levinson, N., Theory of Ordinary Differential Equations, *McGraw-Hill, New York*, 1955.
- [2] Eastham, M. S. P., The Spectral Theory of Periodic Differential Equations, *Scottish Academic Press, Edinburgh and London*, 1973.
- [3] Gel'fand, I. M., Expansion in eigenfunctions of an equation with periodic coefficients, *Dokl. Akad. Nauk SSSR*, **73** (1950), 1117-1120. (Russian)

- [4] Guseinov, G. Sh., Eigenfunction expansions of multiparameter differential and difference equations with periodic coefficients, *Dokl. Akad. Nauk SSSR* , **253** (1980), 1040-1044 = *Soviet Math. Dokl.* **22** (1980), 201-205.
- [5] Guseinov, G. Sh., Spectrum and eigenfunction expansions of a quadratic pencil of Sturm-Liouville operators with periodic coefficients, *Spectral Theory of Operators and Its Applications "Elm", Baku, Azerbaijan* **6** (1985), 56-97. (Russian)
- [6] Guseinov, G. Sh., On spectral analysis of a quadratic pencil of Sturm-Liouville operators, *Dokl. Akad. Nauk SSSR* **285** (1985), 1292-1296 = *Soviet Math. Dokl.* **32** (1985), 859-862.
- [7] Guseinov, G. Sh., On the spectral theory of multiparameter difference equations of second order, *Izv. Akad. Nauk. SSSR Ser. Math.* **51**(1987), 785-811 = *Math. USSR Izvestiya* **31** (1988), 95-120.
- [8] Guseinov, G. Sh., Spectral analysis of multiparameter second-order differential equations with periodic coefficients, *Algebra and Analysis (Sanct-Petersburg)* **5**, No. 4 (1993), 98-121 = *St. Petersburg Math. J.* **5** (1994), 745-764.
- [9] Guseinov, G. Sh. and Kocak, Z. F., On the spectral theory of first-order difference equations with periodic coefficients fractional-linear in the spectral parameter, *Bull. Inst. Math. Acad. Sinica* **24** (1996), 299-311.
- [10] Guseinov, G. Sh. and Yaslan, I., On Hill's equation with piecewise constant coefficient, *Turkish J. Math.* **21** (1997), 461-474.
- [11] Magnus, W. and Winkler, S., Hill's Equation, *Interscience, New York*, 1966.
- [12] Neumark, M. A., Lineare Differentialoperatoren, *Akademie-Verlag, Berlin*, 1967.
- [13] Odeh, F. and Keller, J. B., Partial differential equations with periodic coefficients and Bloch waves in crystals, *J. Math. Phys.* **5** (1964), 1499-1504.
- [14] Titchmarsh, E. C., Eigenfunction Expansions Associated with Second-Order Ordinary Differential Equations, Part II, *Clarendon Press, Oxford*, 1958.

F. ARAS and G. Sh. GUSEINOV  
 Ege University, Science Faculty  
 Department of Mathematics  
 35100 Bornova, İzmir-TURKEY

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