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Surface bundles: some interesting examples

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1. Introduction

It is a well-known fact that Euler characteristics multiply in a fibration. According to classical examples of Atiyah [1] and, independently, Kodaira [6], the signature does not admit this property — there are 4-manifolds with nonzero signature admitting surface bundle structure. After sporadic examples [1, 5, 6], a more systematic study of this phenomenon was initiated by Endo [2]. More recently it has been proved that

Theorem 1.1 ([3]). *For any $h \geq 3$ and $g \geq 9$ there is a genus-h surface bundle $X \to \Sigma_g$ with nonzero signature.* □

Remark 1.2. *It is fairly easy to see that for $g = 0, 1$ a fibration $X \to \Sigma_g$ has vanishing signature.* [1] *It can be proved that the same holds once $h \leq 2$. Therefore Theorem 1.1 almost solves the "geography problem" of surface bundles with nonzero signature, i.e., the determination of pairs $(g, h) \in \mathbb{N} \times \mathbb{N}$ for which a genus-h fibration over the surface of genus g with nonzero signature exists. Notice that a surface bundle with $g = 2$, $h = 3$ and nonzero signature (such a manifold admits a symplectic structure according to* [13]*) would violate the Bogomolov-Miyaoka-Yau inequality $c_1^2(X) \leq 3c_2(X)$. (This inequality is known to be true for complex surfaces and is conjectured to hold for symplectic 4-manifolds.)*

In this note we give two constructions for surface bundles with nonzero signature. In Section 2 a very elementary topological construction of a genus-5 fibration of nonzero signature is given. In Section 3 — by improving techniques already present in [5] — we get surface bundles over the genus-3 surface. Using these examples we can improve bounds on the asymptotic behaviour of the genus function $g_h(k)$. (For the definition of $g_h(k)$ see Section 3 or [3].) We also show a construction which produces rational curves in moduli spaces of complex curves of certain genus. In some of our constructions we will use the correspondence between Lefschetz fibrations and relators in mapping class groups composed by right-handed Dehn twists. (This correspondence is given by the monodromies of the singular fibers; for a more careful treatment of these issues see [3, 4].)

[1] The proof of this fact follows from the equality $\chi(X) = 4(g-1)(h-1)$ and the inequality $b_1(X) \leq 2g + 2h$ for a genus-h fibration over Σ_g. The above inequalities show $|\sigma(X)| \leq b_2(X) \leq 4gh + 2$, hence for $g = 0$ the iterated fiber sum $X \#_f X \#_f X$ has vanishing signature (since $\sigma(X \#_f X \#_f X) = 3\sigma(X)$); this implies $\sigma(X) = 0$. For $g = 1$ the pull-back of $X \to \Sigma_1$ via a $(4h+3)$-fold unramified cover $\pi\colon \Sigma_1 \to \Sigma_1$ shows that $|\sigma(\pi^*(X))| = (4h+3)|\sigma(X)| = 0$.

2. The topological construction

Consider the 4-manifold $\Sigma_2 \times S^2$ (here, as always, Σ_g denotes the Riemann surface of genus g), and fix the singular curve $C = \cup_{i=1}^4 \{q_i\} \times S^2 \bigcup \cup_{j=1}^2 \Sigma_2 \times \{p_j\} \subset \Sigma_2 \times S^2$ (q_i and p_j are distinct points in Σ_2 and S^2 resp., $i = 1, \dots, 4$ and $j = 1, 2$).

Lemma 2.1. *The desingularization Y of the double branched cover of $\Sigma_2 \times S^2$ branched along C is diffeomorphic to $\Sigma_2 \times S^2 \# 8\overline{\mathbb{CP}^2}$.*

Proof. Compose the map $\varphi \colon Y \to \Sigma_2 \times S^2$ with the projection to Σ_2. The resulting map $\eta \colon Y \to \Sigma_2$ equips Y with a fibration over Σ_2; the generic fiber is the double branched cover of S^2 branched in two points — hence it is the sphere S^2 again. Therefore η defines a singular ruling on Y. By taking a closer look at the desingularization process, we will show that Y is diffeomorphic to $\Sigma_2 \times S^2 \# n\overline{\mathbb{CP}^2}$ for some n and also determine the value of n. (It is known that a singular ruling is the blow-up of a ruling [4], but we will not use this fact in our discussion.)

One way of resolving the singularities of a double branched cover of $\Sigma_2 \times S^2$ along C is the following: blow up $\Sigma_2 \times S^2$ in the singular points of C and take the branched cover along the (smooth) proper transform in $\Sigma_2 \times S^2 \# 8\overline{\mathbb{CP}^2}$. The blow-ups turn the trivial S^2-bundle $\Sigma_2 \times S^2 \to S^2$ into an S^2-fibration with four singular fibers — each singular fiber is the plumbing of two disjoint rational (-1)-curves and a rational (-2)-curve which intersects the two exceptional curves transversally once. The (-1)-curves give rise to rational (-2)-curves in the branched cover (since the exceptional curves are not in the branch locus), while the (-2)-curves become rational (-1)-curves upstairs, since the (-2)-curves are in the branch locus. (For more details about the above process see Chapter 7 of [4].) Blowing the four (-1)-curves in Y down we still have a singular ruling on the resulting 4-manifold — each of the four singular fibers consist of two transversally intersecting (-1)-curves now. Blowing down four of these we end up with an honest S^2-fibration over Σ_2, hence Y is diffeomorphic to $\Sigma_2 \times S^2 \# 8\overline{\mathbb{CP}^2}$. □

The composition of $\varphi \colon Y \to \Sigma_2 \times S^2$ with $\mathrm{pr}_2 \colon \Sigma_2 \times S^2 \to S^2$ provides a holomorphic map $h \colon Y \to \mathbb{CP}^1$; after perturbing h slightly we get a genus-5 Lefschetz fibration $g_Y \colon Y \to S^2$ on Y. (For more about Lefschetz fibrations see [3, 4, 8], for example.)

Lemma 2.2. *The Lefschetz fibration $g_Y \colon Y \to S^2$ admits 20 singular fibers.*

Proof. From the handlebody decomposition of a genus-g Lefschetz fibration $X \to S^2$ it is clear that $\chi(X) = 2(2 - 2g) + s$ where s denotes the number of singular fibers in the fibration. In our case $Y = \Sigma_2 \times S^2 \# 8\overline{\mathbb{CP}^2}$, hence computing $\chi(Y)$ in two different ways yields $\chi(Y) = 2(2-4)+8 = 2(2-10)+s$. This implies $s = 20$, concluding the proof. □

Notice that $h \colon Y \to S^2$ has two singular fibers (over the points p_1 and $p_2 \in S^2$ in the above notation), and the two singular fibers admit coinciding monodromies. By perturbing h we can achieve that the critical points of the projection $Y \to S^2$ become Morse critical points and the new map is injective on the set of its critical points. It is not hard to see that the monodromy of each (new) singular fiber is a right-handed Dehn twist along some simple closed curve (see, e.g., [4]). Since we can assume that the perturbations of h agree near the two singular fibers, the word in the mapping class group corresponding to the perturbed fibration $g_Y\colon Y \to S^2$ is of the form $\Pi_{i=1}^m t_i \cdot \Pi_{i=1}^m t_i = 1$. Lemma 2.2 now implies that $m = 10$, hence the relator given by the monodromies of $g_Y\colon Y \to S^2$ is $(\Pi_{i=1}^{10} t_i)^2 = 1$ (where all t_i are right-handed Dehn twists along the appropriate vanishing cycles of the singular fibers). It is a well-known fact that the mapping class group Γ_g is perfect (i.e., $(\Gamma_g)' = \Gamma_g$) once $g \geq 3$, consequently $(\Pi_{i=1}^{10} t_i)^{-1}$ can be written as the product of commutators: $(\Pi_{i=1}^{10} t_i)^{-1} = \Pi_{j=1}^k [a_j, b_j] \in \Gamma_5$. The relator $\Pi_{i=1}^{10} t_i \Pi_{j=1}^k [a_j, b_j] = 1$ now defines a genus-5 Lefschetz fibration $g_Z\colon Z \to \Sigma_k$ over Σ_k with 10 singular fibers (corresponding to the Dehn twists t_i in the defining relator). Fix disks $D_1, D_2 \subset S^2$ and $D_Z \subset \Sigma_k$ containing 10 critical values of g_Y and g_Z respectively (and $D_i \subset S^2$ contains the 10 critical values corresponding to the 10 singular fibers providing monodromy equal to $\Pi_{i=1}^{10} t_i$). It is now easy to see that the fibrations $g_Y^{-1}(D_i) \to D_i$ $(i = 1, 2)$ and $g_Z^{-1}(D_Z) \to D_Z$ are fiber- and orientation-preserving diffeomorphic. By changing the orientation on Z this diffeomorphism becomes orientation-reversing, hence can be used to glue $Y - (g_Y^{-1}(D_1) \cup g_Y^{-1}(D_2))$ and two copies of $Z - g_Z^{-1}(D_Z)$ together along their boundaries. In this way we get a genus-5 surface bundle $f\colon X \to \Sigma_{2k}$ with signature $\sigma(X) = \sigma(Y) - 2\sigma(Z)$. (The negative sign in the formula results from reversing the orientation on Z.) Now we are ready to prove

Theorem 2.3. *For all sufficiently large g there exists a 4-manifold X which carries a surface bundle structure $f\colon X \to \Sigma_g$ with fiber genus 5 and has positive signature.*

Proof. Consider the genus-5 fibration $X \to \Sigma_{2k}$ constructed above. All we need to show is that $\sigma(X) \neq 0$. (If $\sigma(X) < 0$ then reverse the orientation on X; this leaves the surface bundle structure unchanged but changes the sign of the signature.) We will show that $\sigma(X)$ is not divisible by 8 — in particular, it is nonzero. Notice that (based on our previous description) we know that $\sigma(Y) = -8$, hence we only need to examine $\sigma(Z)$. According to a result of Gompf (see [4], for example), a Lefschetz fibration with fiber genus at least 2 admits a symplectic, hence an almost complex structure. Since the Noether formula applies to almost complex manifolds, we get that $\sigma(Z) + \chi(Z)$ is divisible by 4. Now $\chi(Z)$ can be computed as $(2-2\cdot 5)(2-2\cdot k)+10 \equiv 2 \pmod 4$, hence $-2\sigma(Z) \equiv 4 \pmod 8$. This observation shows that $\sigma(X) \equiv 4 \pmod 8$, hence it is not divisible by 8, consequently it is nonzero. Now for any $g \geq 2k$ the fiber sum of the above X with the trivial genus-5 fibration $\Sigma_5 \times \Sigma_{g-2k} \to \Sigma_{g-2k}$ gives the desired fibration since $\sigma(X) = \sigma(X \#_f (\Sigma_5 \times \Sigma_{g-2k}))$. □

Remark 2.4. *According to* [7]*, k can be chosen to be at most 6, hence we can arrange that X (with the properties given by Theorem 2.3) fibers over a Riemann surface of genus at most 12.*

Remark 2.5. *Similar argument works for any fiber genus of the form $4n+1$. For other fiber genera either the modification of the argument or the (tedious) determination of the* *signature $\sigma(Z)$ is needed in order to reach a conclusion similar to the one we proved in Theorem 2.3.*

3. Branched cover constructions

Now we describe our construction of surface bundles with small base genus (allowing the fiber genus to become comparatively large). The main result of this section is summarized by the following theorem:

Theorem 3.1. *There exist smooth algebraic surfaces X_n that have signature $\sigma(X_n) = \frac{8}{3}n(n-1)(n+1)$ and admit two smooth fibrations $X_n \to B$ and $X_n \to \widetilde{B}$ so that the pair (g, h) given by the base genus and fiber genus of the fibrations are $(3, 3n^3 - n^2 + 1)$ and $(2n^2 + 1, 3n)$ respectively.*

In proving Theorem 3.1 we will describe an improved version of Hirzebruch's original branched cover construction given in [5].

Proof. Hirzebruch showed ([5], page 262) that if $D_1, D_2 \subset S$ are smooth, disjoint curves in an algebraic surface S, and the homology class of the divisor $D = D_1 - D_2$ is divisible by n, then there exists an algebraic surface X which is a $\mathbb{Z}/n$ cyclic cover of S, totally ramified over D, and the signature of X is given by

$$\sigma(X) = n\sigma(S) - \frac{n^2-1}{3n}D^2.$$

Let B be a curve of genus 3 with a free involution τ. (Consider, for example an unramified double cover $B \to C$ of a genus-2 curve C and choose τ to be the nontrivial deck transformation.) Let $\Delta \subset B \times B$ denote the diagonal and let Δ' denote its image under $Id \times \tau$. Δ and Δ' are smooth, disjoint curves, but $\Delta - \Delta'$ is not a divisible class so we cannot directly apply Hirzebruch's construction. The solution is to find a certain unramified cover $\pi\colon \widetilde{B} \to B$ and pull Δ and Δ' back to $\widetilde{B} \times B$. We define π as follows.

Note that the image of $1 - \tau_*\colon H_1(B; \mathbb{Z}/n) \to H_1(B; \mathbb{Z}/n)$ has rank two, hence we have

$$0 \to \mathrm{Ker}(1 - \tau_*) \to H_1(B; \mathbb{Z}/n) \to (\mathbb{Z}/n)^2 \to 0.$$

Let $\pi\colon \widetilde{B} \to B$ be the $(\mathbb{Z}/n)^2$ cover corresponding to the surjection $\pi_1(B) \to H_1(B; \mathbb{Z}/n) \to (\mathbb{Z}/n)^2$. Note that the genus of $\widetilde{B}$ is $2n^2 + 1$.

Let $\Gamma_\pi \subset \widetilde{B} \times B$ be the graph of π and let Γ'_π be the image of Γ_π under $Id \times \tau$. Γ_π and Γ'_π are smooth, disjoint curves; let $D = \Gamma_\pi - \Gamma'_\pi$.

Lemma 3.2. *The homology class of D is divisible by n.*

Proof. We need to show that $[D]$ is 0 in $H_2(\widetilde{B} \times B; \mathbb{Z}/n)$. For any unramified map $f\colon Z \to Y$, the class of the graph $\Gamma_f \subset Z \times Y$ is given by $\sum_i f^*(\alpha_i) \times \alpha^i$ where $\{\alpha_i\}$ is a basis for $H_*(Y)$ and α^i is the dual basis so that $\alpha_i \cdot \alpha^j = \delta_{ij}$. (Here $\cdot$ is the intersection product, i.e., the Poincaré dual of the cup product.) Since f is a free cover, if α is a geometric cycle then $f^*(\alpha)$ is just represented by the inverse image $f^{-1}(\alpha)$.

For the sake of concreteness, we pick a geometric basis $\{a, b, c, d, e, f\}$ for $H_1(B; \mathbb{Z}/n)$ such that the intersection form and τ_* are given respectively by

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is simply what one gets if one draws B with its 3 holes in a row, assigns (a, b), (c, d), and (e, f) to the three standard pairs of generators, and lets τ be the involution obtained by $180°$ rotation about the axis passing through the middle hole.

We then have the basis $\{pt, a, b, c, d, e, f, B\}$ for $H_*(B; \mathbb{Z})$, and its dual basis is $\{B, b, -a, d, -c, f, -e, pt\}$. Thus

$$\begin{aligned} \Gamma_\pi = \quad & \pi^*(pt) \times B + \pi^*(B) \times pt \\ & + \pi^*(a) \times b - \pi^*(b) \times a + \pi^*(c) \times d - \pi^*(d) \times c \\ & + \pi^*(e) \times f - \pi^*(f) \times e \end{aligned}$$

and

$$\begin{aligned} \Gamma'_\pi = \quad & \pi^*(pt) \times B + \pi^*(B) \times pt \\ & + \pi^*(a) \times f - \pi^*(b) \times e + \pi^*(c) \times d - \pi^*(d) \times c \\ & + \pi^*(e) \times b - \pi^*(f) \times a \end{aligned}$$

so

$$\begin{aligned} D = \quad & \pi^*(a) \times (b - f) + \pi^*(e) \times (f - b) \\ & - \pi^*(b) \times (a - e) - \pi^*(f) \times (e - a) \\ = \quad & \pi^*(a - e) \times (b - f) - \pi^*(b - f) \times (a - e). \end{aligned}$$

One can use the definition of π to directly check that the elements $\pi^*(b-f)$ and $\pi^*(a-e)$ are 0 modulo n. This proves the lemma. □

Let X_n be the $\mathbb{Z}/n$ cyclic branched cover of $\widetilde{B} \times B$ ramified over D. Let $X_n \to \widetilde{B}$ and $X_n \to B$ be the induced projections. Since $D|_{pt \times B}$ is the divisor $\pi(pt) - \tau(\pi(pt))$, the fibers of $X_n \to \widetilde{B}$ are smooth $\mathbb{Z}/n$ covers of B totally ramified over 2 points, hence have genus $3n$. Similarly, since $D|_{\widetilde{B} \times pt}$ is the divisor $\pi^{-1}(pt) - \tau(\pi^{-1}(pt))$, the fibers of $X_n \to B$ are $\mathbb{Z}/n$ covers of $\widetilde{B}$ ramified over $2n^2$ points, hence have genus $3n^3 - n^2 + 1$.

The signature of X_n is determined by Hirzebruch's formula which, since $\sigma(\widetilde{B} \times B) = 0$, becomes

$$\sigma(X_n) = -\frac{n^2 - 1}{3n} D^2.$$

Now D^2 is easily computed as follows:

$$\begin{aligned} D^2 = &\quad 2\pi^*(a-e)\cdot\pi^*(b-f)\times(b-f)\cdot(a-e) \\ = &\quad 2\pi^*(a\cdot b+e\cdot f)\times(b\cdot a+f\cdot e) \\ = &-8\pi^*(pt)\times pt \\ = &-8n^2. \end{aligned}$$

Thus we have $\sigma(X_n)=\frac{8}{3}(n^3-n)$ which completes the proof of our main theorem. □

Remark 3.3. *One way we improved Hirzebruch's construction in the above was by considering an n^2-fold cover as opposed to the n^6-fold cover taken in* [5]*. (We replaced the kernel of the natural map $\pi_1(B)\to H_1(B;\mathbb{Z}/n)$ by the kernel of the composition $\pi_1(B)\to H_1(B;\mathbb{Z}/n)\to(\mathbb{Z}/n)^2$ when we defined the unramified cover of B.) This makes us able to find fibrations over the same base genus as in* [5] *with smaller fiber genera. The other basic improvement was to observe that both the projections to B and $\widetilde{B}$ induce smooth fibrations, in particular, Hirzebruch's original construction gave a fibration with base genus 3. (This latter fact was also observed by LeBrun* [9]*.)*

4. Concluding remarks

4.1. Asymptotics of the genus function

The question of determining the smallest possible base genus of a surface bundle over a surface with nonzero signature can be refined as follows:

Definition 4.1. *Let $g_h(k)$ be the smallest possible base genus for a smooth genus-h fibration with signature $4k$, i.e.*

$$g_h(k)=\min\{g\mid\exists\ \text{ a } \Sigma_h\text{-bundle } X\to\Sigma_g \text{ with } \sigma(X)=4k\}.$$

It is not hard to see that the sequence $\frac{g_h(k)}{k}$ converges; following [3] *we define G_h as $\lim_{k\to\infty}\frac{g_h(k)}{k}$.*

As a corollary of Theorem 3.1 we have:

Corollary 4.2. *For $h=3n$ the asymptotic value G_h satisfies $G_{3n}\le\frac{3n}{n^2-1}$.*

Proof. The proof follows the standard argument by taking an m-fold unbranched cover $\widetilde{B}_m$ of the base $\widetilde{B}$ and pulling back the family $X_n\to\widetilde{B}$ found in Theorem 3.1. In this way we obtain a genus-$3n$ fibration over a curve of genus $1+2n^2m$, whose total space has signature $\frac{8m}{3}(n^3-n)$. Therefore $g_{3n}(\frac{2m}{3}(n^3-n))\le 1+2n^2m$ and so

$$G_{3n}\le\lim_{m\to\infty}\frac{1+2n^2m}{2m(n^3-n)/3}=\frac{3n}{n^2-1}.$$

This concludes the proof of the corollary. □

Remark 4.3. *Notice that in Definition 4.1 we used the fact that the signature of a surface bundle is divisible by 4. This follows from the Noether formula (because a surface bundle of fiber genus* ≥ 2 *admits an almost complex structure) in the same way as this formula had been used in the proof of Theorem 2.3.*

Remark 4.4. *By setting* $h = 3n$ *we get that* $G_h \leq \frac{9}{h-\frac{9}{h}} \leq \frac{9}{h-2}$ *once* h *is divisible by 3. This bound improves the bound found in* [3] *nearly by 50%.*

4.2. Surfaces in the moduli of curves

By fixing a Riemann metric on a surface bundle $f\colon X \to \Sigma_g$ of fiber genus h the fibers become complex curves, therefore the map f induces a map $\varphi_f\colon \Sigma_g \to \mathcal{M}_h$, where $\mathcal{M}_h$ is the moduli space of genus-h complex curves. It is known that $H^2(\mathcal{M}_h;\mathbb{Z}) \cong \mathbb{Z}$ once $h \geq 5$ and from Meyer's work [11] for a generator a of this cohomology group we have $4\langle a, (\varphi_f)_*[\Sigma_g]\rangle = \pm\sigma(X)$. Therefore a surface bundle with nonzero signature gives a (real) 2-dimensional surface in $\mathcal{M}_h$ representing a nontrivial homology class. Hence the problem of determining the smallest possible base genus of a surface bundle with nonzero signature and fiber genus h is in close connection with the minimal genus problem of 2-dimensional homology classes of $\mathcal{M}_h$. Notice, though, that because of the failure of the existence of the universal curve over $\mathcal{M}_h$, a map $\varphi\colon \Sigma_g \to \mathcal{M}_h$ does not necessarily come from a surface bundle through the above construction.

We conclude the paper by a construction of a holomorphic map $\varphi\colon \mathbb{CP}^1 \to \mathcal{M}_{321}$. Since $\varphi(\mathbb{CP}^1)$ is a complex submanifold, it is nontrivial in homology (because the Kähler form of $\mathcal{M}_{321}$ evaluates nontrivially on it).

Remark 4.5. *It is known* [10] *that* $\pi_1(\mathcal{M}_h) = 0$, *hence* $H_2(\mathcal{M}_h;\mathbb{Z}) \cong \pi_2(\mathcal{M}_h)$, *implying that every second homology class can be represented by a sphere. The novelty of our construction is that it provides an example of a class represented by a holomorphic sphere, i.e. a complete rational curve.*

4.2.1. The construction of the map

Consider an unramified double cover $\tau\colon C' \to C$ of the holomorphic genus-2 curve C. For $c \in C$ we construct a genus-321 holomorphic curve (and therefore an element in $\mathcal{M}_{321}$) in the following way: take the unramified cover of $C' - \tau^{-1}(c)$ corresponding to the subgroup of $\pi_1(C' - \tau^{-1}(c))$ specified by the kernel of the natural surjection $\pi_1(C' - \tau^{-1}(c)) \to H_1(C' - \tau^{-1}(c);\mathbb{Z}/2)$. Let B_c be the compactification of this curve; an easy argument shows that B_c is of genus 321. The complex structure on B_c only depends on the complex structure of C and the point c, therefore B_c is isomorphic to $B_{t(c)}$ where $t\colon C \to C$ is the hyperelliptic involution of C. Since the quotient of C by the action of t is $\mathbb{CP}^1$, we get the desired map $\varphi\colon \mathbb{CP}^1 \to \mathcal{M}_{321}$ by $[c, t(c)] \mapsto [B_c]$.

One can also argue that the classifying map $B \to \mathcal{M}_{3n^3-n^2+1}$ induced by X_n (see Section 3) factors through $B \to C \to \mathbb{CP}^1$ which gives in particular a rational curve in $\mathcal{M}_{21}$. In all of the above cases, it is not hard to show that the family over the genus-3 curve B will not descend to a family over the genus-2 curve C (or $\mathbb{CP}^1$, obviously). Thus, to find a fibration with a base of genus 2 requires a different construction.

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