

## On the Linearity of Certain Mapping Class Groups

*Mustafa Korkmaz*

### Abstract

S. Bigelow proved that the braid groups are linear. That is, there is a faithful representation of the braid group into the general linear group of some field. Using this, we deduce from previously known results that the mapping class group of a sphere with punctures and hyperelliptic mapping class groups are linear. In particular, the mapping class group of a closed orientable surface of genus 2 is linear.

**Key Words:** Mapping class groups, Braid groups, Linear groups

### Introduction

One of the well-known open problem in the theory of mapping class groups is that whether these groups are linear or not (cf. [2], Problem 30, p. 220). A group is called *linear* if it has a faithful representation into  $GL(n, F)$  for some field  $F$  and for some integer  $n$ .

Recently, S. Bigelow [1] proved that the braid groups are linear. The braid group  $B_n$  on  $n$  strings divided out by its center is isomorphic to a finite index subgroup of the mapping class group of a sphere with  $n+1$  marked points. Using this, we observe that the mapping class group of a sphere with marked points and that the hyperelliptic mapping class groups, which are defined below, are linear. In particular, the mapping class group of a closed orientable surface of genus 2 is linear. The linearity of the mapping class group of a surface of genus  $\geq 3$  still remains open.

### Preliminaries

We first set up the notations and state the theorems used in the proof of the results of this paper. Then we prove our results.

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Let  $S$  be a compact connected orientable surface of genus  $g$  with  $r$  marked points (also called *punctures*) contained in the interior of  $S$  and with  $s$  boundary components. The *mapping class group*  $\mathcal{M}_{g,r}^s$  of  $S$  is defined to be the group of isotopy classes of orientation preserving diffeomorphisms of  $S$  which preserve the set of marked points and are the identity on the boundary. The isotopies are assumed to fix each marked point and each boundary point. We denote the group  $\mathcal{M}_{g,0}$  simply by  $\mathcal{M}_g$ .

The braid group  $B_n$  on  $n$  strings is the group which admits a presentation with generators  $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ , and with the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ if } |i - j| \geq 2$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} .$$

In fact, the group  $B_n$  is isomorphic to  $\mathcal{M}_{0,n}^1$ , the mapping class group of a disc with  $n$  marked points. The generator  $\sigma_i$  is the isotopy class of a certain diffeomorphism of  $D_n$  which interchanges  $i$ th and  $i + 1$ st marked points so that its square is a Dehn twist.

S. Bigelow proved the following remarkable theorem in [1].

**Theorem 1** *The braid groups are linear.*

For a group  $G$  and for a subset  $X \subseteq G$ , the centralizer of  $X$  in  $G$  is defined to be

$$C_G(X) = \{y \in G : xy = yx \text{ for every } x \in X\}.$$

The center of  $G$  is  $C_G(G)$  and it is denoted by  $C(G)$ ;

$$C(G) = \{x \in G : xy = yx \text{ for every } y \in G\}.$$

For a field  $F$ , let  $F_n$  denote the space of  $n \times n$  matrices with entries in  $F$ . As usual,  $GL(n, F)$  denotes the group of invertible matrices.

**Theorem 2** ([5], **Theorem 6.2.**) *Let  $G$  be a subgroup of  $GL(n, F) \subseteq F_n$  and  $H$  a normal subgroup of  $G$  such that  $H = C_G(X)$  for some subset  $X$  of  $F_n$ . Then there exists a homomorphism of  $G$  into  $GL(n^2, F)$  with kernel  $H$ .*

**Corollary 3** *If  $G$  is linear, then so is  $G/C(G)$ .*

**Proof.** Take  $X = G$  in Theorem 2. □

The following theorem is probably well known to algebraists and can easily be proved by using the induced representation (cf. [4]).

**Theorem 4** *Let  $G$  be a group and  $H$  be a subgroup of  $G$  of finite index  $n$ . Then any injective homomorphism  $H \rightarrow GL(k, F)$  gives rise to an injective homomorphism  $G \rightarrow GL(kn, F)$ . In particular,  $G$  is linear if and only if  $H$  is linear.*

### The results

We are now ready to state and prove our results of this note.

**Theorem 5** *The mapping class group  $\mathcal{M}_{0,n}$  of a sphere with  $n$  marked points is linear for every  $n$ .*

**Proof.** If  $n \leq 3$ , then  $\mathcal{M}_{0,n}$  is a finite group and hence it is linear. Hence, we assume that  $n \geq 4$ .

Recall that the braid group  $B_{n-1}$  is isomorphic to the mapping class group of a disc  $D_{n-1}$  with  $n-1$  marked points. The center of the braid group  $B_{n-1}$  is the infinite cyclic group generated by a Dehn twist about a simple closed curve isotopic to the boundary component of the disc  $D_{n-1}$  (cf. [2]). Let us glue a disc with one marked point  $x$  to the boundary of  $D_{n-1}$  to get a sphere  $S$  with  $n$  marked points. Extending the diffeomorphisms of  $D_{n-1}$  to  $S$  by the identity gives a homomorphism  $\varphi$  from  $B_{n-1}$  to  $\mathcal{M}_{0,n}$ . The image  $\varphi(B_{n-1})$  of  $\varphi$  is precisely the stabilizer of  $x$  under the action of  $\mathcal{M}_{0,n}$  on the set of marked points, which is of index  $n$ , and the kernel of  $\varphi$  is the center  $C(B_{n-1})$  of  $B_{n-1}$ .

The group  $\varphi(B_{n-1})$  is isomorphic to the quotient group  $B_{n-1}/C(B_{n-1})$ . Since the group  $B_{n-1}$  is linear, so is  $\varphi(B_{n-1})$  by Theorem 2. By Theorem 4 the group  $\mathcal{M}_{0,n}$  is linear.  $\square$

Suppose that a closed connected orientable surface of genus  $g$  is embedded in the  $xyz$ -space as in Figure 1 in such a way that it is invariant under the rotation  $J(x, y, z) = (-x, y, -z)$  about the  $y$ -axis. Let us denote the isotopy class of  $J$  by  $j$ . The hyperelliptic mapping class group of genus  $g$  is defined to be the centralizer  $C_{\mathcal{M}_g}(j)$  of  $j$  in  $\mathcal{M}_g$ . If  $g = 1$  or  $2$ , then the hyperelliptic mapping class group is equal to the mapping class group.

**Theorem 6** *Let  $S$  be a closed connected orientable surface of genus  $g$ . Then the hyperelliptic mapping class group of  $S$  is linear. In particular, the mapping class group of a closed connected orientable surface of genus 2 is linear.*

**Proof.** Since the mapping class group of a torus is isomorphic to  $SL(2, \mathbb{Z})$ , which is linear, we can assume that  $g \geq 2$ .

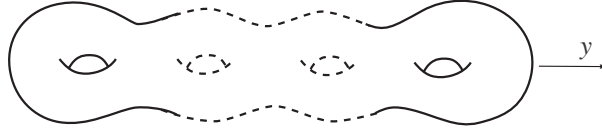


Figure 1. A surface embedded in  $\mathbb{R}^3$  which is invariant under  $J$ .

There is a well known short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow C_{\mathcal{M}_g}(j) \xrightarrow{p} \mathcal{M}_{0,2g+2} \longrightarrow 1,$$

where  $\mathbb{Z}_2$  is the subgroup generated by the involution  $j$  (cf. [3]). Let  $\rho : \mathcal{M}_g \rightarrow Sp(2g, 3)$  the natural homomorphism from the mapping class group to the symplectic group over the finite field with three elements given by the action of  $\mathcal{M}_g$  on the first homology group of  $S$ . Let us denote by  $H$  the intersection of  $C_{\mathcal{M}_g}(j)$  with the kernel of  $\rho$ . Hence,  $H$  is a finite index subgroup of  $C_{\mathcal{M}_g}(j)$ . As  $j$  acts as the minus identity on the first homology, it is not contained in  $H$ . Hence, the restriction of  $p$  to  $H$  is injective. Since  $p(H)$  is of finite index in  $\mathcal{M}_{0,2g+2}$ ,  $H$  is linear. Therefore, the group  $C_{\mathcal{M}_g}(j)$  is linear by Theorem 4.

The second statement follows from the fact that  $\mathcal{M}_2 = C_{\mathcal{M}_2}(j)$ . □

**Remark 7** *Bigelow proves that the braid group  $B_n$  can be embedded in  $GL(\frac{n(n-1)}{2}, \mathbb{Z}[q^{\pm 1}, t^{\pm 1}])$ . Since the ring  $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$  can be embedded in the field  $\mathbb{R}$  of real numbers by assigning to  $q, t$  two algebraically independent nonzero real numbers, the group  $B_n$  embeds into  $GL(\frac{n(n-1)}{2}, \mathbb{R})$ . Using Theorems 2 and 4 and the fact that the order of the group  $Sp(2g, 3)$  is  $3^{g^2} \prod_{i=1}^g (3^{2i} - 1)$ , it can be deduced from the proofs of Theorems 5 and 6 that*

- 1)  $\mathcal{M}_{0,n}$  embeds into  $GL(\frac{n(n-1)^2(n-2)^2}{4}, \mathbb{R})$ ,
- 2) the hyperelliptic mapping class group of genus  $g$  embeds into

$$GL(2(g+1)g^2(2g+1)^2 3^{g^2} \prod_{i=1}^g (3^{2i} - 1), \mathbb{R}).$$

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*In particular,  $\mathcal{M}_2$  embeds into  $GL(2^{10} 3^5 5^3, \mathbb{R})$ .*

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### References

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Mustafa KORKMAZ  
Department of Mathematics,  
Middle East Technical University,  
06531 Ankara-TURKEY  
e-mail: korkmaz@arf.math.metu.edu.tr

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