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Remarks on the Paper "on the Commutant of the Ideal Centre"

Şafak Alpay, Bahri Turan

In memory of Yunus Aran (1976 - 2000)

Abstract

We continue with the work started in [4] and give a new sufficient condition on Riesz spaces having topologically full centres for $Z^{\sim}(E)_C = Orth(E^{\sim})$ to hold.

If E is a Riesz space $E^{\sim}$, the order dual of E will be the Riesz space of all order bounded linear functionals on E. Riesz spaces considered in this note are assumed to have separating order duals. $Z(E)$ will denote the ideal centre, Orth (E) , will denote the orthomorphisms of E. If E is a topological Riesz space E' will denote continuous dual of E. When $T : E \to F$ is an order bounded operator between two Riesz spaces, the adjoint of T carries $F^{\sim}$ into $E^{\sim}$ and it will be denoted by $T^{\sim}$. In all undefined terminology concerning Riesz spaces we will adhere to the definitions in [1], [5] and [8].

When the order dual $E^{\sim}$ separates the points of the Riesz space E, an order bounded operator $T : E \to E$ is an orthomorphism if and only if its adjoint $T^{\sim} : E^{\sim} \to E^{\sim}$ is an orthomorphism. Moreover, the operator $\psi : Orth(E) \to Orth(E^{\sim}); \psi(T) = T^{\sim}$ is a one to one Riesz homomorphism [1]. The image under ψ of the centre $Z(E)$ will be denoted by $Z^{\sim}(E).Z^{\sim}(E)$ is a Riesz subspace of $Z(E^{\sim})$.

Definition *A Riesz space E, is said to have topologically full centre if, for each pair x, y in E with $0 \leq y \leq x$, there exists a net (π_α) in $Z(E)$ with $0 \leq \pi_\alpha \leq I$ for each α, such that $\pi_\alpha x \to y$ in $\sigma(E, E^{\sim})$.*

Banach lattices with topologically full centre were initiated in [7]. These spaces were also studied in [2],[3], [4] and [6]. The class of Riesz spaces and the class of Banach spaces have topologically full centres are quite large. σ-Dedekind complete Riesz spaces have topologically full centres. However, not all Riesz spaces have topologically full centres.

Order bounded maps on the Riesz space E will be denoted by $L_b(E).Z(E)_C$ will denote the commutant of $Z(E)$ in $L_b(E)$. That is, $Z(E)_C = \{T \in L_b(E) : T\pi = \pi T$ for each $\pi \in Z(E)\}$. The Riesz space Orth (E) under composition is an Archimedean f-algebra and therefore it is commutative. Hence Orth $(E) \subset Z(E)_C$.

We have studied the commutant $Z(E)_C$ of the ideal centre $Z(E)$ in the order bounded operators $L_b(E)$ [4]. If E is a Riesz space with topologically full centre, we have identified $Z(E)_C$ with Orth (E).

If E has topologically full centre, it was claimed that $Z^{\sim}(E)_C =$ Orth $(E^{\sim})$. However, as Arenson has pointed out, part of the proof of this claim contains an error. If $E = C(K)$, then we can embed E' into $E''' = C(K)'''$ in two different ways. One of these embeddings is the usual embedding of a Banach space into its bidual as: $\mu \in E' \to \hat{\mu} \in E'''$. Let $\hat{E}'$ denote the image of E' in E'''. For $\psi \in E'''$, we consider $\mu = \psi\mid_E \in \hat{E}'$. For each $\psi \in E''', \psi - \hat{\mu} \in E^o \subset E'''$ with $\mu = \psi \mid_E$. Thus, $\psi = (\psi - \hat{\mu}) + \hat{\mu}$ implies that $E''' = \hat{E}' \oplus E^o$. The correspondence $\psi \to \hat{\mu}$ is a positive operator which fails to be a lattice homomorphism.

On the other hand, $\hat{E}'$ can be identified with the space of order continuous linear functionals on $E'' = C(K)''$. Consequently, $\hat{E}'$ is a band in E''' and there exists an order projection $P : E''' \to \hat{E}'.P$ is an orthomorphism and $E''' = \hat{E}' \oplus (I - P)E'''$. However, $E^o \neq (I - P)E'''$ and $P\psi \neq \psi \mid_E$ as it was erroneously claimed in [4].

The next example of Arenson's (private communication) explains the situation even better.

Example: (Arenson) Let K be a compact Hausdorff space with no isolated points and E be $C(K)$. Then $Z(E) = E$ and $E^{\sim} = Z(E)'$ is the space of measures on K. If Q is the Stone compact space of the Banach lattice $Z(E)'$, we identify $Z(E^{\sim})$ with $C(Q)$. Since $Z(E)$ and $Z^{\sim}(E)$ are isometrically isomorphic, we are able to identify $Z^{\sim}(E)$ with $C(K)$.

Let us note that $Z(E^{\sim}) = C(Q)$ and $Z^{\sim}(E)'' = C(K)''$. Therefore we have $Z(E^{\sim})' = C(Q)' = C(K)''' = Z^{\sim}(E)'''$. Let j be the natural embedding of $Z^{\sim}(E)' = C(K)'$ into $C(K)''' = Z(E^{\sim})'$ and let $H_1 = j(Z^{\sim}(E)'), H_2 = H_1^d.H_1$ is a band of $Z(E^{\sim})'$ as $Z^{\sim}(E)'$ is an AL-space. Therefore $Z(E^{\sim})' = C(Q)' = H_1 \oplus H_2$. It is well known that H_1 is the class of order continuous functionals on $C(Q)$ and therefore:

(1) If $\mu \in H_1$ then the support of μ is a closed and open subset of Q;

(2) If the support of $\mu \in C(Q)'$ is nowhere dense then $\mu \in H_2$.

Under this circumstances $\{Z^{\sim}(E)^0\}^d = \{0\}$ and $P = 0$. To see this, let $S(\mu) = j(\mu\mid_{Z^{\sim}(E)})$ $S : H_2 \to H_1$ be the restriction map. If ϑ is a nonzero measure in H_2 then the measure $\mu = \vartheta - S(\vartheta)$ is in $Z^{\sim}(E)^0$ and $|\mu| \wedge |\vartheta| = |\vartheta| \neq 0$. Therefore $P(\vartheta) = 0$. If μ is a non-zero measure in H_1, then by the following lemma, there exists a measure $\vartheta \in H_2$ with $S(\vartheta) = \mu$. The measure $\eta = \vartheta - \mu$ is an element of $Z^{\sim}(E)^0$ and $|\eta| \wedge |\mu| = |\mu| \neq 0$.

Therefore $P(\mu)=0$.

Let us note that if Q_1 is a nowhere dense closed subset of Q then $C(Q_1)'$ (considered as the space of measures on Q whose supports are contained in Q_1) is contained in H_2. To complete the proof of $(Z^{\sim}(E)^{\circ})^d=0$ we only need to prove the following lemma.

Lemma 1. *There exists a nowhere dense closed subset Q_1 of Q such that $S(C(Q_1)')=H_1$.*

Proof. Let $\varphi: Q\to K$ be the continuous surjection which gives rise the natural embedding $\pi\to\pi\cdot\varphi$ of $C(K)$ into $C(Q)$.

For each $t\in K$, let δ_t be the point evaluation at t. i.e, δ_t is : $\pi\to\pi(t)$ on $C(K)$. Similarly, for each $q\in Q$, let Δ_q be the functional $\pi\to\pi(q)$ on $C(Q)$. If $t=\varphi(q)$, then $\Delta_q\mid_{C(K)}=\delta_t$.

For each $t\in K$, there is a unique point in Q, say $\psi(t)$, such that $j(\delta_k)=\Delta_{\psi(t)}.\psi(t)$ is an isolated point of Q and $\psi: K\to Q$ is discontinuous and maps K onto an open subset $V=\psi(K)$ of Q. Let $Q_1=\overline{V}\backslash V$. Q_1 is nowhere dense and closed. To prove the lemma, it suffices to show that $\varphi(Q_1)=K$. Let $t\in K$. As there are no isolated points in K, there exists a net $\{t_a\}, t_\alpha\neq t$ for each α in K with $t=\lim_{\alpha} t_\alpha$. Let q be a cluster point of the net $\{\psi(t_\alpha)\}$. Then $q\in Q_1$ and $\varphi(q)=t$ as $t_\alpha=\varphi\{\psi(t_\alpha)\}$ for each α. □

Let us mote that the conclusion $Z^{\sim}(E)_C=Orth(E^{\sim})$ remains valid for Arenson's example. The details are below.

We now give a sufficient condition for $Z^{\sim}(E)_C=Orth(E^{\sim})$. We first give a lemma that will be needed.

Lemma 2. *Let E be a Riesz space with topologically full centre and satisfying $(E^{\sim})^{\sim}=(E^{\sim})^{\sim}_n$. Then the bilinear map*

$$(f,F)\to\psi_{f,F} \text{ of } E^{\sim}\times(E^{\sim})^{\sim}\to Z^{\sim}(E)^{\sim} \text{ defined by } \psi_{f,F}(\tilde{\pi})=F(\tilde{\pi}f)$$

is a bi-lattice homomorphism.

Proof. For each $f\in E^{\sim}_{+}$, the map $\psi_f:(E^{\sim})^{\sim}\to Z^{\sim}(E)^{\sim}$ defined by $F\to\psi_{f,F}$ is positive. Hence we have $\psi_f(F)^+\leq\psi_f(F^+)$ for each $F\in(E^{\sim})^{\sim}$. Let $\tilde{\pi}\in Z^{\sim}(E)_+$ be arbitrary, then

$$\psi_f(F^+)(\tilde{\pi})=\psi_{f,F^+}(\tilde{\pi})=F^+(\tilde{\pi}f)=\sup\{F(g):0\leq g\leq\tilde{\pi}f\}$$

If $0\leq g\leq\tilde{\pi}f$, we claim there exists $\{\pi_\alpha\}$ in $Z(E)$ satistying $0\leq\pi_\alpha\leq I$ for each α and $\tilde{\pi}_\alpha(\tilde{\pi}f)\to g$ in $\sigma(E^{\sim},(E^{\sim})^{\sim})$. As $E^{\sim}$ is Dedekind complete, we can find $S\in Z(E^{\sim})$

with $0 \leq S \leq I$ and $S(\tilde{\pi} f) = g$. The Arens homomorphism $m : Z(E)'' \to Z(E^{\sim})$ is surjective and continuous when the domain is equipped with $\sigma(Z(E)'', Z(E)')$ and the range has the $\sigma(E^{\sim}, (E^{\sim})^{\sim}_{n})$ operator topology [6]. Therefore there exists F in $Z(E)''$ with $0 \leq F \leq I$ satisfying $m(F) = S$. Using the fact that $Z(E)$ is $\sigma(Z(E)'', Z(E)')$ dense in $Z(E)''$, we can find a net $\{\pi_\alpha\}$ in $Z(E)$ satisfying $0 \leq \pi_\alpha \leq I$ for each α and $\pi_\alpha \to F$ in $\sigma(Z(E)'', Z(E)')$. Continuity of the map $m : Z(E)'' \to Z(E^{\sim})$ imply that $m(\pi_\alpha) = \tilde{\pi}_\alpha \to m(F) = S$ in $\sigma(E^{\sim}, (E^{\sim})^{\sim}_{n})$ operator topology. This is to say $G(\tilde{\pi}_\alpha h) \to G(Sh)$ for each $h \in E^{\sim}$ and $G \in (E^{\sim})^{\sim}_{n}$. Thus $\tilde{\pi}_\alpha(\pi f) \to g$ in $\sigma(E^{\sim}, (E^{\sim})^{\sim}_{n})$.

$0 \leq \tilde{\pi}_\alpha(\tilde{\pi} f) \leq \tilde{\pi}(f)$ for each α, so that $F(\tilde{\pi}_\alpha(\tilde{\pi} f)) \leq \psi_f(F)^{+}(\tilde{\pi})$
which yields

$F(g) \leq \psi_f(F)^{+}$ for each g with $0 \leq g \leq \tilde{\pi} f$. Hence $\psi_f(F^{+}) \leq \psi_f(F)^{+}$.

We now show that $\psi_F : E^{\sim} \to Z^{\sim}(E)^{\sim}$ is a lattice homomorphism for an arbitrary F in $(E^{\sim})^{\sim}_{+}$. Let $f \wedge g = 0$ in $E^{\sim}$. As I is a strong order unit in $Z^{\sim}(E)$, it suffices to show $[\psi_F(f) \wedge \psi_F(g)](I) = 0$.

$$\begin{aligned}
[\psi_F(f) \wedge \psi_F(g)](I) &= (\psi_{f,F} \wedge \psi_{g,F})(I) \\
&= inf\{\psi_{f,F}(\pi_1) + \psi_{g,F}(\pi_2) : \pi_1, \pi_2 \in Z^{\sim}(E)_{+}; \pi_1 + \pi_2 = I\} \\
&= inf\{F(\pi_1 f) + F(\pi_2 g) : \pi_1, \pi_2 \in Z^{\sim}(E)_{+}; \pi_1 + \pi_2 = I\}
\end{aligned}$$

As $E^{\sim}$ is Dedekind complete, the principal band generated by f, B_f is a projection band and let $P_f : E^{\sim} \to B_f$ be this projection. $P_f \in Z(E^{\sim}), P_f(g) = 0, (I - P_f)(f) = 0$ and $(I - P_f) + P_f = I$. Arguing as above, we can find a net (π_α) in $Z(E), 0 \leq \pi_\alpha \leq I$ and $\tilde{\pi}_\alpha \to P_f$ in $\sigma(E^{\sim}, (E^{\sim})^{\sim}_{n})$ operator topology.

Thus,

$$\begin{aligned}
[\psi_F(f) \wedge \psi_F(g)](I) &\leq F(I - \tilde{\pi}_\alpha) f + F(\tilde{\pi}_\alpha g) \text{ for each } \alpha \\
&\leq F(I - P_f) f + F(P_f g) = 0.
\end{aligned}$$

Proposition. *Let E be a Riesz space with $(E^{\sim})^{\sim} = (E^{\sim})^{\sim}_{n}$ and having topologically full centre. Then $Z^{\sim}(E)_C = Orth(E^{\sim})$.*

Proof. Let $T \in Z^{\sim}(E)_C$ be arbitrary; let $f, g \in E^{\sim}$ satisfying $f \perp g$. For each F, G in $(E^{\sim})^{\sim}$, we have $\psi_{f,F} \perp \psi_{g,G}$ [3].

Thus for $f \in E^{\sim}$ and $F \in (E^{\sim})^{\sim}$,

$$\psi_{Tf,F}(\tilde{\pi}) = F(\tilde{\pi}(Tf)) = F(T(\tilde{\pi} f)) = \tilde{T}(F)(\tilde{\pi} f) = \psi_{f,\tilde{T} f}$$

which yields $|\psi_{Tf,F}| \wedge |\psi_{g,F}| = \psi_{|Tf| \wedge g,F} = 0$. Therefore $F(|Tf| \wedge |g|) = 0$ for each $F \in (E^{\sim})^{\sim}_{+}$ which gives $Tf \perp g$ and T is an orthomorphism. □

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Şafak ALPAY
Department of Mathematics
Middle East Technical University
Ankara-TURKEY

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Bahri TURAN
Department of Mathematics,
Faculty of Sciences,
Gazi University, Beşevler-ANKARA