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Strongly Prime Ideals in CS-Rings

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Abstract

We study and characterize strongly prime right ideals in CS-rings.

1. Introduction

Throughout this paper all rings will be associative with identity and modules will be unital right modules. A ring R is called right *CS-ring(or extending ring)* if every right ideal I in R is essentially contained in a direct summand of R, (see for example [9]). Every right self-injective ring is right CS-ring. A right ideal I of a ring R is called *strongly prime* if for $a, b \in R$, $aIb \leq I$ and $ab \in I$ imply $a \in I$ or $b \in I$. Every maximal right ideal is strongly prime right ideal. Let M be a module and N a submodule of M. N is called *prime submodule* of M if $N \neq M$ and whenever $r \in R$, $m \in M$ and $mr \in N$ then $m \in N$ or $Mr \leq N$. Prime submodules have been extensively studied (see for example[1]-[3]). For a commutative ring R it is well known that a submodule N of M is prime if and only if $P = \{r \in R : Mr \leq N\}$ is a prime ideal of R and the $(R/P) - module$ M/N is torsion free [3,Lemma 1]. Let M be a module. We write $N \leq M$ for a submodule N of M. $N << M$ will stand for N is small submodule of M, equivalently $N + K = M$ for submodule K of M implies $K = M$. A regular ring will mean a von Neumann regular ring [5]. In [10], it is proved that a maximal right ideal, which is projective in a self-injective regular ring, is a direct summand. This result is generalized to strongly prime right ideals in self-injective regular rings in [7]. Let M be a module. If every submodule of M is contained in a maximal submodule in M then M is called coatomic module. By [6, Exercise 9(c),page 239] a module M is semisimple if and only if M is a coatomic and every maximal submodule is direct summand. Since every ring R is a coatomic $R-$module then a ring R is semisimple if and only if every maximal right ideal in R is direct summand. There is a self-injective regular ring having maximal right ideal which is not projective(see namely [11]). Every maximal submodule N of any module M is essential or direct summand. In this vein we prove the following.

2. Results

Lemma 1. *Let R be a right CS-ring. Then any strongly prime right ideal P of R is either essential or a direct summand.*

Proof. Let P be a strongly prime right ideal in R. Then there exists an idempotent $e \in R$ such that $eP = P \leq eR$. Now $(1-e)Pe \leq P$ and $(1-e)e = 0$ imply $1-e \in P$ or $e \in P$. Assume $1-e \in P$ then $(1-e)R \leq P$ and hence $P = R$. If $e \in P$ then $P = eR$. □

The ring R as in [11] which is (commutative) self-injective regular that is not semisimple contains a maximal, therefore strongly prime, right ideal which is not projective. In such a ring not all strongly prime right ideals are a direct summand.

The next Lemma generalises [10,Proposition 1] and [7,Proposition 2].

Lemma 2. *Let R be a regular right CS-ring. Then any projective strongly prime right ideal in R is a direct summand.*

Proof. Let P be a projective strongly prime right ideal of R. Then by Lemma 1, P is either essential or direct summand. Assume P is essential. By [8] and hypothesis $P = \bigoplus_{i\in\Lambda}(e_iR); e_i^2 = e_i \in R$ for all $i \in \Lambda$ for some index set Λ. By [4, Lemma 3.8], there exist orthogonal idempotents $f_i (i \in \Lambda)$ such that $e_iR = f_iR (i \in \Lambda)$. Hence $P = \bigoplus_{i\in\Lambda}(f_iR)$ and $f_i^2 = f_i, f_if_j = f_jf_i = 0$ for $i \neq j; i, j \in \Lambda$. Let $\Lambda = \Lambda_1 \cup \Lambda_2$ be a decomposition of Λ into infinite disjoint subsets Λ_1 and Λ_2 and write $P = P_1 \oplus P_2$ where $P_1 = \bigoplus_{i\in\Lambda_1}(f_iR)$ and $P_2 = \bigoplus_{i\in\Lambda_2}(f_iR)$. Since R is a right CS-ring and regular there exist orthogonal idempotents e, f such that $K_1 = eR$, $K_2 = fR$ and $P_1 \leq_e K_1, P_2 \leq_e K_2, R = K_1 \oplus K_2$. Hence $ef = fe = 0$ and $eP = eP_1 + eP_2 = eP_1 = P_1 \leq P$. Since P is strongly prime $e \in P$ or $f \in P$. Assume $e \in P$. Then $eR = K_1 \leq P$ and $e = \sum_{i\in A} f_ir_i$, for some finite subset

A of Λ. Hence $f_i e = 0$ for all $i \in \Lambda_1 \setminus A$, and since $eR = K_1 = \bigoplus_{i\in\Lambda_1}(f_i R)$,$ef_i = f_i$ for all $i \in \Lambda_1$. Thus $f_i = f_i^2 = (ef_i)^2 = 0$ for all $i \in \Lambda_1 \setminus A$. Hence Λ_1 is finite. This is a contradiction. □

A ring R is called right continuous if R is a right CS-ring and for any right ideal isomorphic to a direct summand of R is also direct summand (or equivalently generated by an idempotent.)

Lemma 3. *Let R be a right continuous ring. Then for any projective strongly prime right ideal I containing the Jacobson radical J of R, there exists an idempotent e such that $I = eR + J$.*

Proof. Let R be a right continuous ring and I a projective strongly prime right ideal containing J. Then it is easy to check that I/J is strongly prime right ideal in R/J. By [4, Prop.3.11] R/J is a right continuous ring and by [4, Prop.3.5] $J = \{a \in R : r(a)$is essential right ideal in $R\}$, where $r(a)$ denotes the right annihilator of a in R. Let $f \in Hom(I, R)$. Since for any $x \in R$, $r(x) \subseteq r(f(x))$, then $f(J) \subseteq J$. By the dual basis Lemma it follows that I/J is a projective right ideal in R. By lemma 2, I/J is a direct summand of R/J. Since idempotents of R/J lift to R by [4, Lemma 3.7], an idempotent e exists in R such that $I = eR + J$. This completes the proof. □

Example 4. Let R be a principal ideal domain which is not a field. Let P be a nonzero maximal ideal in R. Then P is a projective and essential ideal. R is not a regular ring but a CS-ring. Every projective proper ideal is isomorphic to R and is not a direct summand of R.

Proposition 5 generalises 40.1 in [12].

Proposition 5. *Let R be a non-singular commutative $CS-$ring with maximal quotient ring Q. Let I be a nonzero ideal in R. Then I is projective if and only if there exists $a_1, ..., a_t \in I$ and $g_1, ..., g_t \in Hom(I, R)$ such that $a = \sum_{i=1}^{t} a_i g_i(a)$ for all $a \in I$.*

Proof. Let I be a projective ideal in R. Then there are $\{a_\lambda \in I : \lambda \in \Lambda\}$ and $\{f_\lambda \in Hom(I, R) : \lambda \in \Lambda\}$ such that $a = \sum a_\lambda f_\lambda(a)$ and $f_\lambda(a)$ is zero for all but a finite number of $\lambda \in \Lambda$. Let g_λ denote the extension of f_λ from R to Q. Since Q_R is an injective $R-$module, g_λ always exists for each $\lambda \in \Lambda$. Let a be a nonzero element in I. Then $a = \sum_{i=1}^{t} a_i f_i(a) = \sum_{i=1}^{t} a_i g_i(a) = (\sum_{i=1}^{t} a_i g_i(1))a$, and so $(1 - \sum_{i=1}^{t} a_i g_i(1))a = 0$. Since R is commutative and I is essential in R, and then in Q we have $1 = \sum_{i=1}^{t} a_i g_i(1)$. Let $a \in I$ be any element. Then $a = \sum_{i=1}^{t} a_i g_i(a)$. Conversely, assume that there exist $a_1, ..., a_t \in I$,$g_1, ..., g_t \in Hom(R, I)$ such that $a = \sum_{i=1}^{t} a_i g_i(a)$ for all $a \in I$. By the dual basis Lemma I is projective. □

Lemma 6. *Let M be a projective module and N a submodule of M such that M/N has a projective cover. Let $S = End_R M$ and $F(N) = \{\alpha \in S : \alpha N \leq N\}$. Then there exists $\alpha \in F(N)$ such that $\alpha^2 = \alpha$ and $\alpha N << M$.*

Proof. Let M/N have a projective cover (P, f) with $f : P \to M/N$ and $Ker(f) << P$. Since M is projective module there exists $g \in Hom(M, P)$ such that $P = g(M)$ and $M = (Ker(g)) \oplus K$ for some $K \leq M$. Assume $N \leq Ker(g)$. Let α denote the natural projection of M on K then $\alpha N = 0 \leq N$ and $\alpha N << M$. If $N \nleq Ker(g)$, then $M = Ker(g) \oplus K$, $g(N) \leq Ker(f)$ and $g(N)$ is small in P. Since $g(M) = g(K) = P$ is projective there exists $\phi \in Hom(g(M), M)$ such that $g\phi = 1_{g(M)}$, the identity map of $g(M)$, and $\phi g(N)$ is small in M. For all $m \in M$, $\pi\phi g(m) = \pi(m)$. It follows that $\phi g(N) \leq N$. Set $\phi g = \alpha$, then $\alpha^2 = \alpha$ and $\alpha(N) << M$ and $\alpha \in F(N)$. □

Let R be a ring and I a right ideal in R. In the following $N(I)$ will denote the set $\{r \in R : rI \subseteq I\}$. Note that if e is a nonzero idempotent with $e \in N(I) \setminus I$ and I is a strongly prime right ideal in a ring R such that eI is small in R then R/I has a projective cover. The proof of this fact is known. We give the proof for the sake of completeness.

Proof. Let I be a strongly prime ideal in a ring R and $e \in N(I) \setminus I$ such that eI is small in R. Then $eI \leq I$ so $eI(1-e) \leq I$ and $e(1-e) = 0 \in I$. Since I is a strongly prime right ideal then $e \in I$ or $(1-e) \in I$. Assume $e \in I$ then $eR \leq I$. Since $eR \oplus (1-e)R = R$ $eI + (1-e)R = R$ and so $(1-e)R = R$. Thus $eR = 0$ and so $e = 0$. It follows that $1 - e \in I$, then $(1-e)R \leq I$. Now we define $f : eR \to R/I$ by $f(er) = r + I$. Since $(1-e)R \leq I$, then f is well defined and clearly an $R-$module homomorphism and also

$Ker(f) = eI$. Since eR is projective R-module and eI is small in R then R/I has projective cover (eR, f). □

Definition. Let M be a module and S denote the ring of $R-$endomorphisms of M. Let N be a submodule of M. We call N an $S-$prime submodule of M if whenever $f(m) \in N$, for some $f \in S$ and $m \in M$, then $f(M) \leq N$ or $m \in N$. N is called an $S-$strongly prime submodule of M if whenever $f(m) \in N$ for some $f \in S$ and $m \in M$ then $m \in N$. Any $S-$strongly prime submodule is $S-$prime, and for $M = R$ and $I_R \leq R_R$, being I $R-$(strongly)prime submodule of R in the same as being I (strongly) prime right ideal of R. Note that any $S-$prime submodule N of M is prime submodule ([see 3 or 11]) over a commutative ring.

Lemma 7. *Let N be an $S-$prime submodule of a projective module M. Assume that there exists $0 \neq f \in S$ such that $f^2 = f$, $f(N) \leq N$ and $f(N) << M$. Then M/N has a projective cover.*

Proof. Let N be an $S-$prime submodule of M and $f \in S$ such that $f^2 = f$ and $f(N) \leq N$ and $f(N) << M$. Let $m \in M$. Since $f(1-f)(m) = 0 \in N$ and N is $S-$prime $f(M) \leq N$ or $(1-f)(m) \in N$. Assume that $(1-f)(m) \notin N$ for some $m \in M$. Then $f(M) \leq N$ and so $f(M) \leq f(N)$ and $M = f(N) + (1-f)(M)$. Hence $(1-f)(M) = M$ since $f(N) << M$. Thus $f = 0$. If $(1-f)(m) \in N$ for all $m \in M$ then we define $h : f(M) \rightarrow M/N$ by $h(f(m)) = m + N$ for $m \in M$. Since $f(m) = 0$ implies $m = (1-f)(m) \in N$, h is well-defined. Clearly h is an R-homomorphism and $Ker(h) = f(N)$. Since $f(M)$ is projective and $f(N)$ is small in M, $(f(M), h)$ is a projective cover of M/N. □

Corollary 8 *Let I be a right ideal of R. Assume that I is a prime submodule of $R-$module R and a nonzero idempotent e exists in R such that $eI \leq I$ and eI is small in R. Then R/I has a projective cover.*

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