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Floer homology and its continuity for non-compact Lagrangian submanifolds

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1. Introduction

Floer [F1] invented the Floer homology $HF(L_0, L_1)$ of the pair (L_0, L_1) of Lagrangian submanifolds on symplectic manifolds (P, ω) with suitable topological restrictions on the pair. He defined this by considering the (generalized) Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\frac{\partial u}{\partial t} = 0 \\ u(\tau, 0) \in L_0, \;\; u(\tau, 1) \in L_1 \end{cases} \tag{1}$$

for a map $u : \mathbb{R} \times [0, 1] \to P$.

One crucial property of $HF(L_0, L_1)$ for applications to the problems in symplectic topology, is the invariance property under the Hamiltonian deformations of the pair. Floer's original proof [F1] considers the case where $L_1 = \phi^1_H(L_0)$ and $\pi_2(P, L_0) = \{e\}$ where $\phi^1_H : P \to P$ is the time-one map of the Hamiltonian flow of the function $H : P \times [0, 1] \to \mathbb{R}$, and involves some combinatorial study of the changes occurring to the boundary operators when a (generic) degenerate intersection occurs between the pairs during the deformations. Using the fact that generic types of such degenerate intersections are either birth-death or death-birth type, he algebraically analyzed the change. However this study involves a gluing theory of trajectories on degenerate intersections. Although such a gluing theory is believed to be possible by now, details were only sketched in [F1].

Because Floer's analysis in [F1] also uses the fact that the action functional is single-valued in his case (where $\pi_2(P, L_0) = \{e\}$ is assumed), it was not clear to the author at the time of writing [O1] whether this approach can be generalized to more general cases where the action functional is not single-valued. More importantly, Floer's original proof does not give naturality of the chain map. Motivated by Floer's approach taken in [F3] for Hamiltonian diffeomorphisms, the present author [O1] used a variant of (1),

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\frac{\partial u}{\partial t} = 0 \\ u(\tau, 0) \in L, \; u(\tau, 1) \in L_{\rho(\tau)} \end{cases} \tag{2}$$

for the construction of the chain homomorphim from $HF(L, L_0)$ to $HF(L, L_1)$, where $\rho : \mathbb{R} \to [0, 1]$ is a *monotonically increasing* function with $\rho(-\infty) = 0$ and $\rho(+\infty) = 1$. Similar constructions have been also used in our more recent papers [O2,KO1,2] in relation

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to a quantization program of the classical homology theory. In these works, naturality of the chain map is essential for the analysis of change of actions and for the continuity proof of symplectic invariants constructed therein (see [O2] for details). Another way of defining the chain homomorphism is to transform (1) into the dynamical version

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\Big(\frac{\partial u}{\partial t} - X_H(u)\Big) = 0 \\ u(\tau, 0) \in L, \quad u(\tau, 1) \in L_0. \end{cases} \tag{3}$$

The chain map in this set-up can be defined by making the Hamiltonian H depend on τ-variable as in [F3].

However for the cases in [O2,KO1,2] where we consider a family of *conormal varieties* which are non-compact, or more precisely, where the corresponding Hamiltonian isotopy is no longer compactly supported, the crucial C^0-estimates for the equation turned out not to be available for general choices of ρ in neither case of (2) nor (3). The proof of the C^0-estimates works for the perturbed Cauchy-Riemann equation with some particular type of perturbations which are *either compactly supported* as in the most literature on the Floer theory *or directed in certain particular directions* as in [KO2]. In this sense, the present author's paper [O2] contains a gap in that he overlooked the failure of the C^0-estimate which is needed for the proof of continuity of Floer homology $HF(H, S : M)$ under the isotopy of submanifolds $S \subset M$. The proof of this C^0-estimate in [Section 3, O2] works for fixed or compactly supported perturbations of conormal bundles ν^*S, and turns out to work only with a particular choice of the function ρ which should be determined depending on the solution u, for more general types of perturbation. For example, ρ *cannot* be chosen to be constant outside a compact subset unlike in the Floer theory for compact Lagrangian submanifolds.

One purpose of the present paper is to rectify this gap (see Remark 4.3 (1)) by considering a *suspension* of (2). The relevant geometric suspension of Lagrangian submanifolds is a quite natural operation in symplectic geometry which has been used in the literature of symplectic topology (see e.g., [A1, Po]). After we used this suspension to construct the chain map, it became quite apparent to us that the idea of our construction of the chain map applies to more general circumstances, i.e., to certain Lagrangian cobordisms in (P, ω). However, constructing the natural chain map

$$h_{\mathcal{L}} : HF_*(L, L_0) \to HF(L, L_1)$$

and extending invariance property of the Floer homology to the case when L_0 and L_1 are *noncompact* and the Hamiltonian isotopy $\mathcal{L} = \{L_t\}_{0 \le t \le 1}$ is *not compactly supported* is the main purpose of the present paper. Surprisingly, this construction involves the notion of *Lagrangian cobordism* and singular Lagrangian submanifolds of the type that were used by Kasturirangan and the present author in [KO1,2]. This kind of conormal varieties were introduced by mathematicians in the micro-local analysis (see [GM], [KaSc] for example).

For the rest of the paper, we will always assume that (P, ω) is tame: (P, ω) is called *tame* if there exists a compatible complex structure J such that the metric $g_J := \omega(\cdot, J\cdot)$ has bounded sectional curvature and injectivity radius bounded below from zero. We call such almost complex structure J *tame*. It is easy to see that the set of tame almost complex structures is contractible if non-empty. We will need a more restricted class of symplectic manifolds which are *Weinstein at infinity* whose definition is referred to [EG1] or to §2 of this paper. The following is the main theorem whose precise statement will be referred to later sections.

Theorem I. *Let (P, ω) be Weistein at infinity. Let L and $\mathcal{L} = \{L_t\}_{0 \le t \le 1}$ be a (proper) Lagrangian submanifold and an isotopy of proper Lagrangian submanifolds satisfying "suitable" condition at infinity. Suppose $L \cap L_t$ remain compact for all $t \in [0,1]$. Then there exists a canonical isomorphism*

$$h_{\mathcal{L}} : HF(L, L_0) \to HF(L, L_1).$$

An immediate consequence of the present construction is the following intersection theorem of the conormal bundles. A similar intersection result was previously obtained by Eliashberg and Gromov in the name of "deformed conormal bundles" using finite dimensional approach of generating functions [Theorem 0.3.4.1, EG2].

Theorem II. *Let S_1, S_2 be compact submanifolds of M such that S_1 is transverse to S_2. Suppose ϕ is a Hamiltonian diffeomrorphism on T^*M of the types or a composition of them*

(1) *ϕ is obtained by a compactly supported Hamiltonian isotopy, or*

(2) *it is homogeneous symplectomorphic (at infinity) i.e., it is generated by the Hamiltonian of the form $(q,p) \mapsto \langle p, X_t(q) \rangle$ such that S_1 is transverse to $f_t(S_2)$ for all t where $f_t : M \to M$ is the flow of X_t, or*

(3) *it is a fiberwise translation by df where f is a smooth function defined on the base M.*

Then

$$\#(\nu^* S_1 \cap \phi(\nu^* S_2)) \ge \text{rank } H_*(S_1 \cap S_2)$$

provided $\nu^* S_1$ is transverse to $\phi(\nu^* S_2)$. Here $H_*(S_1 \cap S_2)$ is in $\mathbb{Z}$-coefficients in the oriented case and in $\mathbb{Z}_2$-coefficients in general.

We refer to Theorem 7.2 for a more precise statement concerning the Floer homology of the pair $(\nu^* S_1, \nu^* S_2)$.

A special case $S_1 = M$ and $S_2 = S \subset M$ studied in [Oh2] is of particular interest in relation to the gap in [Oh2] mentioned in the beginning. For this case, the transversality hypothesis in Theorem II is automatically satisfied. This leads to complete construction of the chain map and proof of its continuity property which in turn fills the gap in the proof of [Theorem 5.4, Oh2]

Corollary [Theorem 5.4, Oh2] *Denote by $HF_*(S, J : M)$ the Floer homology between $\nu^* S$ and $o_M (= \nu^* M)$. Let S^α and S^β be two isotopic submanifolds of M. Then there is a canonical isomorphism*

$$h_{\alpha\beta} : HF_*(S^\alpha, J^\alpha : M) \to HF_*(S^\beta, J^\beta : M)$$

that preserves the grading.

Next, we like to compare the intersection result in Theorem II or Theorem 7.2 with the conjecture stated in [GM], whose precise meaning ought to be clarified. The results from [KO1,2] and the present paper can be considered as some steps towards this direction.

Conjecture [GM]. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two complex stratifications of a complex manifold X. Assume they are transverse to each other. Let F_1 and F_2 be perverse sheaves constructible with respect to $\mathcal{S}_1$ and $\mathcal{S}_2$. Let $F_1 \otimes F_2$ be the tensor product of F_1 and F_2 on X. Then the global homology groups $H_i(X; F_1 \otimes F_2)$ can be computed as Floer homology of $(-1)_* Ch(\chi F_1)$ and $Ch(\chi F_2)$.*

The case considered in Theorem II is a special case of the Fary functors F_i constructible with respect to the stratifications

$$\mathcal{S}_i = \{S_i, M - S_i\}$$

for $i = 1, 2$ such that their corresponding constructible functions are given by

$$\chi F_i = \begin{cases} 1 & x \in S_i \\ 0 & x \in M - S_i. \end{cases}$$

One can easily check that the characteristic Lagrangian cycle of F_i is nothing but $\nu^* S_i$.

Beside the conormal varieties considered in [O2,KO1,2], good examples to which we can apply the construction of the present paper will be the symplectic manifolds with contact type boundary and proper Lagrangian submanifolds in them. We refer to §5 [KhSe] for some relevant discussions of the latter examples which occur naturally in the study of vanishing cycles of the singularity of holomorphic functions. See also Remark 4.3 of the present paper where our construction is applied to answer some question raised in [KhSe] which concerns naturality of certain isomorphism between the Floer homology of non-compact Lagrangian submanifolds. While this paper was in the stage of completion, we learned from K. Hori (see [HIV]) that some interesting class of non-compact Lagrangian cycles ("wave front trajectories" they call), which are closely related to the vanishing cycles of holomorphic Morse function ("super-potential"), play an important role in the mirror symmetry of open strings in the context of Landau-Ginzburg model through the Picard-Lefschez theory.

2. Hamiltonian deformations and C^0-estimates

In this section, we review the usual construction [F2,O1,2] of the chain map under *compactly supported* Hamiltonian isotopies. Let $j = \{J_t\}_{0 \leq t \leq 1}$ be a family of almost complex structures that is t-independent at infinity, say, $J_t(x) = J_\infty(x)$ at infinity for some almost complex structure J_∞. We denote by Supp j to be the subset

$$\text{Supp } j = \cup_{t \in [0,1]} \overline{\{x \in P \mid J_t(x) \neq J_\infty(x)\}}.$$

Let $\{L_s\}_{0 \leq s \leq 1}$ be a Hamiltonian isotopy associated to a *compactly supported* Hamiltonian

$$H : P \times [0,1] \to \mathbb{R}.$$

We choose a cut-off function $\rho : \mathbb{R} \to [0,1]$ such that

$$\begin{aligned} \rho &= \begin{cases} 0 & \text{for } \tau \leq 0 \\ 1 & \text{for } \tau \geq 1 \end{cases} \\ \rho' &\geq 0, \end{aligned}$$

and $\rho_K(\tau) = \rho\big(\frac{\tau}{K}\big)$. The construction of Floer's chain map

$$h : HF(L, L_0) \to HF(L, L_1)$$

is given by considering either

$$\begin{cases} \frac{\partial u}{\partial \tau} + J(t,u)\frac{\partial u}{\partial t} = 0 \\ u(\tau, 0) \in L, \; u(\tau, 1) \in L_{\rho_K(\tau)} \end{cases} \tag{4}$$

or

$$\begin{cases} \frac{\partial u}{\partial \tau} + J(t,u)\Big(\frac{\partial u}{\partial t} - X_{H^{\rho_K(\tau)}}(u)\Big) = 0 \\ u(\tau, 0) \in L, \;\; u(\tau, 1) \in L_0. \end{cases} \tag{5}$$

This construction works as long as (P, ω) is tame and the deformation $\{L_s\}_{0 \leq s \leq 1}$ can be realized by an ambient Hamiltonian isotopy associated to *compactly supported* Hamiltonian.

Recall from [EG1] that a symplectic manifold (P, ω) is called convex at infinity if it carries a vector field X which is *complete symplectically dilating* at infinity: A vector field X is complete symplectically dilating if the flow $\{X^t\}$ of X is complete and satisfies $(X^t)^*\omega = e^t \omega$. We assume that (P, ω) allows an exhausting pluri-subharmonic funtion at infinity. Following [EG1], we call such manifold *Weinstein* (at infinity). We choose φ an exhausting pluri-subharmonic function with respect to a tame almost complex structure J. We also assume that J is invariant under the flow of X outside a compact set. Then the level set $\varphi^{-1}(R)$ for sufficiently large R carries the induced contact structure (in fact a CR-structure) on it. The following C^0-estimate can be proven by a version of strong maximum principle (See [EHS]).

Theorem 2.1. *Let $j = \{J_t\}_{0 \leq t \leq 1}$ be a family of almost complex structures such that $J_t = J$ outside a compact set. Let $H : P \times [0,1] \to \mathbb{R}$ be a compactly supported Hamiltonian. Suppose that $L_0 \cap L_1$ are compact and L_i's are transverse to the level sets of φ at infinity. Then there exists a compact subset $K = K(P, \omega, \operatorname{supp} j, \varphi) \subset P$ such that*

$$\text{Image } u \subset K$$

for all solutions u of (4) or (5).

Proof. Consider the function $\varphi \circ u : \mathbb{R} \times [0,1] \to \mathbb{R}$. Since this function is subharmonic at infinity with respect to the metric induced by J, it has no interior maximum point outside of Supp j. Suppose that it has a maximum at a boundary point outside Supp j, say, at $(\tau_0, 1)$ and that

$$R_0 := \varphi(u(\tau_0, 1)) > \sup_{L_0 \cap L_1} \varphi.$$

By the strong maximum principle, we have

$$\frac{\partial}{\partial t}(\varphi \circ u)(\tau_0, 1) > 0 \tag{6}$$

unless $\varphi \circ u$ is constant, which is not possible if $R_0 > \sup\{\varphi \circ u(\infty), \varphi \circ u(-\infty)\}$. We note that

$$\begin{aligned} \frac{\partial u}{\partial t}(\tau_0, 1) &= J\frac{\partial u}{\partial \tau}(\tau_0, 1) \in J \cdot TL_1 \\ \frac{\partial}{\partial t}(\varphi \circ u) &= d\varphi(J\frac{\partial u}{\partial \tau}) \end{aligned}$$

On the other hand, we must have

$$d\varphi\Big(\frac{\partial u}{\partial \tau}\Big)(\tau_0, 1) = 0$$

at the maximum point $(\tau_0, 1)$. This implies

$$\frac{\partial u}{\partial \tau}(\tau_0, 1) \in TL_1 \cap T(\varphi^{-1}(R_0))$$

where $R_0 = \varphi(u(\tau_0, 1))$. Since $TL_1 \cap T(\varphi^{-1}(R_0))$ is Legendrian in $\varphi^{-1}(R_0)$ with respect to the induced contact structure (in fact, the induced CR-structure) by the assumption that L_1 is transverse to $\varphi^{-1}(R)$ for sufficiently large R, $J\frac{\partial u}{\partial \tau}(\tau_0, 1)$ is tangent to the contact distribution, which implies

$$\frac{\partial}{\partial t}(\varphi \circ u)(\tau_0, 1) = d\varphi\Big(\frac{\partial u}{\partial t}\Big)(\tau_0, 1) = d\varphi\Big(J\frac{\partial u}{\partial \tau}\Big)(\tau_0, 1) = 0$$

This gives rise to contradiction to (2.3). □

Examples of convex symplectic manifolds include cotangent bundles of compact manifolds. Products of two convex manifolds are also convex. The sum of (exhausting) pluri-subharmonic functions will provide an (exhausting) pluri-subharmonic function on the product.

Note that Theorem 2.1 already takes care of the case when Hamiltonian isotopies are compactly supported. However in relation to the quantization program illustrated by [O2] and [KO1,2], one needs to consider certain deformations $\{L_s\}_{0 \leq s \leq 1}$ of conormal type which cannot be realized by compactly supported Hamiltonians. For example, consider an isotopy $\{S^s\}_{0 \leq s \leq 1}$ of submanifolds $S^s \subset M$. In [O2], we consider the corresponding deformation of the conormal bundles

$$\{\nu^* S^s\}_{0 \leq s \leq 1} \subset T^* M.$$

This deformation is realized by the Hamiltonian

$$H(q,p,s) = \langle p, X_s(q)\rangle \tag{7}$$

where X_s is the vector field realizing the isotopy $\{S^s\}$ i.e. $X_s = \frac{d}{ds}\big|S^s$. Certainly, this Hamiltonian is not compactly supported. If one naively attempts to do the similar construction using (4) or (5) one would immediately encounter a problem in establishing the C^0-estimate. In the next sections, we will carry out construction of the chain map using "suspension" which covers this case as a special case. In hindsight, to get the required C^0 estimates, one has to use a "good" choice of the function ρ in (2) which itself will enter in the Cauchy-Riemann equation and should be determined.

3. Lagrangian cobordism

In this section, we introduce an equivalence relation on the space of Lagrangian embeddings in a given symplectic manifold (P,ω). Compare with [A1,C].

Definition 3.1. We say that two Lagrangian submanifolds L_0 and L_1 are *Lagrangian cobordant* on (P,ω) if there exists a Lagrangian submanifold

$$\beta \subset (P,\omega)\times T^*\mathbb{R}$$

such that
(i) $\partial\beta = L_0 \times \{(1,0)\} - L_1 \times \{(0,0)\}$
(ii) β has flat collars near $\partial\beta$, i.e.,

$$\beta = \begin{cases} L_1 \times \{(s,0)\} & \text{for } 0 \le s \le \varepsilon \\ L_0 \times \{(s,0)\} & \text{for } 1-\varepsilon \le s \le 1. \end{cases}$$

for some $\varepsilon > 0$. We denote by $L_0 \underset{\beta}{\sim} L_1$ if L_0 and L_1 are *Lagrangian cobordant via* β.

Note that $P\times T^*\mathbb{R}$ with the obvious product symplectic structure is tame if (P,ω) is so.

Example 3.1.
(1) Let $L_1 = \phi_H^1(L_0)$ for some Hamiltonian $H: P\times[0,1]\to\mathbb{R}$. We may re-choose H so that $H\equiv 0$ for t near 0 and 1. We define the Lagrangian cobordism

$$\beta_H \subset P\times T^*\mathbb{R}$$

by

$$\beta_H = \{(x,s,a)\in P\times T^*\mathbb{R} \mid x\in L_s,\ a=-H(x,s), 0\le s\le 1\}.$$

One can easily check that β_H is Lagrangian and satisfies both (i) and (ii). Therefore Hamiltonian isotopies are special cases of Lagrangian cobordism.

(2) We would like to separately consider the special case of (1) which was considered in [O2]. Let $\{S^s\}_{0\leq s\leq 1}$ be a smooth family of submanifolds in a smooth manifold M, and $\{\nu^* S^s\}_{0\leq s\leq 1}$ be their conormal bundles. We are given an ambient isotopy $\{\psi^s\}_{0\leq s\leq 1}$ such that

$$S^s = \psi^s(S^0)$$

and $\{X_s\}_{0\leq s\leq 1}$ is its generating vector fields, then the corresponding Lagrangian cobordism is given by

$$\{(q,p,s,a) \in T^*M \times T^*[0,1] \mid q \in S^s, p \in \nu^*_q S^s, a = -\langle p, X_s(q)\rangle \text{ and } 0 \leq s \leq 1\}.$$

One can easily check that this becomes flat if we choose the isotopy to be constant near $s = 0$ and 1. Furthermore, this bordism itself is nothing but the conormal to the suspension

$$\{(q,s) \in M \times [0,1] \mid q \in S^s\}$$

in $T^*(M \times [0,1])$.

4. Construction of chain maps

In this section, we attempt to construct the chain map

$$h_\beta : HF(L, L_0) \to HF(L, L_1)$$

when $L_0 \underset{\beta}{\sim} L_1$. In the beginning, we do not impose any condition on L_1 or L_0. Due to the assumption of flatness near $\partial\beta$, we can smoothly add to β two ends

$$L_0 \times (-\infty, 0] \times \{0\} \amalg L_1 \times [1,\infty) \times \{0\}.$$

We again denote the resulting manifold by β. This β will play a role as the boundary condition at $t = 1$ for the Cauchy-Riemann equation that we will consider. We still need the boundary condition at $t = 0$ which we now describe.

It turns out that the right choice is the following singular Lagrangian submanifold

$$\alpha_L := L \times Ch(1_{[0,1)}) \subset P \times T^*\mathbb{R}$$

where $Ch(1_{[0,1)})$ is the characteristic Lagrangian cycle of the characteristic function $1_{[0,1)}$ of $[0,1)$ on $\mathbb{R}$ in the sense of [GK]. In fact, we can prove (see [KO2] for the general case of standard pairs) that

$$Ch(1_{[0,1)}) = o_{\mathbb{R}}\big|_{(0,1)} \amalg \nu^*_-(\partial[0,1])\big|_0 \amalg \nu^*_+(\partial[0,1])\big|_1$$

where

$$\begin{aligned} \nu^*_-(\partial[0,1])\big|_0 &= \{(s,a) \in T^*\mathbb{R} \mid s = 0, a \geq 0\} \\ \nu^*_+(\partial[0,1])\big|_1 &= \{(s,a) \in T^*\mathbb{R} \mid s = 1, a \geq 0\} \end{aligned}$$

is the negative and positive part (with respect to the induced orientation) of the conormal bundle of $\partial[0,1]$ respectively.

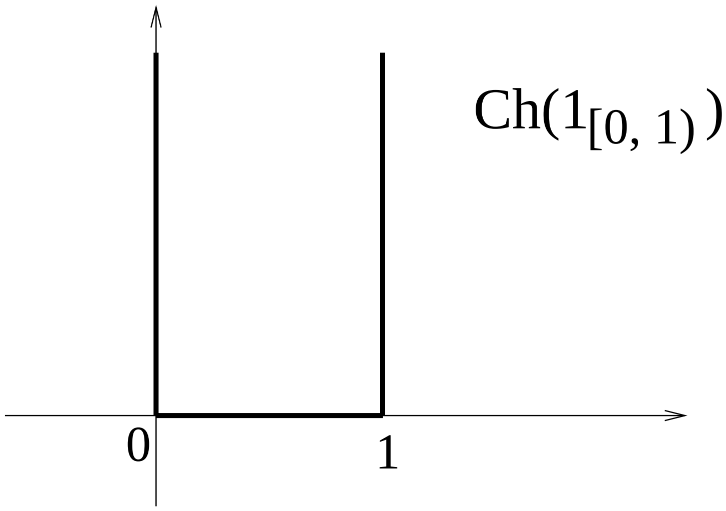

FIGURE 1. $Ch(1_{[0,1)})$

In [KO2], we call $Ch(1_{[0,1)})$ the conormal to the *standard pair* $([0,1],\{1\})$ [GM] and denote it by $\nu^*([0,1],\{1\})$. We refer to [GM] or [KO2] for the definition of standard pairs. Then we consider the following Cauchy-Riemann equation

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial \tau} + \widetilde{J}\Big(\frac{\partial \widetilde{u}}{\partial t}\Big) = 0 \\ \widetilde{u}(\tau,0) \in \alpha_L,\ \widetilde{u}(\tau,1) \in \beta \end{cases} \tag{8}$$

where $\widetilde{J} = J \oplus i$, $\widetilde{u} = (u,v) \subset P \times T^*\mathbb{R}$ where $v = (s,a)$. Since α_L is singular, we need to desingularize α_L in a suitable way as in [KO1,2], which we now describe. We consider

$$\alpha_\varepsilon := L \times \Upsilon_\varepsilon \subset P \times T^*\mathbb{R}$$

where Υ_ε are approximations of $Ch(1_{[0,1)})$ drawn as in Figure 2
Since we assume that β is flat near $\partial\beta$, we can choose $\varepsilon > 0$ so that

$$\beta \cap \alpha_\varepsilon \cap \{0 < s < \varepsilon \ \text{ or } \ 1 - \varepsilon < s < 1, \text{ and } \ a = 0\} = \emptyset.$$

On the other hand, we have

$$\begin{aligned} \partial\beta \cap \alpha_\varepsilon \cap \{s = 0\} &= L \cap L_0 \times \{(0,0)\} \\ \partial\beta \cap \alpha_\varepsilon \cap \{s = 1\} &= L \cap L_1 \times \{(1,0)\}. \end{aligned} \tag{9}$$

If we assume that L is transverse to both L_0 and L_1, we can apply Hamiltonian perturbations in $P \times T^*\mathbb{R}$ of β away from the sets (9) and make β intersect transversely with

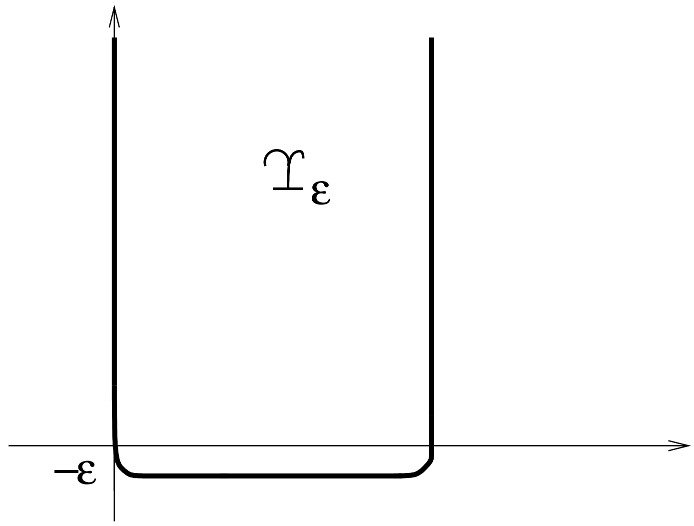

FIGURE 2. Approximation of $Ch(1_{[0,1)})$

α_L. Now for each given $x \in L \cap L_0$ and $y \in L \cap L_1$, we study the equation

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial \tau} + \widetilde{J}\Big(\frac{\partial \widetilde{u}}{\partial t}\Big) = 0 \\ \widetilde{u}(\tau, 0) \in \alpha_L,\ \widetilde{u}(\tau, 1) \in \beta \\ \widetilde{u}(-\infty) = \widetilde{x} = (x, 0, 0),\, \widetilde{u}(+\infty) = \widetilde{y} = (y, 1, 0). \end{cases} \tag{10}$$

Remark 4.1. Let us disseminate (10) for the case $\beta = \beta_H$. In this case, the equation (10) can be re-written as

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\frac{\partial u}{\partial t} = 0 \\ u(\tau, 0) \in L,\ u(\tau, 1) \in L_{s(\tau,1)} \\ u(-\infty) = x,\, u(+\infty) = y \\ \\ \overline{\partial} v = 0 \\ v(\tau, 0) \in Ch([0, 1)),\, a(\tau, 1) = -H(u(\tau, 1), s(\tau, 1)) \\ s(-\infty) = 0,\, s(\infty) = 1 \end{cases}$$

The first part of this equation is nothing but (2) with $\rho(\tau) = s(\tau, 1)$ but s itself must be solved. Furthermore unlike (2), u and s are coupled to each other. It is rather interesting and mysterious to us that our effort obtaining the C^0-estimates of (2) has led us to considering the coupled Cauchy-Riemann equation of u and ρ in the suspended space.

We note that if (P,ω) is Weinstein at infinity, so is $(P,\omega)\times(T^*\mathbb{R},\omega_0)$. If φ is an exhausting pluri-subharmonic function on P, the

$$\widetilde{\varphi}(x,s,a)=\varphi(x)+\frac{1}{2}(s^2+a^2)$$

will be an exhausting pluri-subharmonic function on $(P,\omega)\times(T^*\mathbb{R},\omega_0)$. Furthermore both α_L and β are *fixed* Lagrangian submanifolds and α_L is transverse to the level sets of $\widetilde{\varphi}$ at infinity. Therefore we have the following a priori C^0-estimate for the solutions of (10) from Corollary 2.2.

Proposition 4.1. *Suppose that (P,ω) is Weinstein at infinity. Assume*
(1) *$\alpha_L\cap\beta$ is compact*
(2) *β is transverse to the level sets of $\widetilde{\varphi}$ at infinity.*
Then for any given $\widetilde{x}$, $\widetilde{y}\in\alpha_L\cap\beta$, there exists a compact subset $K=K(\widetilde{x},\widetilde{y},\beta)\subset P\times T^*\mathbb{R}$ such that

$$\text{Image }\widetilde{u}\subset K$$

for all $\widetilde{u}\in\mathcal{M}_\epsilon(\widetilde{J},\beta:\widetilde{x},\widetilde{y})$.

We now study the moduli-space $\mathcal{M}_\varepsilon(\widetilde{J},\beta)$ of solutions of (8) with finite energy. This is decomposed into

$$\mathcal{M}_\varepsilon(\widetilde{J},\beta)=\cup_{\widetilde{x},\widetilde{y}\in\alpha_L\cap\beta}\mathcal{M}_\varepsilon(\widetilde{J},\beta:\widetilde{x},\widetilde{y})$$

where $\mathcal{M}_\varepsilon(\widetilde{J},\beta:\widetilde{x},\widetilde{y})$ is the set of solutions of (10). Note that there is a natural $\mathbb{R}$-action on $\mathcal{M}_\varepsilon(\widetilde{J},\beta:\widetilde{x},\widetilde{y})$ by translations in the τ-direction. We denote

$$\widehat{\mathcal{M}}_\varepsilon(\widetilde{J},\beta:\widetilde{x},\widetilde{y})=\mathcal{M}_\varepsilon(\widetilde{J},\beta:\widetilde{x},\widetilde{y})/\mathbb{R}$$

and

$$n_\varepsilon(x,y:\beta)=\#(\widehat{\mathcal{M}}_\varepsilon(\widetilde{J},\beta:\widetilde{x},\widetilde{y}))$$

when

$$\dim\mathcal{M}_\varepsilon(\widetilde{J},\beta:\widetilde{x},\widetilde{y})=0.$$

By the standard compactness theorem and the dimension counting arguments, the zero-dimensional component of $\widehat{\mathcal{M}}_\varepsilon(\widetilde{J},H,\beta:\widetilde{x},\widetilde{y})$ is compact *under suitable assumptions on L,L_0,L_1 and β*. For example, we may assume that L, L_0 and L_1 are monotone in P and β is monotone in $P\times T^*\mathbb{R}$. (See [O1]). We will always assume these conditions from now on for the simplicity in presenting the main ideas of our construction, although one could consider more general cases using the sophisticated construction employed in [FOOO].

We define a map

$$h_{\beta,\varepsilon}:CF(L,L_0:J,\alpha_\varepsilon)\to CF(L,L_1:J,\alpha_\varepsilon)$$

by

$$h_{\beta,\varepsilon}(x)=\sum_y n_\varepsilon(x,y:\beta)y \tag{11}$$

and study its chain property.

Definition 4.1. Let (L_0, L_1) be a pair of Lagrangian submanifolds transverse to L. Define $\mathcal{B}(L_0, L_1)$ = the set of Lagrangian cobordisms β from L_0 to L_1

$$\mathcal{B}_0(L_0, L_1 : L) = \{\beta \in \mathcal{B}(L_0, L_1) \mid \beta \text{ is transverse to } \alpha_L\}.$$

Lemma 4.2. *The set $\mathcal{B}_0(L_0, L_1 : L)$ is a residual subset of $\mathcal{B}(L_0, L_1)$ in C^1-topology.*

Proof. It is enough to consider Hamiltonian perturbations of given β that are fixed near $\partial\beta$. The proof of this is standard which we omit. ☐

Example 4.2. Consider a Hamiltonian isotopy from L_0 to L_1 and its corresponding Lagrangian cobordism

$$\beta_H = \{(x, s, a) \mid x \in L_s,\ a = -H(x, s)\}.$$

We call this a *Hamiltonian cobordism.* In this case, we note that

$$\beta_H \cap \alpha_L = \{(x, s, a) \in P \times T^*\mathbb{R} \mid x \in L \cap L_s \text{ and } a = -H(x, s) = 0\}$$

and β_H is transverse to α_L if and only if

$$T_xL \oplus T_xL_s = T_xP, \quad \frac{\partial H}{\partial s}(x, s) \neq 0$$

at each $(x, s, a) \in \beta_H \cap \alpha_L$. However in general, we cannot avoid non-transverse intersections for a one parameter family $\{L_s\}_{0\leq s\leq 1}$, which forces us to look at perturbations of β_H on $P \times T^*\mathbb{R}$ to obtain transversal pairs (α_L, β) with β close to β_H.

As usual in the Floer theory, we examine compactness property of the one-dimensional component of $\widehat{\mathcal{M}_\varepsilon}(\widetilde{J}, \beta : \widetilde{x}, \widetilde{y})$ to study the chain property of $h_{\beta,\varepsilon}$, i.e., the identity

$$h_{\beta,\varepsilon} \circ \partial_0 = \partial_1 \circ h_{\beta,\varepsilon} \tag{12}$$

We consider one dimensional components of $\widehat{\mathcal{M}_\varepsilon}(\widetilde{J}, \beta)$ and study structure of the boundary of each one-dimensional component in its compactification. Standard dimension counting argument tells us that the boundary of $\mathcal{M}_\varepsilon(\widetilde{J}, \beta : \widetilde{x}, \widetilde{z})$ consists of the cusp-trajectories of the form $\widetilde{u}_1 \# \widetilde{u}_2$ where $(\widetilde{u}_1, \widetilde{u}_2)$ are elements in $\mathcal{M}_\varepsilon(\widetilde{J}, \beta : \widetilde{x}, \widetilde{y}) \times \mathcal{M}_\varepsilon(\widetilde{J}, \beta : \widetilde{y}, \widetilde{z})$. Here, a priori, $\widetilde{y}$ could be any element in the intersection set $\beta \cap \alpha_{L,\varepsilon}$, not just in the hypersurface of $s = 0$ or 1. This will prevent us from associating a chain homomorphism to general Lagrangian cobordism. ¿From now on, we will mainly concern the case of Hamiltonian cobordism.

Let us first examine the condition (1) from Proposition 4.2 that $\alpha_L \cap \beta$ *is compact.* This is certainly the case if L is compact. For the case of Hamiltonian cobordism β_H, it is easy to see that $\alpha_L \cap \beta_H$ is compact if and only if $L \cap L_t$ is compact for all $t \in [0, 1]$. In general, we introduce the following definition.

Definition 4.2. Let $\mathcal{L} = \{L_t\}_{0\leq t\leq 1}$ be a Hamiltonian isotopy. We say that *intersections do not escape to infinity* if $\cup_{t\in[0,1]}(L \cap L_t)$ is compact.

Under this condition, we prove the following proposition, which will eliminate those intersections $\tilde{y}$ away from $\partial\beta_H \cap \alpha_L$ (i.e., away from $s = 0$ or $s = 1$) that provide the obstruction to the existence of chain property.

Lemma 4.3. *Let $L \subset P$ and $\mathcal{L} = \{L_t\}$ be a Hamiltonian isotopy of L_0 such that the intersections $L \cap L_t$ do not escape to infinity. Let β_H be a Hamiltonian cobordism associated to the Hamiltonian isotopy $\mathcal{L}$. Then we can change H to H' so that $\phi_H^t = \phi_{H'}^t$, and*

$$\beta_{H'} \cap \alpha_L = L \cap L_0 \times \{(0,0)\} \coprod L \cap L_1 \times \{(1,0)\} \tag{13}$$

Proof. We recall

$$\beta_H \cap \alpha_L = \{(x,s,a) \in P \times T^*\mathbb{R} \mid x \in L \cap L_s,\, a = -H(x,s) = 0,\, s \in [0,1]\}$$

Since $\cup_{s\in[0,1]} L \cap L_s$ is compact by hypothesis, Image $H|_{\cup_{s\in[0,1]} L\cap L_s}$ is compact. Therefore we can choose a non-negative function

$$\chi : [0,1] \to \mathbb{R}_+$$

so that

(i) $\chi(s) = 0$ for s near 0 or 1.
(ii) $\chi(s) + H(x,s) > 0$ for (x,s) such that $x \in \cup_{s\in[\delta,1-\delta]} L \cap L_s$ for some small $\delta > 0$.

We just choose $H'(x,s) := H(x,s) + \chi(s)$ as our new Hamiltonian. ☐

¿From now on based on Lemma 4.3 or its proof, we use only the Hamiltonians that satisfy

$$H(x,s) > 0 \quad \text{for } (x,s) \in \cup_{s\in[\delta,1-\delta]} L \cap L_s \tag{14}$$

for the Hamiltonian cobordism β_H when we perform construction of the chain map $h_{\beta_H,\varepsilon}$. We call such Hamiltonians *(positively) admissible to* $(L, \mathcal{L})$. The following lemma is easy to check

Lemma 4.4. *Let L and $\mathcal{L}$ be as in Lemma 4.3. Consider the Hamiltonian cobordisms β_H associated to (positively) adimissible Hamiltonian H. Then two such Hamiltonian cobordisms are Hamiltonian isotopic to each other in $P \times T^*[0,1]$ by an isotopy that is compactly supported in $P \times T^*(0,1)$.*

We now study the condition (2) from Proposition 4.2 that *the Hamiltonian cobordism β_H is transverse to the level sets of $\widetilde{\varphi} = \varphi + \frac{1}{2}(s^2 + a^2)$*. A typical example of such Hamiltonians arise in the following way: Let $\partial P = M$ with its induced contact structure and $L_0 \subset P$ be a proper Lagrangian submanifold with its boundary $R_0 \subset M$. R_0 is a compact Legendrian submanifold of M_0. Consider a Hamiltonian isotopy of L_0 which extends a Legendrian isotopy of $R_0 \subset M$. We choose Hamiltonians which restrict to *contact Hamiltonians* (see [A2] for the definition) on the collar $(1-\epsilon, 1] \times M$ of ∂P, i.e., satisfies

$$H(cm,t) = cH(m,t) \quad \text{for } m \in M,\ c \in \mathbb{R}^+ \tag{15}$$

on the symplectic cone attached to ∂P.

Lemma 4.5. *Let P be Weinstein at infinity and φ be an exhausting pluri-subharmonic function which is super-quadratic over the radial coordinate. Suppose that H satisfy (15) and that L_0 is transverse to the level sets of φ at infinity. Then the induced Hamiltonian cobordism $\beta_H \subset P \times T^*\mathbb{R}$ of L_0 is transverse to the level sets of $\widetilde{\varphi}$ at infinity.*

Proof. Since we extend β_H so that $a = 0$ outside $0 \leq s \leq 1$ which is obviously transverse, it is enough to check the transversality over $0 \leq s \leq 1$. In this region, we may consider the function $\varphi + \frac{1}{2}a^2$ in place of $\widetilde{\varphi}$. Recalling

$$\beta_H = \{(x,s,a) \in P \times T^*\mathbb{R} \mid x \in L_s,\ a = -H(x,s),\ 0 \leq s \leq 1\},$$

it is easy to check that the tangent space of β_H at $(x, s, -H(x,s))$ is spanned by the vectors

$$\vec{v} - H(x,s)dH(\vec{v})\frac{\partial}{\partial a} + c\Big(\frac{\partial}{\partial s} - \frac{\partial H}{\partial s}\frac{\partial}{\partial a}\Big)$$

where d is the differential for x, $\vec{v} \in T_xL_s$ and $c \in \mathbb{R}$. Applying this vector to $\varphi + \frac{1}{2}a^2$, the non-transversal points are characterized by the equation

$$\begin{aligned} d\varphi(x)\big|_{L_s} + H(x,s)dH(x,s)\big|_{L_s} &= 0 \\ H(x,s)\tfrac{\partial H}{\partial s} = 0, \quad a &= -H(x,s) \end{aligned} \tag{16}$$

on the collar or on the symplectic cone attached to ∂P. Since $d\varphi \neq 0$, $H(x,s) \neq 0$ on the cone. On the other hand, since the growth of H is linear and the growth of φ is super-quadratic over the radial coordinate, the first equation of (16) cannot hold at infinity in $P \times T^*[0,1]$. This finishes the proof. $\square$

We now apply Theorem 2.1 to the case $L_0 = \alpha_L$ and $L_1 = \beta_H$ to obtain the C^0-estimate for (10). Once the crucial C^0-estimate is obtained, the standard arguments in the Floer theory prove the following proposition.

Theorem 4.6. *Let $\partial_0 : CF(L, L_0) \to CF(L, L_0)$ and $\partial_1 : CF(L, L_1) \to CF(L, L_1)$ be the Floer boundary maps on (P, ω). Suppose $L, \mathcal{L}$ satisfy the properties required in Definition 4.2 and let H a Hamiltoian generating $\mathcal{L}$ and satisfying (14) and (15). Let $h_{\beta_H,\varepsilon} : CF(L, L_0 : \alpha_\varepsilon) \to CF(L, L_1 : \alpha_\varepsilon)$ be the map defined in (11). Then the identity (12) holds and so $h_{\beta_H,\varepsilon}$'s induce a homomorphism, as $\varepsilon \to 0$,*

$$h_{\beta_H,\varepsilon} : HF(L, L_0 : J) \to HF(L, L_1 : J).$$

Furthermore, this homomorphism is independent of the approximations α_ε and of the choice of H. We denote the common homomorphim by

$$h_{\mathcal{L}} : HF(L, L_0) \to HF(L, L_1) \tag{17}$$

Proof. Under the hypotheses given in the statement, it follows that
$$\widetilde{y} \in \partial\beta \cap \alpha_{L,\varepsilon} = L \cap L_0 \times \{(0,0)\} \amalg L \cap L_1 \times \{(1,0)\}.$$
Once we have this, the standard argument in the Floer theory proves the chain property (4.7).

To prove the independence of $h_{\beta_H,\varepsilon}$ on H and $\varepsilon > 0$, it will be enough to prove that the family of approximations $\{\alpha_\varepsilon\}_{\varepsilon>0}$ and the change of H's satisfying the condition above can be realized by compactly supported Hamiltonian deformations of one another among them. But this follows from the construction of the approximation Υ_ε of $Ch(1_{[0,1)})$. We refer to [KO1,2] for the details of this limiting argument. □

In the next sections, we will prove that our chain map associated to a Hamiltonian isotopy $\mathcal{L}$ is natural and becomes an isomorphism.

Remark 4.3. (1) This will fill the gap present in the construction of the chain isomorphism used in [O2], which the author overlooked in applying the strong maximum principle to get the C^0-estimate for the continuity equation (2) or (3). This C^0-estimate and the isomorphism were crucial in the proof of continuity of the invariants $S \mapsto \rho(H, S)$ under the isotopy of submanifolds S (see the proof of Proposition 6.5 [O2]).

(2) The Hamiltonian isotopies considered in Lemma 4.5 includes the *positive Lagrangian isotopy* of *θ-exact Lagrangian subamnifolds* considered in [KhSe]. In particular, we have provided the recipe of curing the "weakness" mentioned therein in that our construction provides a canonical isomorphism to Lemma 5.11 [KhSe] that was missing therein.

5. Composition rule

In this section, we will prove the following composition rule,
$$h_{\beta_0\#\beta_1} = h_{\beta_1} \circ h_{\beta_0} \tag{18}$$
where $L_0 \underset{\beta_0}{\sim} L_1$, $L_1 \underset{\beta_1}{\sim} L_2$ and $\beta_0\#\beta_1$ denotes the obvious composition of Lagrangian cobordisms β_0 and β_1.

We examine how the Lagrangian boundary conditions are involved. At $t = 1$, we can just take a small perturbation of the elongated $\beta_0\#\beta_1$. At $t = 0$, we need to describe some approximation result for
$$L \times Ch(1_{[0,1)}) \cup L \times Ch(1_{[1,2)}) = L \times (Ch(1_{[0,1)}) \cup Ch(1_{[1,2)})).$$
by a family of Lagrangian submanifolds $\{\Xi_\varepsilon\}_{0<\varepsilon<1}$ as drawn above (See [KO2] for many illustrations of this approximation argument). First, we remark that for any given compact subset of ε in $(0, 1)$, the corresponding Lagrangian submanifolds $L \times \Xi_\varepsilon$ are deformations to one another via *compactly supported* Hamiltonian isotopies $T^*(M \times \mathbb{R})$. Then some modification of standard gluing arguments can be applied to prove the following analytical result (See [KO2] for some relevant discussion).

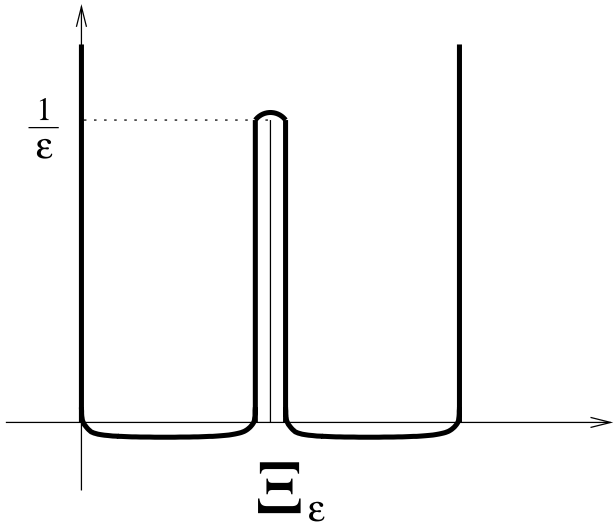

FIGURE 3. Approximation of the cycle $Ch(1_{[0,1)}) \cup Ch(1_{[1,2)})$

Theorem 5.1. *There exists sufficiently small $\varepsilon > 0$ such that we have gluing diffeomorphisms*

$$\mathcal{M}(\widetilde{J}, \alpha_\varepsilon, \beta_0) \times \mathcal{M}(\widetilde{J}, \alpha_\varepsilon, \beta_1) \to \mathcal{M}(\widetilde{J}, L \times \Xi_\varepsilon, \beta_0 \# \beta_1)$$

after a modification of the cobordism $\beta_0 \# \beta_1$ near $s = 1$ as described in the proof of Lemma 4.5. In particular, we have the identity,

$$h_{\beta_1,\varepsilon} \circ h_{\beta_0,\varepsilon} = h_{L\times\Xi_\varepsilon} : HF_*(L, L_0) \to HF_*(L, L_2). \tag{19}$$

Proof. We will be sketchy in the proof because similar gluing arguments have been used many times in the literature by now.

Note that $\beta_0 \# \beta_1$ is again a Hamiltonian cobordism. We choose a Hamiltonian that is positively admissible to $\beta_0 \# \beta_1$. In fact, by adding a bump function supported in a neighborhood of the hypersurface $s = 1$, we can make the corresponding Hamiltonian H so that Graph H is "above" Ξ_ε as in Figure 4. We glue each given pair $u_0 \in \mathcal{M}(\widetilde{J}, \alpha_\varepsilon, \beta_0)$ and $u_1 \in \mathcal{M}(\widetilde{J}, \alpha_\varepsilon, \beta_1)$ with the obvious holomorphic strip in the middle. This gluing is possible, as long as ε is sufficiently small and so $\frac{1}{\varepsilon}$ is sufficiently large and the strip is sufficiently narrow. This finishes the proof. □

After this crucial analytical step, we use the fact, which can be easily checked, that the family $\{L \times \Xi_\varepsilon\}_{0<\varepsilon<1}$ are Hamiltonian deformations to one another via *compactly supported* Hamiltonian isotopy. Therefore we can apply the standard continuation argument in the Floer theory to show that the homomorphisms

$$h_{L\times\Xi_\varepsilon} : HF_*(L, L_0) \to HF_*(L, L_1)$$

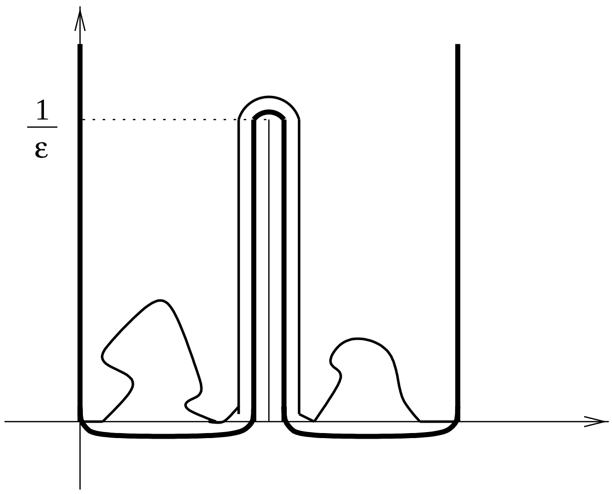

FIGURE 4

are independent of $\varepsilon > 0$. Since we have

$$h_{(\beta_0 \# \beta_1)} = \lim_{\varepsilon \to 0} h_{(L \times \Xi_\varepsilon)}, \tag{20}$$

we have finished proof of (18) combining (19) and (20).

6. Trivial cobordism

In this section we prove the following theorem. This is the place where the power of choosing α_L as we do for the boundary condition at $t = 0$ becomes manifest.

Theorem 6.1. *Consider the trivial product cobordism*

$$\beta_0 = L_0 \times [0,1] \times \{0\} \subset P \times T^*\mathbb{R}.$$

Then the induced homomorphism $h_{\beta_0} : HF_*(L, L_0) \to HF_*(L, L_0)$ is the identity homomorphim.

Proof. Recall $\alpha_\varepsilon = L \times \Xi$. We study the equation

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial \tau} + \widetilde{J}\Big(\frac{\partial \widetilde{u}}{\partial t}\Big) = 0 \\ \widetilde{u}(\tau, 0) \in \alpha_\varepsilon, \ \widetilde{u}(\tau, 1) \in \beta_0 \end{cases} \tag{21}$$

Since $\widetilde{J} = J \oplus i$, $\alpha_\varepsilon = L \times \Upsilon_\varepsilon$ and $\beta_0 = L_0 \times o_{\mathbb{R}}$ all split, (21) splits into

$$\begin{cases} \frac{\partial u}{\partial \tau} + J \frac{\partial u}{\partial t} = 0 \\ u(\tau, 0) \in L, \ \ u(\tau, 1) \in L_0 \end{cases} \tag{22}$$

and

$$\begin{cases} \overline{\partial} v = 0 \\ v(\tau, 0) \in \Upsilon_\varepsilon, \quad v(\tau, 1) \in o_{\mathbb{R}}. \end{cases} \tag{23}$$

Noting that (23) has the unique solution with index 1 (up to translations) and with the asymptotic condition

$$v(-\infty) = (0,0), \quad v(\infty) = (1,0), \tag{24}$$

solutions $\widetilde{u} \in \mathcal{M}_\varepsilon(\widetilde{x}, \widetilde{y} : \widetilde{J}, \alpha_\varepsilon, \beta_0)$ of (21) with index 1 consist of the pairs (u, v) such that u is a solution u of (22) with index 0 and v is the unique solution of (23) satisfying (24). In particular, u must be constant. Hence we have proven that

$$\begin{aligned} n_\varepsilon(\widetilde{x}, \widetilde{x} : \beta_0) &= 1 \quad \text{for all } x \in L \cap L_0 \\ n_\varepsilon(\widetilde{x}, \widetilde{y} : \beta_0) &= 0 \quad \text{if } y \neq x \end{aligned}$$

which in turn implies that the chain map

$$h_{\beta_0, \varepsilon} : CF(L, L_0 : J, \alpha_\varepsilon) \to CF(L, L_0 : J, \alpha_\varepsilon)$$

becomes the identity map. This finishes the proof. □

One immediate corollary of (18) and Theorem 6.1 is the following

Theorem 6.2. *Let H be a positively admissible Hamiltonian to $(L, \mathcal{L})$, and β_H be the Hamiltonian cobordism obtained from the Hamiltonian isotopy $\phi_H^s(L_0)$ from L_0 to $L_1 = \phi_H^1(L_0)$. Then the homomorphism*

$$h_{\beta_H} : HF_*(L, L_0) \to HF_*(L, L_1)$$

is an isomorphism. Hence $h_{\mathcal{L}} : HF_*(L, L_0) \to HF_*(L, L_1)$ is an isomorphism.

Proof. We compose β_H with $\beta_{\overline{H}}$ where

$$\overline{H}(x, s) := -H(\phi_H^s(x), s)$$

which generates the isotopy $\{(\phi_H^s)^{-1}(L_1)\}$. It is immediate to check that the composition $\beta_H \# \beta_{\overline{H}}$ is Hamiltonian isotopic to the product cobordism between L_0 and L_0 via *compactly supported* Hamiltonian isotopy $P \times T^*\mathbb{R}$. Therefore we can apply the standard procedure of using (4) to prove the construction of chain isomorphisms between the cases of the identity cobordism and $\beta_H \# \beta_{\overline{H}}$. This proves the theorem. □

7. Intersection of conormal bundles

In this section, we apply our extended Floer theory to the special case of conormal bundles $\nu^* S_1, \nu^* S_2$ of two smooth submanifolds $S_1, S_2 \subset M$. We would like to compute $HF_*(\nu^* S_1, \nu^* S_2)$, when S_1 is transverse to S_2.

First we note that the intersection of conormals

$$\nu^* S_1 \cap \nu^* S_2 = o_{S_1 \cap S_2}$$

is compact and the following types of deformations or compositions of them leave the intersection set compact:

(1) ϕ_t are compactly supported, or

(2) they are homogeneous symplectomorphisms (at infinity), i.e., it is generated by the Hamiltonian of the form $(q,p) \mapsto \langle p, X_t(q)\rangle$ such that S_1 *is transverse to* $f_t(S_2)$ *for all* t where $f_t : M \to M$ is the flow of X_t, or

(3) they are the fiberwise translations by tdf where f is a smooth function defined on the base M.

One can easily check that any two such $\Phi = \{\phi_t\}$ can be connected by one parameter family $\{\Phi^s\}_{0\leq s\leq 1}$ such that intersections of ν^*S_1 and $\phi_t^s(S_2)$ remain to be compact. Therefore it follows from the discussions in the previous sections that there exist a canonical chain isomorphism

$$h : CF(\nu^*S_1, \phi_1(\nu^*S_2)) \to CF(\nu^*S_1, \phi_2(\nu^*S_2))$$

where, for example, $\Phi_i = \{\phi_i^t\}_{0\leq t\leq 1}$ for $i = 1, 2$ is a Hamiltonian isotopy of T^*M of the above types or a composition of them. Therefore this induces the canonical isomorphism

$$h : HF(\nu^*S_1, \phi_1(\nu^*S_2)) \to HF(\nu^*S_1, \phi_2(\nu^*S_2)).$$

We denote by $HF(\nu^*S_1, \nu^*S_2)$ the common group.

The existence of such isomorphisms for the first two cases is immediate from the discussions in the previous sections. The case (3) follows since we can easily check that the corresponding Hamiltonian cobordism satisfies the hypotheses (1) and (2) from Proposition 4.2.

To compute $HF_*(\nu^*S_1, \nu^*S_2)$, we deform ν^*S_2 to $\phi_f(\nu^*S_2)$ where ϕ_f is the fiberwise translations by df, where the function f on M will be suitably chosen. Since we assume that S_1 is transverse to S_2, $S_1 \cap S_2$ is a smooth submanifold. We choose a smooth Morse function $\widetilde{f} : S_1 \cap S_2 \to \mathbb{R}$ and extend it to M, first quadratically to a tubular neighborhood and then suitably cutting off outside the neighborhood (See [Pz] or [O2]). We denote the extension by $f : M \to \mathbb{R}$.

Proposition 7.1. *Let* f *and* ϕ_f *described as above. Then we have*

(1) ν^*S_1 *is transverse to* $\phi_f(\nu^*S_2)$

(2) $\nu^*S_1 \cap \phi_f(\nu^*S_2)$ *is finite and all lie in the zero section of* T^*M*.*

Proof. We first prove (2). Let $\alpha_1 \in \nu_q^*S_1$. If $\alpha_1 \in \nu_q^*S_1 \cap \phi_f(\nu^*S_2)$, then we should have

$$\alpha_1 = \alpha_2 + df(q) \tag{25}$$

for some $\alpha_2 \in \nu_q^*S_2$. Since $\alpha_1 \in \nu_q^*S_1$, we must have

$$\alpha_2|_{TS_1} = -df(q)|_{TS_1} \tag{26}$$

Since S_1 is transverse to S_2 and $\alpha_2 \in \nu_q^*S_2$, (7.2) uniquely determines α_2 and so α_1. It remains to show that $\alpha_1 = 0$. To show this, it is enough to prove that

$$\alpha_1|_{TS_2/T(S_1\cap S_2)} \equiv 0$$

because $\alpha_1|_{TS_1} \equiv 0$. On the other hand, this follows from (25) noting that $\alpha_2|_{TS_2} \equiv 0$ and that we have extended the Morse function $\widetilde{f}$ on $S_1 \cap S_2$ quadratically to its tubular neighborhood and so $df(q)|_{TM/T(S_1 \cap S_2)} \equiv 0$. This finishes the proof of (2). Once we prove this, (1) immediately follows from transversality of intersections of S_1 and S_2. $\square$

With Propostions 7.1 in our hand, we can repeat the computations from [F2], [Pz] or [O2] to construct one to one correpondence between the moduli space $\mathcal{M}(J, \nu^*S_1, \phi_f(\nu^*S_2))$ of Floer's trajectories and the moduli space $\mathcal{M}^{Morse}(f; S_1 \cap S_2)$ for a suitably chosen almost complex structure J (see [F2], [Pz] for the relevant arguments in a different context). Combining these and construction of orientation of the Floer moduli space from [Oh2], we have proved the following

Theorem 7.2. *Let $S_1, S_2 \subset M$ be transversal compact smooth submanifolds and ν^*S_1, ν^*S_2 be their conormal bundles. Then there exists a canonical chain isomorphim*

$$C^{Morse}(f; S_1 \cap S_2) \to CF(\nu^*S_1, \phi_f(\nu^*S_2))$$

which induces an isomorphism

$$h : H_*(S_1 \cap S_2; \mathbb{Z}_2) \to HF(\nu^*S_1, \phi_f(\nu^*S_2)) \simeq HF(\nu^*S_1, \nu^*S_2)$$

in $\mathbb{Z}_2$-coefficients in general. When S_1, S_2 and M are oriented, then this isomorphism holds in $\mathbb{Z}$-coefficients.

This combined with the invariance property of the Floer homology under the Hamiltonian isotopy of the types, e.g., (1), (2) and (3) above, immediately gives rise to the following intersection theorem.

Corollary 7.3. *Let S_1, S_2 be as before. Suppose ϕ is a Hamiltonian diffeomorphism on T^*M of the types above or a composition of them. Then*

$$\#(\nu^*S_1 \cap \phi(\nu^*S_2)) \geq \text{rank } H_*(S_1 \cap S_2)$$

provided ν^*S_1 is transverse to $\phi(\nu^*S_2)$. Here $H_*(S_1 \cap S_2)$ is in $\mathbb{Z}$-coefficients in the oriented case and in $\mathbb{Z}_2$-coefficients in general.

We would like to compare Theorem 7.2 with the conjecture stated in the end of [GM]. It would be very interesting to generalize the construction in [KO1,2] to the general stratified case to give a precise meaning of the statement of the conjecture [GM].

8. Further discussions

In [Po], Polterovich introduced the notion of *Lagrangian pseudo-isotopy* and in [C], Chekanov introduced that of (connected) *monotone Lagrangian cobordism.* If we restrict to the case of monotone Lagrangian submanifolds for which the Floer homology can be easily constructed without any sophisticated machinery, the construction we have carried out in the previous sections also applies to the monotone Lagrangian cobordism, in particular to the Lagrangian pseudo-isotopy. Therefore we have proved that for any monotone Lagrangian cobordism β from L_0 and L_1, there exists a natural homomorphism

$$h_\beta : HF(L, L_0) \to HF(L, L_1).$$

For more complicated cobordism, we do not expect such homomorphims but expect only some "correspondences".

In fact, this construction works for the case of Lagrangian pseudo-isotopy as long as the Floer homology $HF(L, L_0)$ for the given Lagrangian submanifold L and L_0 can be constructed (We refer to [FOOO] for the most general construction of Floer homology upto now). Unlike the case of Hamiltonian isotopy, the corresponding chain map is not expected to be an isomorphism and so can provide an obstruction to Lagrangian pseudo-isotopy being a Hamiltonian isotopy. It would be an extremely interesting problem to find a nontrivial Lagrangian pseuo-isotopy, when there is.

One very interesting problem is to study the change of $HF(L, L')$ when the isotopy $\{L_t\}_{0\leq t\leq 1}$ of L' undergoes the process of losing the intersections to infinity. A model case to study will be the one of symplectic manifolds with contact type boundary and its proper Lagrangian submanifolds. In this case, the corresponding family of boundary Legendrian submanifolds will have intersections at a finite number of t's in (0,1) with L. In particular, it would be interesting to describe the change of $HF(\nu^*S_1, \nu^*S_2^t)$ at the time t_0, where the intersection pattern of $S_1 \cap S_2^t$ changes through a degenerate intersection. This will be a subject of future study.

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