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The Pitch and the Angle of Pitch of a Closed Nonnull Ruled Hypersurface Whose Generator is Spacelike in R_1^{k+2}

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Abstract

In this paper, the pitch and the angle of pitch of a closed nonnull ruled hypersurface whose generators are spacelike are calculated in R_1^{k+2}.

1. Introduction

A hypersurface in k+2- dimensional Minkowski space $R_1^{k+2} = (R^{k+2}, \sum_{i=1}^{k+1} dx_i - dx_{k+2})$ is called a spacelike (timelike) hypersurface if the induced metric tensor on the hypersurface is a positive definite Riemannian metric (Lorentz metric) [2].If the tangent vector at every point of a given curve in R_1^{k+2} is a spacelike (timelike) vector,then the given curve is called a spacelike(timelike) curve [6].

Let $\eta : I \to R_1^{k+2}$ be a spacelike curve in R_1^{k+2} where $I \subset R$, and let $\{e_1(t), e_2(t), \ldots, e_k(t)\}$ be a given orthonormal vector defined at each point $\eta(t)$. The set $\{e_1(t), e_2(t), \ldots, e_k(t)\}$ spans a space of the tangent space $T_{\eta(t)} R_1^{k+2}$ at the point $\eta(t)$ of R_1^{k+2}. Let us denote this space by $E_k(t)$. The set $M = \cup_{t \in I} E_k(t)$ is a hypersurface of R_1^{k+2}. A parametrization for this hypersurface is

$$\varphi : I \times R^k \to R_1^{k+2}, \varphi(t, v_1, \ldots, v_k) = \eta(t) + \sum_{i=1}^{k} v_i e_i(t) \tag{1}$$

if

$$rank(\varphi_t, \varphi_{v_1}, \ldots, \varphi_{v_k}) = rank(\eta'(t) + \sum_{i=1}^{k} v_i e_i'(t), e_1(t), \ldots, e_k(t)) = k+1$$

then the hypersurface M is said to be a ruled hypersurface, the space $E_k(t)$ is called the generator space of the ruled hypersurface at $\eta(t)$ and the curve η is called base curve of the ruled hypersurface. The line, whose director vector is $e_i(t)$, that passes through $\eta(t)$ is said to be i^{th} generator line of the hypersurface.

For real numbers $p > 0$, the condition $\eta(t+p) = \eta(t)$ is satisfied, then the surface is called a closed ruled hypersurface [4,p.518]. The smallest $p > 0$ satisfying $\eta(t+p) = \eta(t)$ is called the period of the closed ruled hypersurface. A curve which intersects each space $E_k(t)$ orthogonally is said to be an orthogonal trajectory of M.

Adapting the algorithm of [5] to R_1^{k+2}, if the initial basis $\{e_1(t_0), e_2(t_0), \ldots, e_k(t_0)\}$ is given, then the basis $\{e_1(t), e_2(t), \ldots, e_k(t)\}$ satisfying

$$< e_i(t), e_j(t) >= \epsilon_i \delta_{ij} \quad \text{and} \quad < e_i'(t), e_j(t) >= 0, \quad 1 \leq i, j \leq k \tag{2}$$

is uniquely determined, where we donete the derivative of vector field e_ν along the curve η by e_ν', and $<,>$ donetes the scalar prodoct in R_1^{k+2}.

In [3], H. Frank and O.Giering computed the k- number of the pitchs and the angles of pitch for a simple closed $C^2 - (k+1)$ dimensional ruled surface. That is not the generalization of the pitch and the angle of pitch. Desired generalization has been done in A.Altin [1].

In this paper,the results obtained by [1] is investigated in R_1^{k+2}.

2. The Pitch of a Closed Nonnull Ruled Hypersurface in R_1^{k+2}

Theorem 2.1. *Let M be a nonnull ruled hypersurface in R_1^{k+2}. There exists a unique orthogonal trajectory at each point of M.*

Proof. An orthogonal trajectory, if exists, is of the form:

$$\beta : I \to M, \quad \beta(s) = \eta(s) + \sum_{i=1}^{k} f_i(s) e_i(s) \tag{3}$$

where f is a function from I into R. □

Differentiating (3), we obtain

$$\beta'(s) = \eta'(s) + \sum_{i=1}^{k} f_i'(s)e_i(s) + \sum_{i=1}^{k} f_i(s)e_i'(s). \tag{4}$$

Since the curve β is orthogonal to generator space $E_k(s)$, we have

$$< \beta'(s), e_j(s) >= 0, \quad \text{for all} \quad j = 1, \ldots, k. \tag{5}$$

Replacing (4) in (5), we obtain

$$< \eta'(s), e_j(s) > + \sum_{i=1}^{k} f_i'(s) < e_i(s), e_j(s) > + \sum_{i=1}^{k} f_i(s) < e_i'(s), e_j(s) >= 0.$$

Using (2) leads to

$$< \eta'(s), e_j(s) > + \epsilon_j f_j'(s) = 0.$$

Thus we have

$$f_j(s) = -\int \epsilon_j < \eta'(s), e_j(s) > ds + c_j.$$

If we denote

$$-\int \epsilon_j < \eta'(s), e_j(s) > ds = F_j(s) \tag{6}$$

then we have

$$f_j(s) = F_j(s) + c_j \quad 1 \leq j \leq k. \tag{7}$$

Since c_j is chosen arbitrarily, there are many curves satisfying (5). However, there exists a unique orthogonal trajectory passing through each $p_0 \in M$ which is of the from

$$p_0 = \varphi(s_0, v_{1_0}, \ldots, v_{k_0}) = \eta(s_0) + \sum_{i=1}^{k} v_{i_0} e_i(s_0).$$

Definition 2.1. *Let M be a closed nonnull ruled hypersurface in R_1^{k+2} and let $\{e_1(t), e_2(t), \ldots, e_k\ (t)\}$ be the orthonormal frame of M at the point $\eta(t)$. Let β denote the orthogonal trajectory at $\eta(t)$. The distance between $\beta(t)$ and $\beta(t+p)$ is called the pitch of M.*

Theorem 2.2. *Assume that L_t is the pitch of a closed nonnull ruled hypersurface M whose generator η is a spacelike curve in R_1^{k+2}.*

If M is a spacelike then

$$L_t = \sqrt{L_{1_t}^2 + L_{2_t}^2 + \cdots + L_{k_t}^2}.$$

If M is a timelike then

$$L_t = \sqrt{\mid L_{1_t}^2 + L_{2_t}^2 + \cdots + L_{(k-1)_t}^2 - L_{k_t}^2 \mid}.$$

Proof. Let M be a closed spacelike ruled hypersurface in R_1^{k+2}, that is, e_i $(1 \leq i \leq k)$ is a spacelike vector field in R_1^{k+2}.

If we choose p_0 of Theorem 2.1 as $\eta(t)$ then we have $\nu_{i_t} = 0$, subsequently $f_i(t) = 0$ and thus $c_i = -F_i(t)$. From (7), for $s = t + p$, we have

$$f_i(t+p) = F_i(t+p) - F_i(t)$$

and

$$f_i(t+p) - f_i(t) = F_i(t+p) - F_i(t).$$

This, together with (6), implies that

$$f_i(t+p) - f_i(t) = -\int_t^{t+p} \epsilon_i < \eta'(s), e_i(s) > ds. \tag{8}$$

Now, let

$$f_i(t+p) - f_i(t) = L_{i_t} \quad 1 \leq i \leq k.$$

L_{i_t} is called the pitch on the i^{th} generator at the point $q(t)$. Clearly,

$$\beta(t)\vec{\beta(t+p)} = L_{1_t}e_1 + L_{2_t}e_2 + \cdots + L_{k_t}e_k. \tag{9}$$

The length of this vector renders the pitch of the closed spacelike ruled hypersurface M at $\eta(t)$. We have

$$L_t = \sqrt{L_{1_t}^2 + L_{2_t}^2 + \cdots + L_{k_t}^2}.$$

Let M be a closed timelike ruled hypersurface in R_1^{k+2}, that is the normal vector field of M is spacelike. Then one of the e_i $(1 \leq i \leq k)$ must be a timelike vector field. Assume that e_k is a timelike vector field.

From (9) we have

$$L_t = \sqrt{| \, L_{1_t}^2 + L_{2_t}^2 + \cdots + L_{(k-1)_t}^2 - L_{k_t}^2 \, |}.$$

The pitch L_t, by (8), is independent of the choice of orthogonal trajectory.

3. The Angle of Picth of a Closed Nonnull Ruled Hypersurface in R_1^{k+2}

Definition 3.1. *Let M be a closed nonnull ruled hypersurface in R_1^{k+2} and let $e_1(t), e_2(t), \ldots, e_k(t)$ be unit director vectors at the point $\eta(t)$ of the generator η. Let $e_{k+1}(t)$ denote the unit tangent vector of orthogonal trajectory at $\eta(t)$. The angle between $e_{k+1}(t)$ and $e_{k+1}(t+p)$ is called the angle of pitch of M where $e_{k+1}(t+p)$ is the tangent vector of the orthogonal trajectory at $\eta(t+p)$ and p is the period of the closed curve η.*

Suppose that M is a closed nonnull ruled hypersurface in R_1^{k+2}. Let $\{\overline{e}_1(t), \overline{e}_2(t), \ldots, \overline{e}_{k+2}(t)\}$ be an orthonormal frame at the initial point $\beta(t) = \eta(t)$. For all $s \in [t, t+p]$, an orthonormal frame $\{e_1(s), \ldots, e_{k+2}(s)\}$ at the point $\beta(s)$ can be written as

$$e_i(s) = \sum_{j=1}^{k+2} a_{ij}(s)\overline{e}_j, \qquad 1 \leq i \leq k+2. \tag{10}$$

Since β is an orthogonal trajectory, then, at $s = t+p$, we have

$$e_\mu(s) = \overline{e}_\mu, \qquad 1 \leq \mu \leq k$$

$$e_{k+1}(s) = a_{(k+1)(k+1)}(s)\overline{e}_{k+1} + a_{(k+1)(k+2)}(s)\overline{e}_{k+2} \tag{11}$$

$$e_{k+2}(s) = a_{(k+2)(k+1)}(s)\overline{e}_{k+1} + a_{(k+2)(k+2)}(s)\overline{e}_{k+2}$$

Since M is a closed spacelike ruled hypersurface whose generator η is spacelike in R_1^{k+2}, the orthogonal trajectory must be a spacelike curve. Thus, the angle between $\overline{e}_{k+1}(s)$ and $e_{k+1}(s+p)$ is defined.

Assume that M is a closed a timelike ruled hypersurface whose generator η is spacelike in R_1^{k+2}. Then the normal vector field of M is spacelike. Since e_k is a timelike vector field, $\overline{e}_{k+1}(s)$ and $e_{k+1}(s+p)$ must be two spacelike vectors.

Theorem 3.1. *Let M be a closed nonnull ruled hypersurface whose generator η is spacelike in R_1^{k+2}. The angle of pitch of M is*

$$\theta_t = -\int_t^{t+p} < e'_{k+1}(s), e_{k+2}(s) > ds.$$

Proof. We can divide the proof in two cases: □

1.case.

Let M be a closed spacelike ruled hypersurface whose η is spacelike in R_1^{k+2}. Thus, $e_i(s)$ and $\overline{e_i}$ $1 \le i \le k+1$ are spacelike vectors, and $e_{k+2}(s), \overline{e}_{k+2}$ are timelike vectors. Hence, $\{\overline{e}_1(t), \overline{e}_2(t), \ldots, \overline{e}_{k+2}(t)\}$ is the orthonormal frame at the initial point $\eta(t) = \beta(t)$ and the orthonormal frame is $\{e_1(s), e_2(s), \ldots, e_{k+2}(s)\}$ at the point $\beta(s)$ for all $s \in [t, t+p]$, where β is the orthogonal trajectory of M. Let us denote the angle between $\overline{e}_{k+1}$ and $e_{k+1}(s)$ by $\theta(s)$.From (11) we can choose

$a_{(k+1)(k+1)}(s) = ch\theta(s), \quad a_{(k+1)(k+2)}(s) = sh\theta(s),$

$a_{(k+2)(k+1)}(s) = sh\theta(s)$ and $a_{(k+2)(k+2)}(s) = ch\theta(s).$

Hence, we get

$$\begin{aligned} e_\mu(s) &= \overline{e}_\mu, \qquad 1 \le \mu \le k \\ e_{k+1}(s) &= ch\theta(s)\overline{e}_{k+1} + sh\theta(s)\overline{e}_{k+2} \\ e_{k+2}(s) &= sh\theta(s)\overline{e}_{k+1} + ch\theta(s)\overline{e}_{k+2}. \end{aligned} \tag{12}$$

Using (12) we have

$$e'_{k+1}(s) = e_{k+2}(s)\theta'(s) \quad \text{and} \quad -< e'_{k+1}(s), e_{k+2}(s) >= \theta'(s).$$

Thus,we have

$$\theta_t = \int_t^{t+p} d\theta = -\int_t^{t+p} < e'_{k+1}(s), e_{k+2}(s) > ds.$$

2.case.

Let M be a closed timelike ruled hypersurface whose η is spacelike in R_1^{k+2}. Thus, $e_k(s)$, and $\overline{e_k}$ are timelike vectors, and $e_i(s)$, and $\overline{e}_i$ $1 \leq i \leq k+2$ $(i \neq k)$ are spacelike vectors.

Hence, $\{\overline{e}_1(t), \overline{e}_2(t), \ldots, \overline{e}_{k+2}(t)\}$ is the orthonormal frame at the initial point $\alpha(t) = \beta(t)$ and the orthonormal frame is $\{e_1(s), e_2(s), \ldots, e_{k+2}(s)\}$ at the point $\beta(s)$ for all $s \in [t, t+p]$, where β is the orthogonal trajectory of M. Let us denote the angle between $\overline{e}_{k+1}$ and $e_{k+1}(s)$ by $\theta(s)$.From (11) we can choose

$a_{(k+1)(k+1)}(s) = cos\theta(s)$, $a_{(k+1)(k+2)}(s) = -sin\theta(s)$,

$a_{(k+2)(k+1)}(s) = sin\theta(s)$ and $a_{(k+2)(k+2)}(s) = cos\theta(s)$.

Hence, we get

$$\begin{aligned} e_\mu(s) &= \overline{e}_\mu, \qquad 1 \leq \mu \leq k \\ e_{k+1}(s) &= cos\theta(s)\overline{e}_{k+1} - sin\theta(s)\overline{e}_{k+2} \\ e_{k+2}(s) &= sin\theta(s)\overline{e}_{k+1} + cos\theta(s)\overline{e}_{k+2}. \end{aligned} \tag{13}$$

Using (13) we have

$$e'_{k+1}(s) = -e_{k+2}(s)\theta'(s) \quad \text{and} \quad -< e'_{k+1}(s), e_{k+2}(s) >= \theta'(s).$$

Thus,we have

$$\theta_t = \int_t^{t+p} d\theta = -\int_t^{t+p} < e'_{k+1}(s), e_{k+2}(s) > ds.$$

which completes the proof.

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