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On subspaces isomorphic to ℓ^q in interpolation of quasi Banach spaces

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Abstract

We show that every sequence $\{x_n\}_{n=1}^{\infty}$ in a real interpolation space $(E_0, E_1)_{\theta,q}$, $0 < \theta < 1$, $0 < q < \infty$, of quasi Banach spaces E_0, E_1, which is $0-$convergent in $E_0 + E_1$ but $\inf_n \|x_n\|_{(E_0,E_1)_{\theta,q}} > 0$, has a subsequence which is equivalent to the standard unit basis of ℓ^q.

Key Words: real intepolation method, quasi Banach spaces.

1. Introduction

The classical theory of interpolation of Banach spaces by the real method has been extended to interpolation of quasi Banach spaces by Sagher (see [4]). An important theorem about the geometric structure of such Banach spaces $(E_0, E_1)_{\theta,q}$, $1 \leq q < \infty$, $0 < \theta < 1$, concerning to the existence of subspaces isomorphic to ℓ^q on it is established by Levy in [3]. On this paper we extend Levy's result to the case of interpolation of quasi Banach spaces by the real method, although we present our result in a sligtly different way. We recall that a quasi Banach space is a vector space E over the field $\mathbb{K}$ of real or complex numbers which is complete under the metric $d(x, y) = \|x - y\|$ where $\|.\| : E \to [0, \infty[$ is a quasinorm, i.e. a function with properties

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1) $\|x\| = 0$ if and only if $x = 0$.

2) $\forall\, x \in E,\ \ \forall\, \lambda \in \mathbb{K} \quad \|\lambda x\| = |\lambda|\, \|x\|$.

3) There is $K \geq 1$ such that for all $x, y \in \mathbb{K} \quad \|x + y\| \leq K(\|x\| + \|y\|)$.

By a theorem of Aoki and Rolewicz, if $(E, \|.\|)$ is a quasi Banach space, there is $0 < r \leq 1$ and an equivalent quasinorm $\|.\|_E$ on E which is indeed an r-norm, i.e. the properties 1), 2) and

4) $\forall\, x, y \in E \quad \|x + y\|_E^r \leq \|x\|_E^r + \|y\|_E^r$,

are verified by $\|.\|_E$.

We shall use the following elementary consequence of Hölder's inequality: if $\alpha \geq 1$, $A \in \mathbb{R}$ and $B \in \mathbb{R}$, then

$$|A + B|^\alpha \leq 2^{\alpha-1}(|A|^\alpha + |B|^\alpha) \tag{1}$$

and hence

$$|A - B|^\alpha \geq \frac{|A|^\alpha}{2^{\alpha-1}} - |B|^\alpha. \tag{2}$$

2. Main result

Theorem 1 *Let $0 < q < \infty$ and let $E_i,\ i = 0, 1$ be quasi Banach spaces. Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in the interpolation space $(E_0, E_1)_{\theta,q}$, such that $lim_{n\to\infty}\, x_n = 0$ in $E_0 + E_1$ but there is $\varepsilon > 0$ such that $\varepsilon < \|x_n\|_{(E_0,E_1)_{\theta,q}}$ for each $n \in \mathbb{N}$. Then there is a subsequence $\{x_{k_r}\}_{r=1}^\infty$ such that its closed linear span in $(E_0, E_1)_{\theta,q}$ is isomorphic to ℓ^q.*

Proof. Let B_i, $i = 0, 1$ be the closed unit ball of the space E_i. Let $\|.\|_h$, $h \in \mathbb{Z}$ be the Minkowski functional of the set $e^{-\theta h}B_0 + e^{(1-\theta)h}B_1 \subset E_0 + E_1$. The quasinorm of $(E_0, E_1)_{\theta,q}$ is equivalent to

$$|||x||| = \left(\sum_{h\in\mathbb{Z}} \|x\|_h^q\right)^{\frac{1}{q}}.$$

(The proof is given in §4, proposition 4 of chapter I in [1] in the case of Banach spaces, but the proof is also valid in the quasi Banach case). In consequence, we can suppose that for some $M > 0$, $\varepsilon > 0$

$$\forall n \in \mathbb{N} \quad M > |||x_n||| > \varepsilon. \tag{3}$$

There are numbers $0 < r_i \leq 1$, $i = 0, 1$ such that every E_i is an r_i-Banach space respectively. Fix $0 < r \leq \min\{r_0, r_1, q\}$. Suppose we have found *strictly increasing* finite sequences $(k_i)_{i=1}^n$ and $(t_i)_{i=1}^n$ in $\mathbb{N}$ with $t_1 = 1$ such that for every $1 \leq j \leq n$

$$\sum_{|h| \leq k_j} \|x_{t_j}\|_h^q > \varepsilon^q, \qquad \sum_{|h| > k_j} \|x_{t_j}\|_h^q \leq \frac{\varepsilon^q}{2^{j+4+\frac{(q-r)(j+2)}{r}}} \tag{4}$$

and

$$\forall\, j = 2, 3, ..., n \quad \sum_{|h| \leq k_{j-1}} \|x_{t_j}\|_h^q \leq \frac{\varepsilon^q}{2^{j+4+\frac{(q-r)(j+2)}{r}}}. \tag{5}$$

Since $(\sum_{|h| \leq k_n} \|x\|_h^q)^{1/q}$ is an equivalent quasi norm to the quasi norm of $E_0 + E_1$, (see [1]) and $\lim x_n = 0$ in $E_0 + E_1$, there is $t_{n+1} > t_n$ such that (5) holds for $j = n+1$. Now, by (3), there is $k_{n+1} > k_n$ such that (4) holds for $j = n+1$ and the process can be repeated indefinitely. Remark that, by (4) and (5), for every $n \in \mathbb{N}$ we have

$$\forall n \in \mathbb{N} \quad \sum_{|h| = k_{n-1}+1}^{k_n} \|x_{t_n}\|_h^q > \varepsilon^q - \frac{\varepsilon^q}{2} = \frac{\varepsilon^q}{2}. \tag{6}$$

Take a finite scalar sequence $\{\lambda_n\}_{n=1}^s$ and put $k_0 := -1$. Since each E_i, $i = 0, 1$ is an $r-$Banach space, every $\|.\|_h$, $h \in \mathbb{Z}$ is an $r-$norm (see section 6.3 of [2]) . Moreover $q/r \geq 1$. Then we have

$$\left|\left|\left|\sum_{n=1}^s \lambda_n x_{t_n}\right|\right|\right|^q \leq \sum_{h \in \mathbb{Z}} \left(\sum_{n=1}^s |\lambda_n|^r \|x_{t_n}\|_h^r \right)^{\frac{q}{r}} =$$

$$\sum_{i=1}^s \sum_{|h|=k_{i-1}+1}^{k_i} \left(\sum_{n=1}^s |\lambda_n|^r \|x_{t_n}\|_h^r \right)^{\frac{q}{r}} + \sum_{|h| > k_s} \left(\sum_{n=1}^s |\lambda_n|^r \|x_{t_n}\|_h^r \right)^{\frac{q}{r}} \leq$$

and applying (1) several times and by (3) and (4)

$$\sum_{i=1}^s \sum_{|h|=k_{i-1}+1}^{k_i} 2^{\frac{q-r}{r}} \left(|\lambda_i|^q \|x_{t_i}\|_h^q + \left(\sum_{n=1, n \neq i}^s |\lambda_n|^r \|x_{t_n}\|_h^r \right)^{\frac{q}{r}} \right) +$$

$$\sum_{|h|>k_s} \sum_{n=1}^{s} 2^{\frac{(q-r)(n+1)}{r}} |\lambda_n|^q \|x_{t_n}\|_h^q \leq$$

$$\sum_{i=1}^{s} 2^{\frac{q-r}{r}} |\lambda_i|^q \sum_{|h|=k_{i-1}+1}^{k_i} \|x_{t_i}\|_h^q + \sum_{i=1}^{s} \sum_{|h|=k_{i-1}+1}^{k_i} \sum_{n=1, n\neq i}^{s} 2^{\frac{(q-r)(n+1)}{r}} |\lambda_n|^q \|x_{t_n}\|_h^q +$$

$$\frac{\varepsilon^q}{4} \sum_{n=1}^{s} |\lambda_n|^q \leq$$

$$\left(M^q \, 2^{\frac{q-r}{r}} + \frac{\varepsilon^q}{4} \right) \sum_{i=1}^{s} |\lambda_i|^q + \sum_{i=1}^{s} \sum_{|h|=k_{i-1}+1}^{k_i} \sum_{n=1, n\neq i}^{s} 2^{\frac{(q-r)(n+1)}{r}} |\lambda_n|^q \|x_{t_n}\|_h^q. \tag{7}$$

Now remark that

$$\sum_{i=1}^{s} \sum_{|h|=k_{i-1}+1}^{k_i} \sum_{n=1, n\neq i}^{s} 2^{\frac{(q-r)(n+1)}{r}} |\lambda_n|^q \|x_{t_n}\|_h^q =$$

$$\sum_{i=1}^{s} \sum_{|h|=k_{i-1}+1}^{k_i} \sum_{n=1}^{i-1} 2^{\frac{(q-r)(n+1)}{r}} |\lambda_n|^q \|x_{t_n}\|_h^q +$$

$$\sum_{i=1}^{s} \sum_{|h|=k_{i-1}+1}^{k_i} \sum_{n=i+1}^{s} 2^{\frac{(q-r)(n+1)}{r}} |\lambda_n|^q \|x_{t_n}\|_h^q =$$

$$\sum_{i=1}^{s} \sum_{|h|=k_{i-1}+1}^{k_i} \sum_{n=1}^{s} b_{inh} + \sum_{i=1}^{s} \sum_{|h|=k_{i-1}+1}^{k_i} \sum_{n=1}^{s} c_{inh}$$

where

$$b_{inh} = \begin{cases} 2^{\frac{(q-r)(n+1)}{r}} |\lambda_n|^q \|x_{t_n}\|_h^q & \text{if } \; n \leq i-1, \;\; k_{i-1} < |h| \leq k_i \\ 0 & \text{if } \; n > i-1, \;\; k_{i-1} < |h| \leq k_i \end{cases}$$

and

$$c_{inh} = \begin{cases} 2^{\frac{(q-r)(n+1)}{r}} |\lambda_n|^q \|x_{t_n}\|_h^q & \text{if } \; n > i, \;\; k_{i-1} < |h| \leq k_i \\ 0 & \text{if } \; n \leq i, \;\; k_{i-1} < |h| \leq k_i \; . \end{cases}$$

Then

$$\sum_{i=1}^{s}\sum_{|h|=k_{i-1}+1}^{k_i}\sum_{n=1}^{s} b_{inh} + \sum_{i=1}^{s}\sum_{|h|=k_{i-1}+1}^{k_i}\sum_{n=1}^{s} c_{inh} =$$

$$\sum_{n=1}^{s}\sum_{i=1}^{s}\sum_{|h|=k_{i-1}+1}^{k_i} b_{inh} + \sum_{n=1}^{s}\sum_{i=1}^{s}\sum_{|h|=k_{i-1}+1}^{k_i} c_{inh} =$$

$$\sum_{n=1}^{s}|\lambda_n|^q \sum_{i=n+1}^{s}\sum_{|h|=k_{i-1}+1}^{k_i} 2^{\frac{(q-r)(n+1)}{r}}\|x_{t_n}\|_h^q +$$

$$\sum_{n=1}^{s}|\lambda_n|^q \sum_{i=1}^{n-1}\sum_{|h|=k_{i-1}+1}^{k_i} 2^{\frac{(q-r)(n+1)}{r}}\|x_{t_n}\|_h^q \leq$$

and by (4) and (5)

$$\frac{\varepsilon^q}{2^{3+\frac{q-r}{r}}}\sum_{i=1}^{s}|\lambda_i|^q + \frac{\varepsilon^q}{2^{3+\frac{q-r}{r}}}\sum_{i=1}^{s}|\lambda_i|^q = \frac{\varepsilon^q}{2^{2+\frac{q-r}{r}}}\sum_{i=1}^{s}|\lambda_i|^q \leq \frac{\varepsilon^q}{4}\sum_{i=1}^{s}|\lambda_i|^q. \tag{8}$$

In definitive, by (7)

$$\left|\left|\left|\sum_{n=1}^{s}\lambda_n x_{t_n}\right|\right|\right|^q \leq \left(M^q 2^{\frac{q-r}{r}} + \frac{\varepsilon^q}{2}\right)\sum_{i=1}^{s}|\lambda_i|^q.$$

On the other hand, using (2) several times, (6) and (8)

$$\left|\left|\left|\sum_{n=1}^{s}\lambda_n x_{t_n}\right|\right|\right|^q \geq \sum_{i=1}^{s}\sum_{|h|=k_{i-1}+1}^{k_i}\left\|\sum_{n=1}^{s}|\lambda_n| x_{t_n}\right\|_h^q \geq$$

$$\sum_{i=1}^{s}\sum_{|h|=k_{i-1}+1}^{k_i}\left||\lambda_i|^r\|x_{t_i}\|_h^r - \left\|\sum_{n=1,n\neq i}^{s}\lambda_n x_{t_n}\right\|_h^r\right|^{\frac{q}{r}} \geq$$

$$\sum_{i=1}^{s}\sum_{|h|=k_{i-1}+1}^{k_i}\left(\frac{1}{2^{\frac{q-r}{r}}}|\lambda_i|^q\|x_{t_i}\|_h^q - \left\|\sum_{n=1,n\neq i}^{s}\lambda_n x_{t_n}\right\|_h^q\right) \geq$$

$$\sum_{i=1}^{s} \sum_{|h|=k_{i-1}+1}^{k_i} \left(\frac{1}{2^{\frac{q-r}{r}}} |\lambda_i|^q \|x_{t_i}\|_h^q - \left(\sum_{n=1, n\neq i}^{s} |\lambda_n|^r \|x_{t_n}\|_h^r \right)^{\frac{q}{r}} \right) \geq$$

and using (1) repeatedly

$$\sum_{i=1}^{s} \sum_{|h|=k_{i-1}+1}^{k_i} \left(\frac{1}{2^{\frac{q-r}{r}}} |\lambda_i|^q \|x_{t_i}\|_h^q - \sum_{n=1, n\neq i}^{s} 2^{\frac{(q-r)(n+1)}{r}} |\lambda_n|^q \|x_{t_n}\|_h^q \right) \geq$$

and by (6) and (8)

$$\frac{\varepsilon^q}{2^{1+\frac{q-r}{r}}} \sum_{i=1}^{s} |\lambda_i|^q - \frac{\varepsilon^q}{2^{2+\frac{q-r}{r}}} \sum_{n=1}^{s} |\lambda_n|^q = \frac{\varepsilon^q}{2^{2+\frac{q-r}{r}}} \sum_{n=1}^{s} |\lambda_n|^q$$

which proves the theorem. ■

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