

New and Old Types of Homogeneity

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Abstract

We introduce new types of homogeneity ; namely : locally homogeneity and closed homogeneity .Several results are included discussing some relations between these types and the old ones. Some characterization and decomposition theorems are obtained. Relevant examples and counterexamples are discussed throughout this paper.

Key Words: Homogeneity, countable dense homogeneity, strong local homogeneity, representable, n-homogeneity, weakly n-homogeneity.

1. Introduction

If X is a topological space, then $H(X)$ denotes the group of all autohomeomorphisms on X . Recall that a topological space X is *weakly n -homogeneous* ($n \in \mathbb{N}$) if for any A and B two n -element subsets of X , there is a homeomorphism $h \in H(X)$ such that $h(A) = B$. A space X is *n -homogeneous* means that if $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ are two n element subsets of X then there is a homeomorphism $h \in H(X)$ such that $h(a_i) = b_i$ for all $i = 1, \dots, n$. A space X is called *homogeneous* if it is 1-homogeneous (equivalently, if it is weakly 1-homogeneous). Let \sim be the relation defined on X by $x \sim y$ if there is an $h \in H(X)$ such that $h(x) = y$. This relation turns out to be an equivalence relation

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on X whose equivalence classes C_x will be called *homogeneous components* determined by $x \in X$. C_x is indeed a homogeneous subspace of X . It is clear that X is homogeneous if and only if it has only one homogeneous component. The connected component of X determined by x will be denoted by K_x while the quasi-component will be denoted by Q_x .

The homogeneity concept was introduced by W. Sierpinski [11] in 1920. Seven years earlier, L.E.J. Brouwer [3] in his development of dimension theory had shown that if A and B are two countable dense subsets of the n -dimensional Euclidean space R^n , then there is an autohomeomorphism on R^n that takes A to B . Following R.B. Bennett [2], we use the term *countable dense homogeneous* (CDH) to denote a separable space provided it has this property. After J.W. Bales, a topological space X is called *SLH* (*strongly locally homogeneous*) at $p \in X$, respectively, *representable* at $p \in X$, provided that if $p \in U$ and U is open in X , then there is an open set V in X such that $p \in V \subseteq U$ and such that, if $q \in V$, there is $h \in H(X)$ such that $h(p) = q$ and h is the identity on $X - V$, respectively, $X - U$. A topological space X is called SLH, respectively, representable, provided it is SLH, respectively, representable, at each p in X . J.W. Bales [1] showed that a topological space X is SLH if, and only if, X is representable. A topological space X is called *bihomogeneous* if for any x, y in X there is an $h \in H(X)$ such that $h(x) = y$ and $h(y) = x$. Finally, a space X is called *1/n-homogeneous* (see [10]) if the action on X of the group $H(X)$ has exactly n orbits, i.e. if there are n subsets A_1, \dots, A_n of X such that $X = \bigcup_{i=1}^n A_i$ and, for any $x \in A_i, y \in A_j$, there is an $h \in H(X)$ mapping x to y if and only if $i = j$. In fact, one can easily see that X is 1/n-homogeneous if and only if X has precisely n homogeneous components.

If X is a subset of the reals R , then $\tau_{l.r.}, \tau_u, \tau_{dis}, \tau_{ind}$ will denote the left-ray, the usual (Euclidean) subspaces of R , the discrete and the indiscrete topologies on X , respectively. $|Y|$ denotes the cardinality of the set Y .

2. Locally Homogeneous Spaces

Let us start this section with the following definition.

Definition 2.1. *A topological space X is called LH (locally homogeneous) at p in X*

provided there exists an open set U in X containing p such that for any $q \in U$ there is $h \in H(X)$ such that $h(p) = q$. A topological space X is called an LH space if it is locally homogeneous at each p in X .

It is clear that every SLH at p must be LH at p . The following results clarify our new notion (their proofs are omitted because they are straightforward).

Proposition 2.2. *A topological space X is locally homogeneous at p if and only if C_p is open in $X(p \in X)$.*

Corollary 2.3. *A topological space X is locally homogeneous if and only if each C_p is clopen (i.e. closed and open,, simultaneously) in X .*

(Remember that the collection $\{C_p : p \in X\}$ forms a partition on X).

Using Corollary 2.3, one can notice that “ $K_x \subseteq Q_x \subseteq C_x$ for every $x \in X$ ” holds in any locally homogeneous space X .

Proposition 2.4 *i) Every homogeneous space is locally homogeneous.
ii) Every strongly locally homogeneous space is locally homogeneous.*

Proposition 2.5. *A connected LH space must be homogeneous.*

Proposition 2.5 shows the good behavensness of LH spaces like other types of homogeneity (see [4]). It is worth noticing that an LH space which is not homogeneous can not have the fixed point property because such a space is disconnected.

Example 2.6. *The Sierpinski space $X = \{0, 1\}$, $\tau_{l.r.} = \{\phi, X, \{0\}\}$ is clearly LH at 0 because $C_0 = \{0\}$ is open in X , but not LH at 1. Notice that $C_1 = \{1\}$ is not open in X . Clearly, X is 1/2-homogeneous. It can be easily seen that X is not SLH at 1.*

As one may notice that X ; in Example 2.6; has a small cardinality. One can still have a similar result with higher separation axioms (but; of course; X should have greater cardinality than before).

Example 2.7. *The usual Euclidean space $X = [0, 2)$ is clearly an LH space at each $x \in (0, 2)$ but not LH at 0. It is also clear that X is 1/2-homogeneous and not SLH at 0.*

Using Corollary 2.3 one can get the following result.

Proposition 2.8. *If X is a locally homogeneous space and $x \in X$, then there exists an open set V in X containing x such that for any y in V , there is $h \in H(X)$ such that $h(x) = y$ and $h(t) = t$ for all $t \in X - V$.*

(Indeed we may take $V = C_x$ which is clopen in X .)

Although Proposition 2.8 implies that LH spaces are close to SLH spaces but the fact says that they are still (indeed) very far from them. To see this more closely, we have the following illustration.

Example 2.9. *There exists a connected CDH space which is not an SLH space. The example (\mathbb{R}^2, Γ') of Fitzpatrick and Zhou [5], and the “0-angle” space of Watson and Simon [12] can serve this purpose. The latter has the advantage of being regular. According to [4], both spaces are homogeneous and hence LH spaces.*

The following decomposition theorem is our next main goal.

Proposition 2.10. *Every locally connected LH space X can be uniquely partitioned as a disjoint union of homogeneous connected clopen subspaces.*

Proof. Since X is a locally homogeneous space, therefore each C_p is a clopen set in X . Since X is locally connected, so the connected components $K_{x,p}$ of C_p are themselves clopen in X . It is not difficult to check that our required decomposition is composing of $\{K_{x,p} : x \in C_p, p \in X\}$. \square

It is important to notice that if we release the connectedness property on subspaces, we may end up without uniqueness property as in the case of (\mathbb{R}, τ_{dis}) .

3. Old Types of Homogeneity

We start this section with the following result.

Lemma 3.1. *Let X be a topological space and let C_p be any homogeneous component of X such that $\text{Int } C_p \neq \emptyset$. Then C_p is open in X .*

Proof. Let $x \in C_p$. Since $\text{Int } C_p \neq \emptyset$, therefore there exists $t \in \text{Int } C_p$ which implies that there exists an open set V in X such that $t \in V \subseteq C_p$. Since $x, t \in C_p$, therefore

there exists $h \in H(X)$ such that $h(t) = x$. It is easy to observe that $h(C_p) = C_p$ and hence $x \in h(V) \subseteq C_p$. This completes the proof that C_p is an open set in X . \square

Proposition 3.2. *Let X be a $1/2$ -homogeneous space and let C_p, C_q be the two homogeneous components of X . Then we have one of the following two cases: (i) either one of C_p, C_q is an open set and the other is closed; or (ii) both C_p and C_q are dense sets in X without any interior point.*

Proof. Suppose (i) does not hold. Then both C_p and C_q are not open sets. Consequently, by Lemma 3.1, $\text{Int } C_p = \text{Int } C_q = \phi$ and hence $\text{Int } (X - C_p) = \text{Int } (X - C_q) = \phi$ which yields that both C_p and C_q are dense sets in X . \square

Example 2.6 shows that case (i) in Proposition 3.2 is possible. However, the next example shows that case (ii) is still possible.

Example 3.3. *By letting $X = [0, \infty)$, the space $(X, \tau_{l,r})$ (in this case; of course; $\tau_{l,r} = \{\phi, X, [0, a) : a > 0\}$) has the following property: $C_1 = (0, \infty)$ and $C_0 = \{0\}$. In fact, $(X, \tau_{l,r})$ is a $1/2$ -homogeneous space in which case (ii) of Proposition 3.2 holds.*

The following result is a nice characterization of n -homogeneous spaces.

Proposition 3.4. *A topological space X is n -homogeneous if and only if for any x, y in X and for any $(n - 1)$ -element subset $A \subseteq X - \{x, y\}$, there is a homeomorphism $h \in H(X)$ such that $h(x) = y$ and h is the identity on A .*

Proof. (\Rightarrow) Let X be an n -homogeneous space and let $x, y \in X$. Let $A = \{x_1, \dots, x_{n-1}\}$ be an $(n - 1)$ -element subset of $X - \{x, y\}$. Since $A_1 = A \cup \{x\}$, $A_2 = A \cup \{y\}$ are two n -element subsets of the n -homogeneous space X , therefore there exists $h \in H(X)$ such that $h(x_i) = x_i$ for all $i = 1, \dots, n - 1$ and $h(x) = y$

(\Leftarrow) Conversely, let X be a topological space such that for any x, y in X and for any $(n - 1)$ -element subset A of $X - \{x, y\}$, there is a homeomorphism $h \in H(X)$ such that $h(x) = y$ and h is the identity on A . To prove X is n -homogeneous, let $B_1 = \{x_1, \dots, x_n\}$, $B_2 = \{y_1, \dots, y_n\}$ be any two n -element subsets of X . We may

first assume that $B_1 \cap B_2 = \phi$. By successive application of the hypothesis, there exist $h_i \in H(X)$ ($i = 1, \dots, n$) such that $h_1(x_1) = y_1$ and $h_1(x_i) = x_i$ for $i = 2, \dots, n$; $h_2(x_2) = y_2$ and $h_2(y_1) = y_1$, $h_2(x_i) = x_i$ for $i = 3, \dots, n$; $h_3(x_3) = y_3$ and $h_3(y_i) = y_i$ ($i = 1, 2$), $h_3(x_j) = x_j$ for $j = 4, \dots, n$; ...; $h_n(x_n) = y_n$ and $h_n(y_i) = y_i$ for $i = 1, 2, \dots, n-1$. Letting $h = h_n \circ \dots \circ h_3 \circ h_2 \circ h_1$, then $h \in H(X)$ and satisfying the property that $h(x_i) = y_i$ for each $i = 1, \dots, n$. In the case $B_1 \cap B_2 \neq \phi$, choose $B_3 \subseteq X$ such that $|B_3| = n$ and $B_3 \cap (B_1 \cup B_2) = \phi$. Write $B_3 = \{z_1, \dots, z_n\}$. Then; by what we had just proved; since $B_1 \cap B_3 = \phi$ therefore there exists $\varphi_1 \in H(X)$ such that $\varphi_1(x_i) = z_i$ for $i = 1, 2, \dots, n$. Similarly, since $B_3 \cap B_2 = \phi$ therefore there exists $\varphi_2 \in H(X)$ such that $\varphi_2(z_i) = y_i$ for $i = 1, 2, \dots, n$. It follows that $\varphi_2 \circ \varphi_1 = \varphi \in H(X)$ and $\varphi(x_i) = y_i$ for $i = 1, 2, \dots, n$. Henceforth, X is n -homogeneous. \square

Notice that according to Proposition 3.16, if X is finite then (X, τ) can not be 2-homogeneous unless τ is the discrete or the indiscrete topology. Thus; in Proposition 3.4; we may always assume that X has a large cardinality.

To obtain our next result we need the following definition.

Definition 3.5. *A topological X is said to be closed-homogeneous provided that for any x, y in X and for any K closed subset of $X - \{x, y\}$, there is $h \in H(X)$ such that $h(x) = y$ and h is the identity on K . If we add the condition that $h(y) = x$, we get the definition of a closed-bihomogeneous space. Similarly, we can have the definitions of compact-homogeneous and finite-homogeneous spaces.*

Using Proposition 3.4 one can have the following result.

Proposition 3.6. *A topological space X is finite-homogeneous if and only if X is n -homogeneous for all $n \in \mathbb{N}$.*

The following result is easy to observe.

Proposition 3.7. *(i) If X is a T_1 closed-homogeneous space then X is finite-homogeneous.
(ii) If X is a T_2 closed-homogeneous space then X is compact-homogeneous.
(iii) Let X be a compact T_2 -space. Then in X , closed-homogeneity is equivalent to compact-homogeneity.*

(iv) If X is closed-homogeneous (compact-homogeneous, finite-homogeneous; respectively) then for any K closed (compact, finite; respectively) subset of X ; $X-K$ is a homogeneous space.

Proposition 3.8. *Every zero-dimensional T_0 -homogeneous space X is closed-bihomogeneous.*

Proof. Let $x, y \in X$ and let K be any closed subset of $X - \{x, y\}$. If $x = y$ then the identity mapping serves our purpose. But, if $x \neq y$, then there are two disjoint clopen sets V_1, V_2 in X such that $x \in V_1, y \in V_2$ and $K \cap (V_1 \cup V_2) = \emptyset$. Since X is a homogeneous space, there exists $h \in H(X)$ such that $h(x) = y$. Now, one can find two clopen sets U_1, U_2 such that $x \in U_1 \subseteq V_1, y \in U_2 \subseteq V_2$ and $h(U_1) = U_2$. Now define $\varphi : X \rightarrow X$ by $\varphi(t) = h(t)$ if $t \in U_1, \varphi(t) = h^{-1}(t)$ if $t \in U_2$; and $\varphi(t) = t$ if $t \in X - (U_1 \cup U_2)$. Then φ is clearly a homeomorphism on X satisfying the property that $\varphi(x) = y$ and $\varphi(y) = x$. Moreover, φ is the identity on an open neighbourhood of K . \square

The following result is easy to observe as a direct conclusion of Proposition 3.7 and 3.8

Corollary 3.9. *If the one points compactification \hat{X} of a space X is a homogeneous 0-dimensional T_1 space then X itself is a finite-homogeneous space (in particular, it is bihomogeneous).*

To present our next result we need the following proposition.

Proposition 3.10. *Every CDH space with countably many points is discrete.*

Proof. Let X be a CDH space and assume that X is a nonempty countable set. Let D be any dense subset of X . Since X, D are both countable dense sets in the CDH space X , therefore there exists $h \in H(X)$ such that $h(X) = D$. The fact that h is onto will imply that $D = X$. Now, since the only dense set in X is X itself, it follows that X has the discrete topology (indeed, for any $x \in X, X - \{x\}$ is not dense in X , so $cl(X - \{x\}) = X - \{x\} = X - \{x\}$, i.e. $X - \{x\}$ is a closed set in x which implies that $\{x\}$ is open in X). \square

The following examples illustrate the powerfullness of our last theorem.

Examples 3.11. (i) A 0-dimensional T_0 -homogeneous space need not be CDH as in the rational space (Q, τ_u) . (See Proposition 3.10),

(ii) The Euclidean space (R, τ_u) is a homogeneous metrizable space which is not even 3-homogeneous. This shows that “0-dimensionality” is heavily needed in Proposition 3.8.

(iii) To show the usefulness of T_0 in our Proposition 3.8, let $X = \{1, 2, 3, 4\}$, $\beta = \{\{1, 2\}, \{3, 4\}\}$. Then the topological space $(X, \tau(\beta))$; generated by the base β ; is indeed a 0-dimensional homogeneous space which is not even weakly 2-homogeneous (for if $A = \{1, 2\}$ and $C = \{1, 3\}$, then there is no $h \in H(X)$ such that $h(A) = C$).

It is important to notice that SLH and closed-homogeneity are two different notions as the following example illustrates.

Example 3.12. There exists an SLH space X which is not closed-homogeneous. In fact, if we take $X = R - \{0\}$, $\beta = \{\{x\} : x < 0\} \cup \{(a, b) : 0 < a < b\}$. Then the topological space $(X, \tau(\beta))$; generated by the base β ; is indeed an SLH metrizable space which is not closed-homogeneous. In fact, X is not even homogeneous.

The following result shows that SLH is a weaker notion than closed-homogeneity.

Proposition 3.13. Every closed-homogeneous space X is SLH.

Proof. Let U be any open set in X and $x \in U$. Taking $V = U$, and letting any $y \in V$. Since $X - V$ is a closed set and $X - V \subseteq X - \{x, y\}$, therefore there exists $h \in H(X)$ such that $h(x) = y$ and h is the identity on $X - V$. \square

In the lights of Proposition 3.13 and Example 3.12, our result; Proposition 3.8; is considered as a strengthen of Theorem 5 of [6] (see page 29) and the last apparently was first observed by J. Van Mill [9].

It is known (see [4]) that components of CDH spaces are CDH and are, if nontrivial, open sets. It is easy to verify (see [7], page 3) that the components of a CDH space are the quasi components of the space. Concerning CDH spaces, we have the following result.

Proposition 3.14. (i) If X is a CDH space then X has at most countably many homogeneous components.

(ii) Every CDH compact space is $1/n$ -homogeneous for some $n \in \mathbb{N}$.

Proof. (i) Since X is a CDH space, so it is separable, i.e., X has a countable dense subset $D \subseteq X$. We claim that $X = \cup\{C_p : p \in D\}$. To prove our claim, let $t \in X$. Since $D \cup \{t\}$ and D are both countable dense subsets of X , there exists $h \in H(X)$ such that $h(D) = D \cup \{t\}$. It follows that there exists $q \in D$ such that $h(q) = t$. Hence $t \in C_q$.

(ii) Since $\phi = \{C_p : p \in D\}$ is an open cover for X according to Theorem 2.5 of [7], therefore ϕ has a finite minimal subcover $\phi' = \{C_j : j = 1, \dots, n\}$. Consequently, X is a $1/n$ -homogeneous space. \square

Fora and Al-Bsoul proved in their paper [8] the following result.

Theorem 3.15. *Let X be a topological space which contains a nonempty open indiscrete subset (in the induced topology). Then the following are equivalent:*

- (i) X is a homogeneous space.
- (ii) X is a disjoint union of indiscrete topological spaces all of which are homeomorphic to one another.

A topological space satisfying Theorem 3.15 will be called a *homogeneous partition space* because it has a base consisting of a partition for X . It is now the time to state our last result concerning n -homogeneous partition spaces.

Proposition 3.16 *Let (X, τ) be a homogeneous partition space. Then the following are equivalent:*

- (i) (X, τ) is n -homogeneous for all $n \geq 3$.
- (ii) (X, τ) is 2-homogeneous.
- (iii) (X, τ) is weakly 2-homogeneous.
- (iv) $\tau = \tau_{dis}$ or $\tau = \tau_{ind}$.

Proof. The implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are easy by assuming X to have enough number of elements.

To prove (iii) \Rightarrow (iv), suppose not, i.e, that (X, τ) is weakly 2- homogeneous and $\tau_{ind} \neq \tau \neq \tau_{dis}$. Then there exists U a basic open set such that $|U| > 1$ and there exists also V a nonempty open set such that $U \cap V = \emptyset$ (in fact U and V are homeomorphic).

Let $a, b \in U$ and $c \in V$, where $a \neq b$. Now, it is impossible to find $h \in H(X)$ such that $h(\{a, b\}) = \{a, c\}$. Hence (X, τ) is not weakly 2-homogeneous, a contradiction.

The implication (iv) \Rightarrow (i) is very clear. □

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