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On Conjugation in the Mod-p Steenrod algebra

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Abstract

In this paper we prove a formula involving the canonical anti-automorphism χ of the mod-p Steenrod algebra.

Key Words: Steenrod algebra, anti-automorphism, Milnor basis

1. Introduction and Main Result

Let $\mathcal{A}$ be a mod-p Steenrod algebra. Let $R = (r_1, r_2, \dots)$ be a sequence of nonnegative integers with finitely many nonzero terms. Let $\mathcal{P}(R)$ denote the corresponding Milnor basis element in $\mathcal{A}$ so that the elements $\mathcal{P}(R)$ form an additive basis for the subalgebra A_p of $\mathcal{A}$ generated by the Steenrod powers $\mathcal{P}^i$, $i \geq 0$. We define $|R| = \sum_{i=1}^{\infty} \left(p^i - 1\right) r_i$ and $e(R) = \sum_{i=1}^{\infty} r_i$. Thus, considered as a mod-p cohomology operation, $\mathcal{P}(R)$ raises the dimension of a cohomology class by $2\,|R|$ and has excess $2e(R)$. The anti-automorphism χ of A_p plays a fundamental part in our argument, and we find it convenient to write

$$\widehat{\theta} = (-1)^{dim\theta} \chi(\theta)$$

for every element $\theta \in A_p$.

We are interested in an explicit conjugation formula for the Steenrod operations of A_p in the form

$$X(k,n) = \mathcal{P}(p^k n)\mathcal{P}(p^{k-1}n) \cdots \mathcal{P}(pn)\mathcal{P}(n),$$

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where k and n are nonnegative integers. So the following formula is the mod-p analogue of Theorem 3.1 in [6].

Theorem 1.1 *For all positive integers j and i, we have*

$$\widehat{X}(j, p^{i+1} - 1) = X(i, p^{j+1} - 1).$$

We will introduce the following useful notation: each natural number a has a unique p-adic expansion

$$a = \sum_{i=0}^{\infty} \alpha_i(a) p^i$$

with $0 \leq \alpha_i(a) < p$. It is a fact that

$$\binom{a}{b} \equiv \prod_{i=0}^{\infty} \binom{\alpha_i(a)}{\alpha_i(b)}. \tag{1}$$

Using Davis' method [1] we can derive the following formulae.

Proposition 1.2

$$\mathcal{P}(u) \cdot \widehat{\mathcal{P}}(v) = \sum_R \binom{|R| + e(R)}{pu} \mathcal{P}(R) \tag{2}$$

$$\widehat{\mathcal{P}}(u) \cdot \mathcal{P}(v) = \sum_R \binom{e(R)}{v} \mathcal{P}(R), \tag{3}$$

where the sum is taken over all R for which $|R| = (p-1)(u+v)$.

Proof. See [2] for the proof of (2) and look at [4] for the proof of (3). □

Using these formulae, we can prove the following proposition.

Proposition 1.3 *For nonnegative integers k, l, m, n and $k > l$, suppose that*

(1) $m + n = p^k - p^l$

(2) $m < p^{k-1}$

(3) $m \equiv 0 \mod p^l$.

Then

(i) *When* $l = 0$, *we have*

$$\mathcal{P}(m) \cdot \widehat{\mathcal{P}}(n) = \widehat{\mathcal{P}}(n-(p-1)m-1)\mathcal{P}(pm+1)$$

(ii) *When* $l > 0$, *we have*

$$\mathcal{P}(m)\cdot\widehat{\mathcal{P}}(n)=\widehat{\mathcal{P}}(n-(p-1)(m+p^l))\mathcal{P}(pm+(p-1)p^l)-\sum_{w=1}^{p-1}\binom{p-1}{w}\mathcal{P}(m+wp^{l-1})\cdot\widehat{\mathcal{P}}(n-wp^{l-1}).$$

Proof.

(i) Let $l = 0$. Using Proposition 1.3, we have

$$\mathcal{P}(m) \cdot \widehat{\mathcal{P}}(n) = \sum_R \binom{|R| + e(R)}{pm} \mathcal{P}(R),$$

and

$$\widehat{\mathcal{P}}(n-(p-1)m-1) \cdot \mathcal{P}(pm+1) = \sum_R \binom{e(R)}{pm+1} \mathcal{P}(R),$$

where $|R| = (p-1)(p^k-1)$ and $1 \leq e(R) \leq p^k - 1$. In order to prove these sums are equivalent in mod-p, we need to show that their binomial coefficients are equivalent in mod-p, i.e.

$$\binom{|R| + e(R)}{pm} \equiv \binom{e(R)}{pm+1} \mod p.$$

We know that $|R| = \sum_{i=1}^{\infty} r_i(p^i - 1)$ and $e(R) = \sum_{i=1}^{\infty} r_i$. Using these facts, we have

$$\begin{aligned} p^k - 1 = \frac{|R|}{p-1} &= r_1 + \sum_{i=2}^{\infty} r_i(p^{i-1} + p^{i-2} + \cdots + p + 1) \\ &= e(R) - \sum_{i=2}^{\infty} r_i + \sum_{i=2}^{\infty} r_i(p^{i-1} + p^{i-2} + \cdots + p + 1) \\ &= e(R) + \sum_{i=2}^{\infty} r_i p(p^{i-2} + p^{i-3} + \cdots + p + 1). \end{aligned}$$

Since $\sum\limits_{i=2}^{\infty} r_i p(p^{i-2}+p^{i-3}+\cdots+p+1) \equiv 0 \mod p$, $e(R) \equiv p-1 \mod p$. Using Equation (1), and the upper bounds of $e(R)$ and m, we have

$$\binom{|R|+e(R)}{pm} = \binom{(p-1)(p^k-1)+e(R)}{pm} \equiv \binom{e(R)}{pm+1} \mod p.$$

This completes the proof of part (i).

(ii) Let $l > 0$. Again using Proposition 1.3, we have

$$\mathcal{P}(m) \cdot \widehat{\mathcal{P}}(n) = \sum_R \binom{|R|+e(R)}{pm} \mathcal{P}(R),$$

$$\mathcal{P}(m+wp^{l-1}) \cdot \widehat{\mathcal{P}}(n-wp^{l-1}) = \sum_R \binom{|R|+e(R)}{pm+wp^l} \mathcal{P}(R),$$

and

$$\widehat{\mathcal{P}}(n-(p-1)(m+p^l)) \cdot \mathcal{P}(pm+(p-1)p^l) = \sum_R \binom{e(R)}{pm+(p-1)p^l} \mathcal{P}(R)$$

where $|R| = (p-1)(p^k-p^l)$ and $1 \le e(R) \le p^k - p^l$. In order to prove the sums in part (ii) are equivalent, we need to show that

$$\binom{|R|+e(R)}{pm} + \sum_{w=1}^{p-1} \binom{p-1}{w} \binom{|R|+e(R)}{pm+wp^l} \equiv \binom{e(R)}{pm+(p-1)p^l} \mod p.$$

Case 1: $0 \le \alpha_l(e(R)) < p-1$. Then $\binom{e(R)}{pm+(p-1)p^l}$ are equivalent to zero in mod-p. So it is enough to show that

$$\binom{|R|+e(R)}{pm} + \sum_{w=1}^{p-1} \binom{p-1}{w} \binom{|R|+e(R)}{pm+wp^l} \equiv 0 \mod p$$

i.e.

$$1 + \sum_{w=1}^{p-1} \binom{p-1}{w} \binom{1+\alpha_l(e(R))}{w} \equiv 0 \mod p$$

for $0 \leq \alpha_l(e(R)) < p-1$. The following equivalent holds

$$\sum_{w=0}^{p-1} \binom{p-1}{w} \binom{1+\alpha_l(e(R))}{w} \equiv \binom{p+\alpha_l(e(R))}{p-1} \mod p$$

by considering coefficient of x^{p-1} in the binomial expansion of

$$(x+1)^{p+\alpha_l(e(R))} = (x+1)^{p-1}(x+1)^{1+\alpha_l(e(R))}.$$

Since $0 \leq \alpha_l(e(R)) < p-1$, $\binom{p+\alpha_l(e(R))}{p-1}$ is equivalent to zero in mod-p. Hence

$$\sum_{w=0}^{p-1} \binom{p-1}{w} \binom{1+\alpha_l(e(R))}{w} \equiv 0 \mod p$$

i.e.

$$1 + \sum_{w=1}^{p-1} \binom{p-1}{w} \binom{1+\alpha_l(e(R))}{w} \equiv 0 \mod p$$

So the result holds.

Case 2: $\alpha_l(e(R)) = p-1$. Then

$$\sum_{w=1}^{p-1} \binom{p-1}{w} \binom{|R|+e(R)}{pm+wp^l} \equiv 0 \mod p$$

and

$$\binom{|R|+e(R)}{pm} \equiv \binom{e(R)}{pm+(p-1)p^l} \mod p.$$

Therefore the result holds. □

2. Proof of Main Result

Proof of Theorem 1.1 We are going to prove the theorem by induction on i under the assumption that $i \leq j$. For $i = 0$ and all j , $\widehat{X}(j, p-1) = \mathcal{P}(p^{j+1}-1) = X(0, p^{j+1}-1)$ by Davis' formula in [1]. Assume that for all $\hat{i} \leq i-1$ and all j, and for $\hat{i} = i$ and $\hat{j} \leq j-1$,

$$\widehat{X}(\hat{i}, p^{\hat{j}+1}-1) = X(\hat{j}, p^{\hat{i}+1}-1).$$

The inductive proof will draw on the following remark: under the above assumptions,

$$\widehat{\mathcal{P}}(cp^{l-1}) \cdot X(l-1, p^{i+1}-1) = 0 \tag{4}$$

where c is a unit in mod-p and $1 \leq l \leq i$. Indeed, by induction on l, we have

$$\begin{aligned}\widehat{\mathcal{P}}(cp^{l-1}) \cdot X(l-1, p^{i+1}-1) &= \widehat{\overbrace{[\widehat{X}(l-1, p^{i+1}-1) \cdot \mathcal{P}(cp^{l-1})]}} \\ &= \widehat{\overbrace{[X(i, p^{l}-1) \cdot \mathcal{P}(cp^{l-1})]}} \\ &= \widehat{\overbrace{[X(i-1, p(p^{l}-1))\mathcal{P}(p^{l}-1) \cdot \mathcal{P}(cp^{l-1})]}}.\end{aligned}$$

By Adem relations, $\mathcal{P}(p^l - 1) \cdot \mathcal{P}(cp^{l-1}) = 0$. Therefore this verifies Equation (4).

We claim that for $0 \leq l \leq j$

$$X(l, p^i - 1) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-1)) = \widehat{\mathcal{P}}(p^i(p^{j+1}-p^{l+1})) \cdot X(l, p^{i+1}-1). \tag{5}$$

The case $l = 0$ follows from Proposition 1.4 (i). Suppose that the statement is true for $l - 1$. Then by induction on l and Proposition 1.4 (ii), we have

$$\begin{aligned}X(l,p^i - 1) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-1)) &= \mathcal{P}(p^l(p^i-1)) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-p^l)) \cdot X(l-1, p^{i+1}-1) \\ &= [\widehat{\mathcal{P}}(p^i(p^{j+1}-p^{l+1})) \cdot \mathcal{P}(p^l(p^{i+1}-1))] \cdot X(l-1, p^{i+1}-1) \\ &- [\sum_{w=1}^{p-1} \binom{p-1}{w} \mathcal{P}(p^l(p^i-1)+wp^{l-1}) \cdot \widehat{\mathcal{P}}(p^{i+j+1}-p^{l+i}-wp^{l-1})] \cdot X(l-1, p^{i+1}-1).\end{aligned}$$

From Equation (4),

$$\widehat{\mathcal{P}}(p^{i+j+1}-p^{l+i}-wp^{l-1}) \cdot X(l-1, p^{i+1}-1) = 0$$

for every $w = 1, 2, 3, \ldots, p-1$. Hence we have

$$X(l, p^i - 1) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-1)) = \widehat{\mathcal{P}}(p^i(p^{j+1}-p^{l+1})) \cdot X(l, p^{i+1}-1).$$

This proves our claim. Finally, taking $l = j$, we find that

$$\begin{aligned}\widehat{X}(i, p^{j+1}-1) &= \widehat{X}(i-1, p^{j+1}-1)) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-1)) \\ &= X(j, p^i - 1) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-1)) \\ &= \widehat{\mathcal{P}}(0) \cdot X(j, p^{i+1}-1).\end{aligned}$$

This completes the proof. □

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