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The p-Stirling Numbers

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Dedicated to the memory of Henry C. Diehl.

Abstract

The purpose of this article is to introduce p-Stirling numbers of the first and second kinds.

Key Words: Binomial coefficient; Character; General linear group; Partittion; Representation; Stirling number; Symmetric function.

1. Introduction

Pronounced "m-choose-n", $C(m,n)$ is the number of n-element subsets of $\{1,2,\ldots,m\}$. Let C_m be the m-by-m matrix whose (i,j)-entry is $C(i,j)$. Then, for example,

$$C_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 \\ 4 & 6 & 4 & 1 & 0 \\ 5 & 10 & 10 & 5 & 1 \end{pmatrix}.$$

Because $\det(C_m) = 1$, not only is C_m invertible, but C_m^{-1} is an integer matrix. Indeed, among the many wonderful properties of binomial coefficients is the fact that C_m^{-1} can be obtained from C_m by inserting a few well chosen minus signs: The (i,j) -entry of C_m^{-1} is $(-1)^{i+j}C(i,j)$. Thus

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$$C_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3 & -3 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 \\ 5 & -10 & 10 & -5 & 1 \end{pmatrix}.$$

One way to insert this "checkerboard array" of minus signs is by means of matrix multiplication:

$$C_m^{-1} = D_m C_m D_m, \tag{1}$$

where $D_m = D_m^{-1} = \mathrm{diag}(-1, 1, -1, \ldots, (-1)^m)$ is the $m - by - m$ diagonal matrix of ∓ 1's.

The Stirling number of the second kind, $S(m, n)$, is the number of ways to partition $\{1, 2, \ldots, m\}$ into a disjoint union of n nonempty subsets caled the *parts* of the partition. The 2-part partitions of $\{1, 2, 3, 4\}$ are

$$\{1\} \cup \{2,3,4\}, \quad \{2\} \cup \{1,3,4\}, \quad \{3\} \cup \{1,2,4\}, \quad \{4\} \cup \{1,2,3\},$$
$$\{1,2\} \cup \{3,4\}, \quad \{1,3\} \cup \{2,4\}, \quad and \quad \{1,4\} \cup \{2,3\}$$

Thus, $S(4, 2) = 7$. More colorfully, $S(m, n)$ is the number of ways to distribute m distinguishable (labeled) cows among n identical (unlabeled) pastures, with each pasture containing at least one cow. So, $S(m, m) = 1 = S(m, 1)$, $m \geq 1$, and $S(m, n) = 0$ if $m \geq 1 > n$ or $n > m$. Moreover,

$$S(m+1, n) = S(m, n-1) + nS(m, n), \quad m \geq 1. \tag{2}$$

(While the author is most familiar with [7], standart facts like this can be found in almost any undergraduate combinatorics text.) Let T_m (T for "two") be the m-by-m matrix whose (i, j)-entry is $S(i, j)$. Then, for example,

$$T_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{pmatrix}.$$

Given Stirling numbers of the second kind, one naturally expects there to be other kinds. The (unsigned) Stirling number of the first kind, $s(m,n)$, is the number of permutations of degree m whose disjoint cycle factorizations consist of n cycles (including cycles of length one). Of the permutations of degree 4, three have "cycle type" [2,2], namely, (12) (34), (13) (24), and (14) (23), and eight have cycle type [3,1], namely, the 3-cycles (123), (124), ... Thus, $s(4,2) = 3 + 8 = 11$

In general, $s(m,1) = (m-1)!$, $s(m,m) = 1$, $m \geq 1$, and $s(m,n) = 0$ if $m \geq 1 > n$ or $n > m$. Moreover,

$$s(m+1,n) = s(m,n-1) + ms(m,n), \ m \geq 1. \tag{3}$$

Let F_m be the m-by-m matrix whose (i,j)-entry is $s(i,j)$. Then,

$$F_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 6 & 11 & 6 & 1 & 0 \\ 24 & 50 & 35 & 10 & 1 \end{pmatrix}.$$

Inverting F_m is not as simple as sprinkling minus signs among its entries; they must be sprinkled among the entries of T_m! That is,

$$F_m^{-1} = D_m T_m D_m \tag{4a}$$

or, because $D_m^{-1} = D_m$,

$$T_m^{-1} = D_m F_m D_m. \tag{4b}$$

In particular,

$$F_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 \\ -1 & 7 & -6 & 1 & 0 \\ 1 & -15 & 25 & -10 & 1 \end{pmatrix}.$$

2. Extensions

Suppose a rancher has p bulls and m cows that he plans to graze on $p+n$ identical pastures. In how many ways can he distribute the cattle among the pastures so that no pasture is left emty and no two bulls occupy the same pasture? If we assume he knows his cattle well enough to distinguish one from another, then the answer is a *p-Stirling number of the second kind.* Denote by $S_p(m,n)$ the number of ways to partition $\{1,2,\ldots,p,p+1,\ldots,p+m\}$ into a disjoint union of $p+n$ nonempty parts, subject to the condition that no two elements of $\{1,2,\ldots,p\}$ belong to the same part. Then $S_0(m,n)=S(m,n), S_p(m,0)=p^m,\ p>0$, and $S_p(m,m)=1$.

Example 2.1. *Suppose $m=5$, $n=2$, and p is arbitrary. Once the bulls have been distributed among the pastures, we can isolate*

(i) all 5 cows from the bulls in $S(5,2)=15$ ways;

(ii) 4 cows from the bulls in $5\times p\times S(4,2)=35p$ ways;

(iii) 3 cows from the bulls in $C(5,2)\times p^2\times S(3,2)=30p^2$ ways; and

(iv) 2 cows from the bulls in $C(5,3)\times p^3\times S(2,2)=10p^3$ ways.

Because no pasture can be left empty, this exausts all the possibilities, i.e., $S_p(5,2)=10p^3+30p^2+35p+15$.

Because $C(m,m-k)=C(m,k)$, the argument illustrated in Example 2.1 proves the following:

Theorem 2.2. *Let $p\neq 0$ be fixed but arbitrary. If $1\leq n\leq m$ then*

$$S_p(m,n)=\sum_{k=1}^{m}C(m,k)S(k,n)p^{m-k}. \tag{5}$$

Because $S(k,n)=0$, $k<n$, $S_p(m,n)$ is a polynomial of degree $m-n$ in p with leading coefficient $C(m,n)$ and constant coefficient $S(m,n)$. That $S_1(m,n)=S(m+1,n+1)$ follows from (5) and the well known relation (see, e.g., [7, p. 76])

$$S(m+1,n+1)=\sum_{k=1}^{m} C(m,k)S(k,n).$$

Theorem 2.3. *Let $p \neq 0$ be fixed but arbitrary. Define $S_p(0,0)=1$, and $S_p(m,n)=0$ if $n<0$ or $n>m$. Then*

$$S_p(m+1,n)=S_p(m,n-1)+(p+n)S_p(m,n), \quad m \geq 1. \tag{6}$$

Proof. There are $S_p(m,n-1)$ distributions that leave the $(m+1)$ *st* cow alone in a pasture. In the $S_p(m,n)$ ways to distribute the first m cows without leaving an empty pasture, the $(m+1)$ *st* cow can join any of the resulting $p+n$ herds. □

Theorem 2.3 can also be proved using Theorem 2.2 and the rcurrence relations for binomial coefficients and Stirling numbers of the secod kind.

Predictably, there is a family of p-Stirling numbers of the first kind. Denote by $s_p(m,n)$ the number of permutations of degree $p+m$ whose disjoint cycle factorizations consist of $p+n$ cycles, subject to the condition that no two elements of $\{1,2,\ldots,p\}$ belong to the same cycle. Then, $s_0(m,n)=s(m,n)$ and $s_p(m,m)=1,\ m\geq 1$..

Example 2.4. *Suppose $p>0$. Imagine modifying the identity permutation of degree p by successively inserting $p+1,\ p+2,\ldots,p+m$ into what is initially a product of the 1-cycles, (i), $1\leq i\leq p$, according to the following rule: Having previously inserted k integers, the next one, $p+k+1$, may be placed in any one of $p+k$ places, namely, immediately following any of $1,2,\ldots,p+k$. The number of different ways to insert all m integers is*

$$\begin{aligned} s_p(m,0) &= p(p+1)(p+2)\cdots(p+m-1) \\ &= p^{(m)}, \end{aligned} \tag{7}$$

the so-called "rising factorial function".

If $0<n\leq k\leq m$, then each of the $C(m,k)$ k-element subsets of $\{p+1,p+2,\ldots,p+m\}$ can be distributed among n empty cycles in $s(k,n)$ ways, and the remaining $m-k$ integers

among the cycles already containing the first p positive integers in $p^{(m-k)}$ ways. Hence (because $s(k,n)=0,\ k<n$),

$$s_p(m,n)=\sum_{k=1}^{m} C(m,k)s(k,n)p^{(m-k)}. \tag{8}$$

Theorem 2.5. *Let $p \neq 0$ be fixed but arbitrary. Define $s_p(0,0)=1$ and $s_p(m,n)=0$ if $n<0$ or $n>m$. Then*

$$s_p(m+1,n)=s_p(m,n-1)+(p+m)s_p(m,n), \quad m \geq 1. \tag{9}$$

Proof. Among the $s_p(m+1,n)$ permutations under consideration, $s_p(m,n-1)$ fix $m+1$. If it is not left fixed, $m+1$ immediately follows one of the other $p+m$ integers. That is, $(p+m)s_p(m,n)$ of the permutations do not fix $m+1$. □

While (8) may be regarded as an analog of (5), it is more useful to have $s_p(m,n)$ expressed as a polynomial in ordinary powers of p.

Theorem 2.6 *Let $p \neq 0$ be fixed but arbitrary. If $m \geq n \geq 1$ then*

$$s_p(m,n)=\sum_{k=1}^{m} s(m,k)C(k,n)p^{k-n}. \tag{10}$$

Because $C(k,n)=0$, $k<n$, $s_p(m,n)$ is a polynomial of degree $m-n$ in p with leading coefficient $C(m,n)$ and constant coefficient $s(m,n)$. That $s_1(m,n)=s(m+1,n+1)$ follows from (10) and the well known relation (see, e.g., [7, p. 111])

$$s(m+1,n+1)=\sum_{k=1}^{m} s(m,k)C(k,n).$$

The appearance of (10) suggests a proof that splits the enumeration into $m-n+1$ cases (as in the derivation of (8)). If such a revealing argument exists, it has escaped the author. (See, e.g., [8 p. 232] for a recurrence argument proving the equivalence of (8) and (10); the existence of a combinatorial argument is not the issue.)

Proof. The proof is by induction on m. If $m = 1$ then both sides of (10) are equal to 1. By (9) and the induction hypothesis,

$$
\begin{aligned}
s_p(m+1,n) &= s_p(m,n-1) + (p+m)s_p(m,n) \\
&= \sum_{k=1}^{m} s(m,k)C(k,n-1)p^{k-n+1} + (p+m)\sum_{k=1}^{m} s(m,k)C(k,n)p^{k-n} \\
&= \sum_{k=1}^{m} s(m,k)[C(k,n-1)+C(k,n)]p^{k-n+1} + m\sum_{k=1}^{m} s(m,k)C(k,n)p^{k-n} \\
&= C(m+1,n)p^{m+1-n} + \sum_{k=1}^{m-1} s(m,k)C(k+1,n)p^{k+1-n} \\
&\quad + \sum_{k=1}^{m} ms(m,k)C(k,n)p^{k-n} \\
&= C(m+1,n)p^{m+1-n} + \sum_{k=2}^{m} s(m,k-1)C(k,n)p^{k-n} \\
&\quad + \sum_{k=1}^{m} ms(m,k)C(k,n)p^{k-n} \\
&= s(m+1,m+1)C(m+1,n)p^{m+1-n} \\
&\quad + \sum_{k=2}^{m}[s(m,k-1)+ms(m,k)]C(k,n)p^{k-n} + ms(m,1)C(1,n) \\
&= \sum_{k=1}^{m+1} s(m+1,k)C(k,n)p^{K-n},
\end{aligned}
$$

by the recurrence relations for binomial coefficients and Stirling numbers of the first kind, an the facts that $s(m,m) = 1 = s(m+1,m+1)$ and $s(m+1,1) = ms(m,1)$. □

Corollary 2.7. *Suppose p is fixed but arbitrary. Let A_m and B_m be the m-by-m matrices whose (i,j)-entries are $s_p(i,j)$ and $S_p(i,j)$, respectively. Then*

$$A_m^{-1} = D_m B_m D_m \tag{11a}$$

and

$$B_m^{-1} = D_m A_m D_m. \tag{11b}$$

Proof. Because $D_m^{-1} = D_m$, (11a) and (11b) are equivalent. If $p = 0$, they become (4a) and (4b), respectively. Otherwise, let P_m be the matrix whose (i, j)-entry is p^{i-j}, $1 \le i, j \le m$. By Theorem 2.2, $B_m = (C_m \circ P_m)T_m$, where $C_m \circ P_m$ is the Hadamard (entry-wise) product of C_m and P_m, and T_m is the "T for two" matrix from Section 1. Thus, $B_m^{-1} = T_m^{-1}(C_m \circ P_m)^{-1}$. Observe that the (i, j)-entry of $[D_m(C_m \circ P_m)D_m][C_m \circ P_m]$ is

$$\begin{aligned}\sum_{k=1}^{m}(-1)^{i+k}C(i,k)p^{i-k}C(k,j)p^{k-j} &= p^{i-j}\sum_{k=1}^{m}(-1)^{i+k}C(i,k)C(k,j)\\ &= p^{i-j}\delta_{i,j}\end{aligned}$$

by (1). Thus, $(C_m \circ P_m)^{-1} = D_m(C_m \circ P_m)D_m$. Together with (4b), this yield

$$\begin{aligned}B_m^{-1} &= T_m^{-1}(C_m \circ P_m)^{-1}\\ &= [D_m F_m D_m][D_m(C_m \circ P_m)D_m]\\ &= D_m[F_m(C_m \circ P_m)]D_m\\ &= D_m A_m D_m,\end{aligned}$$

by Theorem 2.6. □

3. Generating Functions

Define exponential generating functions

$$f_n^p(x) = \sum_{m \ge 0} \frac{s_p(m,n)}{m!} x^m \tag{12}$$

and

$$g_n^p(x) = \sum_{m \geq 0} \frac{S_p(m,n)}{m!} x^m. \tag{13}$$

(Because $s_p(m,n) = 0 = S_p(m,n)$, $m < n$, the first n terms of both summations are zero.)

Theorem 3.1. *If p is a nonnegative integer and $n \geq 1$, then*

$$g_n^p(x) = e^{px}(e^x - 1)^n/n!,$$

and

$$f_n^p(x) = \frac{[-\ln(1-x)]^n}{n!(1-x)^p}.$$

Proof. It is well known (see, e.g., [7, pp 208-209]) that $g_n^0(x) = (e^x - 1)^n/n!$ and $f_n^0 = [-\ln(1-x)]^n/n!$. If $p \neq 0$ then by (5),

$$\begin{aligned}
g_n^p(x) &= \sum_{m \geq 0} \frac{S_p(m,n)}{m!} x^m \\
&= \sum_{m \geq 0} (\frac{1}{m!} \sum_{k=1}^{m} C(m,k) S(k,n) p^{m-k}) x^m \\
&= \sum_{m \geq 0} (\sum_{k=0}^{m} \frac{S(k,n)}{k!} \; \frac{p^{m-k}}{(m-k)!}) x^m \\
&= (\sum_{m \geq 0} \frac{S(m,n)}{m!} x^m)(\sum_{m \geq 0} \frac{p^m}{m!} x^m) \\
&= g_n^0(x) e^{px}.
\end{aligned}$$

Similarly, using (8),

$$\begin{aligned} f_n^p(x) &= \sum_{m\geq 0} \frac{s_p(m,n)}{m!} x^m \\ &= \sum_{m\geq 0} (\frac{1}{m!} \sum_{k=1}^{m} C(m,k) s(k,n) p^{(m-k)}) x^m \\ &= \sum_{m\geq 0} (\sum_{k=0}^{m} \frac{s(k,n)}{k!} \ \frac{p^{(m-k)}}{(m-k)!}) x^m \\ &= (\sum_{m\geq 0} \frac{s(m,n)}{m!} x^m)(\sum_{m\geq 0} \frac{p^{(m)}}{m!} x^m) \\ &= f_n^0(x) \sum_{m\geq 0} C(p+m-1,m) x^m \\ &= f_n^0(x)(1-x)^{-p}, \end{aligned}$$

because $C(p+m-1,m) = (-1)^m C(-p,m)$ (see, e.g., [7, p. 196]). □

Consider the function

$$h_m^p(x) = \sum_{n=0}^{m} s_p(m,n) x^n. \tag{14}$$

Then, for example,

$$\begin{aligned} h_1^p(x) &= s_p(1,0) + s_p(1,1)x \\ &= p + x \end{aligned} \tag{15}$$

and

$$\begin{aligned} h_2^p(x) &= s_p(2,0) + s_p(2,1)x + s_p(2,2)x^2 \\ &= p^{(2)} + (p + [p+1])x + x^2 \\ &= (p+x)([p+1]+x) \\ &= (p+x)^{(2)}, \end{aligned} \tag{16}$$

because $s_p(2,1) = s_p(1,0) + (p+1)s_p(1,1)$.

Theorem 3.2. *The function* $h_m^p(x) = (p+x)^{(m)}$.

Proof. Equations (15)-(16) start a proof by induction. From Equation (9),

$$\begin{aligned} h_{m+1}^p(x) &= \sum_{n=0}^{m+1} s_p(m+1,n)x^n \\ &= \sum_{n=1}^{m+1} s_p(m,n-1)x^n + \sum_{n=0}^{m}(p+m)s_p(m,n)x^n \\ &= x\sum_{n=0}^{m} s_p(m,n)x^n + (p+m)\sum_{n=0}^{m} s_p(m,n)x^n \\ &= (x+[p+m])h_m^p(x), \end{aligned}$$

because $s_p(m,n) = 0$ if $n < 0$ or $n > m$. $\square$

4. Symmetric Functions.

The r*th* elementary symmetric function of the variables $x_1, x_2, \ldots, x_n$ is $E_0(x_1, x_2, \ldots, x_n) = 1$, if $r = 0$, and

$$E_r(x_1, x_2, \ldots, x_n) = \sum_{\alpha \in Q_{r,n}} \prod_{t=1}^{r} x_{\alpha(t)},$$

$1 \leq r \leq n$, where $Q_{r,n}$ is the set of all $C(n,r)$ strictly increasing functions from $\{1, 2, \ldots, r\}$ to $\{1, 2, \ldots, n\}$. In other words, $E_r(x_1, x_2, \ldots, x_n)$ is the sum of all products of the x's taken r at a time. In particular, $E_r(x_1, x_2, \ldots, x_n) = 0$ if $r < 0$ or $r > n$. Moreover,

$$\begin{aligned} E_r(x_1, x_2, \ldots, x_n) = &E_r(x_1, x_2, \ldots, x_{n-1}) \\ &+ x_n E_{r-1}(x_1, x_2, \ldots, x_{n-1}), 1 \leq r \leq n. \end{aligned} \tag{17}$$

Theorem 4.1. *If p is fixed but arbitrary, the p-Stirling number of the first kind is given by the formula*

$$s_p(m,n) = E_{m-n}(p, p+1, \ldots, p+m-1). \tag{18}$$

When $p = 0$, (18) reduces to the well known identity

$$\begin{aligned} s(m,n) &= E_{m-n}(0,1,\ldots,m-1) \\ &= E_{m-n}(1,2,\ldots,m-1), \end{aligned}$$

for Stirling numbers of the first kind.

Proof. From Theorem 3.2, $s_p(m,n)$ is the coefficient of x^n in the polynomial

$$\begin{aligned} h_m^p(x) &= (p+x)^{(m)} \\ &= (p+x)([p+1]+x)\cdots([p+m-1]+x). \end{aligned}$$

On the other hand, the coefficient of x^n in the generic polynomial

$$(x+r_1)(x+r_2)\cdots(x+r_m)$$

is $E_{m-n}(r_1,r_2,\ldots,r_m)$ □

Theorem 4.1 can also be proved using Equations (9) and (17). The analog of Theorem 4.1 for p-Stirling numbers of the second kind involves homogeneous symmetric functions. Let $H_0(x_1,x_2,\ldots,x_n) = 1$ and

$$H_r(x_1,x_2,\ldots,x_n) = \sum_{\alpha\in G_{r,n}} \prod_{t=1}^{r} x_{\alpha(t)},$$

$r \geq 1$, where $G_{r,n}$ is the set of all $C(n+r-1,r)$ nondecreasing functions from $\{1,2,\ldots,r\}$ to $\{1,2,\ldots,n\}$. In other words, $H_r(x_1,x_2,\ldots,x_n)$ is the sum of all monomials of (total) degree r in the x's. In particular, $H_r(x_1,x_2,\ldots,x_n) = 0$ if $r < 0$, and

$$\begin{aligned} H_r(x_1,x_2,\ldots,x_n) &= H_r(x_1,x_2,\ldots,x_{n-1}) \\ &\quad + x_n H_{r-1}(x_1,x_2,\ldots,x_n), \quad r \geq 1. \end{aligned} \tag{19}$$

Theorem 4.2. *If p is fixed but arbitrary, the p-Stirling number of the second kind is given by the formula*

$$S_p(m,n) = H_{m-n}(p,p+1,\ldots,p+n). \tag{20}$$

When $p = 0$, (20) reduces to the well known identity

$$\begin{aligned} S(m,n) &= H_{m-n}(0,1,\ldots,n) \\ &= H_{m-n}(1,2,\ldots,n), \end{aligned}$$

for Stirling numbers of the second kind.

Proof. Using (19) one sees that the arrays defined on either side of (20) obey the same recurrence. Because $H_m(p) = p^m = S_p(m,0)$, the proof is complete. □

Corollary 4.3. *The ordinary generating function for p-Stirling numbers of the second kind is*

$$\sum_{m\geq 0} S_p(m,n)x^m = \prod_{t=0}^{n} \frac{x^n}{1-(p+t)x}.$$

Proof. From Theorem 4.2,

$$\begin{aligned} \sum_{m\geq 0} S_p(m,n)x^m &= \sum_{m\geq 0} H_{m-n}(p,p+1,\ldots,p+n)x^m \\ &= x^n \sum_{m\geq 0} H_{m-n}(p,p+1,\ldots,p+n)x^{m-n} \\ &= x^n \sum_{m\geq 0} H_m(p,p+1,\ldots,p+n)x^m. \end{aligned}$$

The result now follows from the well known fact (see, e.g., [5, p. 21] or [8, p. 285]) that

$$\sum_{m\geq 0} H_m(x_0,x_1,\ldots,x_n)z^m = \sum_{t=0}^{n}(1-x_t z)^{-1}. \tag{21}$$

□

Note that the right-hand sides of (18) and (20) are valid for any real p. This suggests an extension of the definitions of p-Stirling numbers

Definition 4.4. *Let $p \in \mathbb{R}$ be fixed but arbitrary. For integers m and n satisfying $m \geq n \geq 0$, the p-Stirling number of the first kind is $s_p(m,n) = E_{m-n}(p, p+1, \ldots, p+m-1)$, and the p-Stirling number of the second kind is $S_p(m,n) = H_{m-n}(p, p+1, \ldots, p+n)$.*

What are the implications of this definition for the results in Sections 2 and 3? Answer: None! Theorems 2.3 and 2.5 are the recurrence relations that led to (and are reflected in) the extended definitions. For fixed m and n, Theorems 2.2 and 2.6 are polynomial identities of degree $m-n$ in p, valid for all positive integers p. Thus, they remain valid for all $p \in \mathbb{R}$. Finally, because they depend only on previous results (and Newton's Binomial Theorem [7, p. 196]), the proofs of Corollary 2.7 and Theorems 3.1 and 3.2 are unaffected. Restated in terms of symmetric functions, Equation (11b) becomes:

Corollary 4.5. *For any $p \in \mathbb{R}$,*

$$\sum_{r=0}^{m} (-1)^{m+r} E_{m-r}(p, p+1, \ldots, p+m-1) H_{r-n}(p, p+1, \ldots, p+n) = \delta_{m,n}, \qquad (22)$$

where m and n are positive integers.

Because the numbers of variables are not the same, Corollary 4.5 is not an obvious consequence of the well known identity

$$\sum_{r=0}^{m} (-1)^r E_r(x_0, x_1, \ldots, x_n) H_{m-r}(x_0, x_1, \ldots, x_n) = 0, \quad m > 0.$$

(See, e.g., [5, p. 21], [7, p. 204] or [8, p. 286].) It is , however, a special case of the following,

Theorem 4.6. *Let m be a positive integer. If $x_0, x_1, \ldots, x_m$ are independent variables, then*

$$\sum_{r=0}^{m} (-1)^{i+r} E_{i-r}(x_0, x_1, \ldots, x_{i-1}) H_{r-j}(x_0, x_1, \ldots, x_j) = \delta_{i,j},$$

$1 \leq i, j \leq m$.

The author is indebted to the anonymous referee of an earlier effort for the following simple proof of Theorem 4.6.

Proof. Because $E_{i-r}(x_0, x_1, \ldots, x_{i-1})H_{r-j}(x_0, x_1, \ldots, x_j) = 0$ unless $i \geq r \geq j$ and the product is 1 when $i = r = j$, we may assume $i > j$. From Equation (21) and the relation between the roots of a generic monic polynomial and its coefficients (see the proof of Theorem 4.1),

$$\sum_{r \geq 0}(-1)^r E_r(x_0, x_1, \ldots, x_{i-1})z^r \sum_{s \geq 0} H_s(x_0, x_1, \ldots, x_j)z^s$$
$$= \frac{(1-x_0z)\cdots(1-x_{i-1}z)}{(1-x_0z)\cdots(1-x_jz)}$$
$$= (1-x_{j+1}z)\cdots(1-x_{i-1}z), \tag{23}$$

where (23) should be interpreted as 1 when $i = j+1$. Because (23) is a polynomial in z of degree $i-j-1$, the coefficient of z^{i-j} is zero. □

It should be observed that analogous results can be proved for the "q-Stirling numbers" [2] that can be defined as

$$s[m, n] = E_{m-n}(1, 1+q, 1+q+q^2, \ldots, 1+q+q^2+\cdots+q^{m-2})$$

and

$$S[m, n] = H_{m-n}(1, 1+q, 1+q+q^2, \ldots, 1+q+q^2+\cdots+q^{n-1}).$$

Indeed, both the p and the q-Stirling numbers (not to be confused with p, q Stirling numbers [10]) are special cases of "Comtet numbers" [1], [11].

5. Applications

Because it is the sum of its proper divisors, 6 = 1 + 2 + 3 is said to be *perfect.* In this context, 1 + 2 + 3 is the same as 3 + 2 + 1, but different from 4 + 2. In expressing the "perfection" of 6, what interests us is the "partition" [3,2,1].

A *partition* of m of *length* n is an unordered collection of n positive integer *parts* that sum to m. Because a partition is unordered, we may arrange its parts any way we like. The notation $[\pi_1, \pi_2, \ldots, \pi_n] \vdash m$ means $\pi_1 + \pi_2 + \cdots \pi_n = m$, where $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_n \geq 1$. The 3-part partitions of 6 are [4,1,1], [3,2,1], and [2,2,2]. (The words "partition" and "part" were used previously in a different way.)

The "Ferrers diagram" of $\pi = [\pi_1, \pi_2, \ldots, \pi_n] \vdash m$ is an array of n rows of "boxes" $B_{ij}, 1 \le j \le \pi_i$, $1 \le i \le n$. If box B_{ij} contains the expression $x - i + j$, then the product of the contents of all m boxes is the monomial

$$f_\pi(x) = \prod_{i=1}^{n} \prod_{j=1}^{\pi_i} (x - i + j).$$

Figure 1 illustrates the computation for $f_{[3,2]}(x) = x^2(x^2 - 1)(x + 2)$.

$$\begin{array}{l} [\ x\ ][x+1][x+2] \\ [x-1][\ x\ ] \end{array}$$

Figure 1.

Corresponding to each $\pi \vdash m$ is an "irreducible character", χ_π, of the symmetric group S_m. For our surposes, it suffices to think of χ_π as a function from S_m into the integers. The following formula arises in the context of the polynomial representations of the general linear group (see, e.e., [6]). If n is a fixed but arbitrary nonnegative integer, then

$$n^k = \sum_{\pi \vdash m} (\chi_\pi(e_m)\chi_\pi(\tau)/m!) f_\pi(n), \quad m \ge 1, \tag{24}$$

where e_m is the identity, and τ is any permutation in S_m whose disjoint cycle factorization consists of k cycles. If, for example, τ has just one cycle in its disjoint cycle factorization, then

$$\chi_\pi(\tau) = \begin{cases} (-1)^{m-r}, & \text{if} \quad \pi = [r, 1^{m-r}] \\ \\ 0, & \textit{otherwise} \end{cases} \tag{25}$$

where $[r, 1^{m-r}]$ is shorthand for $[r, 1, 1, \ldots, 1] \vdash m$. If $\pi = [r, 1^{m-r}]$, then $\chi_\pi(e_m) = m!/[(r-1)!(m-r)!m]$. Substituting these values into (24), we obtain

$$n = \sum_{r=1}^{m} (-1)^{m-r} C(n + r - 1, m) C(m - 1, r - 1), \quad m \ge 1,$$

a well known combinatorial identity. (See, e.g., [9, p.8].) More significant in the present context is the identity

$$\begin{aligned} f_\pi(x) &= \sum_{j=1-r}^{m-r}(x-j) \\ &= \sum_{n=1}^{m}(-1)^{m-n}E_{m-n}(1-r,2-r,\ldots,m-r)x^n \\ &= \sum_{n=1}^{m}(-1)^{m-n}s_{1-r}(m,n)x^n, \end{aligned} \tag{26}$$

when $\pi = [r, 1^{m-r}]$. Observing that the right-hand side of (26) is valid whether or not $m \geq r$, we define

$$f_{m,r}(x) = \sum_{n=1}^{m}(-1)^{m-n}s_{1-r}(m,n)x^n, \tag{27}$$

for all positive integers m and r.

m \ n	0	1	2	3	4	5
1	-3	1				
2	6	-5	1			
3	-6	11	-6	1		
4	0	-6	11	-6	1	
5	0	-6	5	5	-5	1

Figrue 2. $s_{-3}(m-n)$

Example 5.1. *Suppose* $r = 4$ *so that the coefficient of* x^n *in* $f_{m,r}(x)$ *is* $(-1)^{m-n}s_{-3}(m,n)$. *From* $s_{-3}(m,m) = 1$, $s_{-3}(m,0) = (-3)^{(m)}$, *and* $s_{-3}(m+1,n) = s_{-3}(m,n-1) + (m-3)s_{-3}(m,n)$, *the table in Figure 2 is easily constructed. Together with (27), this table* *yields*

$$\begin{aligned}
f_{1,4}(x) &= x \\
f_{2,4}(x) &= -(-5)x + x^2 = x(x+5) \\
f_{3,4}(x) &= 11x - (-6)x^2 + x^3 = x(x^2+6x+11) \\
f_{4,4}(x) &= -(-6)x + 11x^2 - (-6)x^3 + x^4 = x(x+1)(x+2)(x+3) \\
f_{5,4}(x) &= -6x - 5x^2 + 5x^3 - (-5)x^4 + x^5 = (x-1)x(x+1)(x+2)(x+3),
\end{aligned}$$

and so on. Notice that

$$\begin{aligned}
f_{m,4}(x) &= \prod_{j=-3}^{m-4} (x-j) \\
&= f_{[4,1^{m-4}]}(x), \quad m \geq 4
\end{aligned}$$

Equation (27) expresses $f_{m,r}(x)$ as a linear combination of x^n, $1 \leq n \leq m$. The inverse is an analog of a result of Kramer [3] concerning single-hook class sums and the center of the group algebra of S_m.

Theorem 5.2. *Let r be a fixed but arbitrary positive integer. Then*

$$x^m = \sum_{k=1}^{m} S_{1-r}(m,k) f_{k,r}(x). \tag{28}$$

Proof. For any $m \geq k$, we have from (27) that

$$S_{1-r}(m,k) f_{k,r}(x) = S_{1-r}(m,k) \sum_{n=1}^{k} (-1)^{k-n} s_{1-r}(k,n) x^n.$$

summing both sides on k yields

$$\sum_{k=1}^{m} S_{1-r}(m,k) f_{k,r}(x) = \sum_{k=1}^{m} S_{1-r}(m,k) \sum_{n=1}^{m} (-1)^{k-n} s_{1-r}(k,n) x^n$$

because $s_p(k,n) = 0$ for $n > k$. Interchanging the order of summation,

$$\sum_{n=1}^{m}(-1)^{m-n}(\sum_{k=1}^{m}(-1)^{m+k}S_{1-r}(m,k)s_{1-r}(k,n))x^n = \sum_{n=1}^{m}\delta_{m,n}x^n = x^m$$

by (11b). $\square$

Example 5.3. *When $r = 1$, $f_{m,1}(x) = f_{[1^m]}(x) = x(x-1)\cdots(x-m+1)$, the "falling factorial function", and $s_{1-1}(m,n) = s_0(m,n) = s(m,n)$, a stirling number of the first kind. In this case (28) becomes one of the classical facts about Stirling numbers of the second kind.*

Our final result is an analog of a classical identity equating the Jacobi-Trudi and Nägelsbach-Kostka formulas for Schur functions. The transpose of the Ferrers diagram of $\pi \vdash m$ is the Ferrers diagram of the conjugate partition $\pi^* \vdash m$. Thus, π^* is the partition of length π_1 whose ith part is $\pi_i^* = \circ(\{j : \pi_j \geq i\})$.

Corollary 5.4. *Suppose $\pi = [\pi_1, \pi_2, \ldots, \pi_r]$ is partition of m. Let $\pi^* = [\pi_1^*, \pi_2^*, \ldots, \pi_s^*]$ be its conjugate partition. Then, abbreviating $E_r(x_0, x_1, \ldots, x_k)$ as $E_r(, x_k)$ and $H_r(x_0, x_1, \ldots, x_k)$ as $H_r(, x_k)$*

$$\det(H_{\pi_i-i+j}(, x_{s-\pi_i+i})) = \det((-1)^{i+j}E_{\pi_i^*-i+j}(, x_{s+\pi_i^*-i})). \tag{29}$$

Using the Theorem 4.6 and Jacobi's Identity [8, p. 237], the proof of (29) is identitical to the proof in the classical case [8, p. 287]. (While he hasn't seen it, the author has been told that a result equivalent to Corollary 5.4 can be found in [4].)

Example 5.5. *Suppose $m = 5$ and $\pi = [3,2]$, so $r = 2, s = 3$, and $\pi^* = [2,2,1]$. Setting $x_0 = v$, $x_1 = w$, $x_2 = x$, $x_3 = y$, and $x_4 = z$ the lefthand side of (29) becomes*

$$\begin{aligned} &det\begin{pmatrix} H_3(v,w) & H_4(v,w) \\ H_1(v,w,x,y) & H_2(v,w,x,y) \end{pmatrix} \\ &= H_3(v,w)H_2(v,w,x,y) - H_4(v,w)H_1(v,w,x,y) \end{aligned}$$

which, after some computation, can be expressed as

$$(v^3 + v^2w + vw^2 + w^3)\ (x^2 + xy + y^2) + (v^3w + v^2w^2 + vw^3)\ (x + y) + v^2w^2(v + w)$$

After substituting appropriate values, the righthand side of (29) becomes

$$\det \begin{pmatrix} E_2(v,w,x,y,z) & -E_3(v,w,x,y,z) & E_4(v,w,x,y,z) \\ -E_1(v,w,x,y) & E_2(v,w,x,y) & -E_3(v,w,x,y) \\ 0 & -E_0(v,w) & E_1(v,w) \end{pmatrix}$$
$$\begin{aligned} &= E_2(v,w,x,y,z)[E_2(v,w,x,y)E_1(v,w) - E_3(v,w,x,y)] \\ &\quad + E_1(v,w,x,y)[E_4(v,w,x,y,z) - E_3(v,w,x,y,z)E_1(v,w)] \\ &= (v^3 + v^2w + vw^2 + w^3)(x^2 + xy + y^2) + (v^3w + v^2w^2 + vw^3)(x + y) + v^2w^2(v + w). \end{aligned}$$

(The coefficient of z in the penultimate expression is

$$\begin{aligned} &E_1(v,w,x,y)[E_2(v,w,x,y)E_1(v,w) - E_3(v,w,x,y)] \\ &+ E_1(v,w,x,y)[E_3(v,w,x,y) - E_2(v,w,x,y)E_1(v,w)] = 0. \end{aligned}$$

While the (common) value of the two determinants is symmetric in v and w, and in x and y, unlike the Jacobi-Trudi and Nägelsbach-Kostka determinants, it is not symmetric in all four variables, much less in v, w, x, y, and z.

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