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QR-Submanifolds and Almost Contact 3-Structure

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Abstract

In this paper,QR-submanifolds of quaternion Kaehlerian manifolds with $\dim \nu^{\perp} = 1$ has been considered. It is shown that each QR-submanifold of quaternion Kaehlerian manifold with $\dim \nu^{\perp} = 1$ is a manifold with an almost contact 3-structure. We apply geometric theory of almost contact 3-structure to the classification of QR-submanifolds (resp.Real hypersurfaces) of quaternion Kaehler manifolds (resp.IR^{4m}, $m > 1$). Some results on integrability of an invariant distribution of a QR-submanifold and on the immersions of its leaves are also obtained.

Key Words: Quaternion Kaehler Manifold,QR-Submanifold, Almost Contact 3-Structure

1. Introduction

The geometry of QR-submanifolds of a quaternion Kaehlerian manifolds was firstly reported by Bejancu[1]. Among all submanifolds of a quaternion Kaehlerian manifold, QR-submanifolds have been intensively studied from different points of view by many authors [1],[2],[4].

In case of $\dim \nu^{\perp} = 1$, the study of QR-submanifolds has a significant importance. We show that QR-submanifolds of quaternion Kaehler manifolds with $\dim \nu^{\perp} = 1$ admit an almost contact 3-structure.

2. Preliminaries

Let $\overline{M}$ be a $4n-$ dimensional manifold and g be a Riemannian metric on $\overline{M}$. Then $\overline{M}$ is said to be quaternion Kaehlerian manifold (see, [1]) if there exists a $3-$dimensional vector bundle of tensors of type(1.1) with local basis of almost Hermitian structures

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J_1, J_2, J_3 satisfying

$$J_1 o J_2 = -J_2 o J_1 = J_3 \tag{II.1}$$

and

$$\overline{\bigtriangledown}_X J_a = \sum Q_{ab}(X) J_b, a = 1, 2, 3 \tag{II.2}$$

for all vector fields X tangent to $\overline{M}$,where $\overline{\bigtriangledown}$ is the Levi-Civita connection determined by g on $\overline{M}$ and Q_{ab} are certain $1-$forms locally defined on $\overline{M}$ such that $Q_{ab} + Q_{ba} = 0$.

Let M be $(4m + 3)-$dimensional differenntiable manifold and (ϕ_a, ξ_a, η_a) be three almost contact structures on M i.e. We have

$$\begin{array}{c} \phi_a^2(X) = -X + \eta_a(X)\xi_a, \phi_a(\xi_a) = 0 \\ \eta_a(\xi_a) = 1, \eta_a o \phi_a = 0 \end{array} \tag{II.3}$$

where X tangent to M. Suppose that the almost contact structures satisfy the following conditions

$$\begin{array}{c} \eta_a(\xi_b) = 0, a \neq b, \phi_a(\xi_b) = -\phi_b(\xi_a) = \xi_c \\ \eta_a o \phi_b = -\eta_b o \phi_a = \eta_c \\ (\phi_a o \phi_b)(X) - \xi_a(\eta_b(X)) = (\phi_b o \phi_a)(X) - \xi_b(\eta_a(X)) = \phi_c(X) \end{array} \tag{II.4}$$

for any cyclic permutation $(a, b, c)$of $(1, 2, 3)$. Then, we say that M is endowed with an almost contact 3-structure. If M is a Riemannian manifold, then we can choose a Riemannian metric g on M such that we have

$$\begin{array}{c} (\phi_a X, \phi_a Y) = g(X, Y) - \eta_a(X)\eta_a(Y) \\ \eta_a(X) = g(X, \xi_a) \end{array} \tag{II.5}$$

for any $X, Y \in \chi(M)$. In this case we say that $(\phi_a, \xi_a, \eta_a), a = 1, 2, 3$ define an almost contact metric structure (See [5]). Taking account of (II.3) and (II.5), we obtain

$$g(\phi_a X, Y) + g(X, \phi_a Y) = 0, a = 1, 2, 3$$

for any X, Y tangent to M.

Let M be an $m-$dimensional Riemannian manifold isometrically immersed in $\overline{M}$. We denote by TM (resp. $TM^{\perp}$) the tangent (resp. normal) bundle to M. Then we say that M is a quaternion-real submanifold (QR-Submanifold) if there exists a vector bundle υ of the normal bundle such that we have

$$J_a(\upsilon_x) = \upsilon_x \tag{II.6}$$

and

$$J_a(\upsilon_x^{\perp}) \subset T_M(x) \tag{II.7}$$

for $x \in M$ and $a = 1, 2, 3$, where $\upsilon^{\perp}$ is the complementary orthogonal bundle to υ in $TM^{\perp}$. Let M be a QR-submanifold of $\overline{M}$. For sake of shortness we use D_{ax} for $D_{ax} = J_a(\upsilon_x^{\perp})$, $a = 1, 2, 3$. We consider $D_{1x} \oplus D_{2x} \oplus D_{3x} = D_x^{\perp}$ and $3s-$dimensional distribution $D^{\perp} : x \longrightarrow D_x^{\perp}$ globally defined on M. Where $s = \dim \upsilon_x^{\perp}$. Also we have, for each $x \in M$,

$$J_a\left(D_{ax}\right) = \upsilon_x^{\perp}, J_a\left(D_{bx}\right) = D_{cx}, \tag{II.8}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. Next, we denote by D the complementary orthogonal distribution to $D^{\perp}$ in TM, we see that D is invariant with respect to the action of J_a. i.e. we have

$$J_a(D_x) = D_x \tag{II.9}$$

for any $x \in M$. D is called quaternion distribution . Also note that D_{1x}, D_{2x}, D_{3x} are mutually orthogonal vector spaces of $T_M(x)$ (see [1]).

In [3] D.E.Blair introduced the concept cosymplectic structure as it follows. An almost contact metric structure (ϕ, ξ, η, g) is a cosymplectic structure if and only if

$$(\nabla_X \phi)\, Y = 0, (\nabla_X \eta)\, Y = 0,$$

where ∇ is Levi-civita connection on M.

Definition 2.1 *An almost 3-contact structure* (ϕ_a, ξ_a, η_a) *is* *a) a 3-cosymlectic structure if*

$$(\nabla_X \phi_a) Y = 0, \quad (\nabla_X \eta_a) Y = 0 \tag{II.10}$$

b)a 3-Sasakian structure if

$$(\nabla_X \phi_a) Y = \eta_a(Y)X - g(X,Y)\xi_a \tag{II.11}$$

c) a nearly 3-Sasakian structure if

$$(\nabla_X \phi_a) X = \eta_a(X)X - g(X,X)\xi_a \tag{II.12}$$

where ∇ denotes the Levi-Civita connection and X, Y, Z are arbitrary vector fields on M.

For $Y \in \chi(M)$, we decompose as follows

$$J_a Y = P_a Y + F_a Y, a = 1, 2, 3 \tag{II.13}$$

where $P_a Y$ and $F_a Y$ the tangential and normal parts of $J_a Y$, respectively.

The formula Gauss and Weingarten are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{II.14}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{II.15}$$

for any vector fields X, Y tangent to M and any vector field V normal to M. Where ∇ is the induced Riemann connection in M, h is the second fundemental form, A_V is fundemental tensor of Weingarten with respect to the normal section V and $\nabla^{\perp}$ normal connection on M. Moreover we have the relation

$$g(A_V X, Y) = g(h(X,Y), V) \tag{II.16}$$

3. QR-Submanifolds with dim $\upsilon^{\perp} = 1$

Let M be a QR-submanifold of a quaternion Kaehlerian manifold $\overline{M}$ such that the dimension $\upsilon^{\perp}$ is equal to one. In this case $\upsilon^{\perp}$ is generated by unit vector field, say N.

We shall show in the sequel that N is precisely determined with one of the almost contact 3-structure. Let $-J_a(N) = \xi_a, a = 1, 2, 3$ and hence the distributions D_a are generated by the vector fields ξ_a. It is natural expect that a QR-submanifold of quaternion Kaehlerian manifold with $\dim \upsilon^{\perp} = 1$ is almost contact 3-structure, we describe it as follows;

Proposition 3.2 *Let $\overline{M}$ be a quaternion Kaehlerian manifold and M be a QR-submanifold of $\overline{M}$. Then M is a manifold with almost contact 3-structure.*

Proof. For any $X \in \Gamma(TM)$, put

$$\phi_a X = P_a X, F_a(X) = \eta_a(X)N. \tag{III.1}$$

Then $g(F_a X, N) = \eta_a(X)$ and

$$\eta_a(X) = g(X, \xi_a). \tag{III.2}$$

Moreover $g(P_a X, P_a Y) = g(J_a X, J_a Y) - g(F_a X, F_a Y)$ implies

$$g(\phi_a X, \phi_a Y) = g(X, Y) - \eta_a(X)\eta_a(Y) \tag{III.3}$$

and

$$J_a^2 X = J_a P_a X + J_a F_a X$$

or

$$-X = P_a^2 X + \eta_a(X) J_a N.$$

Hence

$$\phi_a^2 X = -X + \eta_a(X)\xi_a. \tag{III.4}$$

On the other hand, from (III.2) and (III.4) we obtain $\phi_a(\xi_a) = 0$ and $\eta_a(\xi_a) = 1$, respectively. Moreover

$$(\eta_a o \phi_a) X = \eta_a(\phi_a X) = g(\phi_a X, \xi_a) = 0$$

for any $X \in \Gamma(TM)$. From (III.1) and (III.2)

$$\eta_a(\xi_b) = 0$$

and

$$\begin{aligned}\phi_a(\xi_b) &= J_a(\xi_b) - F_a(\xi_b)\\ &= -Ja(J_bN)\\ &= -J_cN = \xi_c.\end{aligned}$$

By using (II.13) and (III.4), we obtain

$$\begin{aligned}(\eta_a o \phi_b)(X) &= \eta_a(\phi_b X)\\ &= g(\phi_b X, \xi_a)\\ &= \eta_c(X)\end{aligned}$$

and

$$\begin{aligned}(\phi_a o \phi_b)(X) - \xi_a(\eta_b(X)) &= (\phi_b o \phi_a)(X) - \xi_b(\eta_a(X))\\ &= (\phi_b(\phi_a(X)) - \xi_b(\eta_a(X))\\ &= \phi_c(X).\end{aligned}$$

This shows that on a QR-submanifold of a quaternion Kaehlerian manifold $\dim \upsilon^{\perp} = 1$ there exists a natural almost contact 3-structure. i.e. tensor field ϕ_a of type (1,1), 1-form η_a and unit vector field ξ_a satisfy (II.3),(II.4) and (II.5).

From now on will denote by M a QR-submanifold with $\dim \upsilon^{\perp} = 1$.

Theorem 3.3 *Let M be a QR-submanifold of a quaternion Kaehlerian manifold. If for any $X, Y \in \Gamma(TM)$, $h(X,Y)$ has no component in $\Gamma(\upsilon^{\perp})$ and $D_a, a = 1,2,3$ are parallel in M, then M is a manifold with cosymplectic 3-structure.*

Proof. For any $X, Y \in \Gamma(TM)$, from (II.2) we have

$$\overline{\nabla}_X J_a Y = Q_{ab}(X) J_b Y + Q_{ac}(X) J_c Y + J_a \overline{\nabla}_X Y.$$

Taking account of (II.13),(II.14),(II.15) and (III.1) we obtain

$$\begin{aligned}(\nabla_X \phi_a)\, Y + (\nabla_X \eta_a(Y))\, N + h(X, \phi_a Y)& \\ -\eta_a(Y) A_N X + \eta_a(Y) \nabla^{\perp}_X N &= Q_{ab}(X)\phi_b Y + Q_{ab}(X)\eta_b(Y)N \\ &+ Q_{ac}(X)\phi_c Y + Q_{ac}(X)\eta_c(Y)N \\ &+ B_a h(X,Y) + C_a h(X,Y).\end{aligned}$$

Comparing the tangential and normal parts both of sides of this equation, we have

$$\begin{aligned}(\nabla_X \phi_a)\, Y = &\ \eta_a(Y) A_N X + Q_{ab}(X)\phi_b Y + Q_{ac}(X)\phi_c Y \\ &+ B_a h(X,Y)\end{aligned} \tag{III.5}$$

and

$$\begin{aligned}(\nabla_X \eta_a(Y))\, N = &-h(X, \phi_a Y) + \eta_a(Y)\nabla^{\perp}_X N \\ &+ Q_{ab}(X)\eta_b(Y)N + Q_{ac}(X)\eta_c(Y)N \\ &+ C_a h(X,Y).\end{aligned} \tag{III.6}$$

Now, we decompose $h(X,Y)$ into components $h^1(X,Y)$ and $h^2(X,Y)$ in $\upsilon^{\perp}$ and υ respectively as

$$h(X,Y) = h^1(X,Y) + h^2(X,Y)$$

where we can put $h^1(X,Y) = \alpha\,(X,Y)\,N$ for some scalar valued bilinear function α. Thus (III.5) and (III.6) gives

$$\begin{aligned}(\nabla_X \phi_a)\, Y = &\ \eta_a(Y) A_N X + Q_{ab}(X)\phi_b Y + Q_{ac}(X)\phi_c Y \\ &- \alpha(X,Y)\xi_a\end{aligned} \tag{III.7}$$

and

$$(\nabla_X \eta_a)\,(Y) = -\alpha(X, \phi_a Y) + Q_{ab}(X)\eta_b(Y) + Q_{ac}(X)\eta_c(Y). \tag{III.8}$$

On the other hand, since $\xi_a = -J_a N$ we have

$$\overline{\nabla}_X\xi_a = -(\overline{\nabla}_X J_a)N - J_a\overline{\nabla}_X N$$

for any $X \in \Gamma(TM)$. Thus by using (II.2),(II.8),(II.15) and taking tangential parts we obtain

$$\nabla_X\xi_a = Q_{ab}(X)\xi_b + Q_{ac}(X)\xi_c + \phi_a A_N X$$

for any $X \in \Gamma(TM)$. Thus we have

$$g(\nabla_X\xi_a, \xi_b) = Q_{ab}(X) + g(\phi_a A_N X, \xi_b).$$

From (II.4) and (II.16) we derive

$$\eta_b(\nabla_X\xi_a) + \alpha(X, \xi_c) = Q_{ab}(X) \tag{III.9}$$

Similarly, we get

$$\eta_c(\nabla_X\xi_a) - \alpha(X, \xi_b) = Q_{ac}(X) \tag{III.10}$$

Combining (III.7) and (III.8) with (III.9) and (III.10) we have

$$\begin{aligned}(\nabla_X\phi_a)Y = \eta_a(Y)A_N X &+ (\eta_b(\nabla_X\xi_a) + \alpha(X, \xi_c))\phi_b Y \\ &+ (\eta_c(\nabla_X\xi_a) - \alpha(X, \xi_b))\phi_c Y - \alpha(X, Y)\xi_a\end{aligned} \tag{III.11}$$

and

$$\begin{aligned}(\nabla_X\eta_a)(Y) = -\alpha(X, \phi_a Y) &+ (\eta_b(\nabla_X\xi_a) + \alpha(X, \xi_c))\eta_b(Y) + \\ &(\eta_c(\nabla_X\xi_a) - \alpha(X, \xi_b))\eta_c(Y)\end{aligned} \tag{III.12}$$

for any $X, Y \in \Gamma(TM)$. From (II.16) and (III.11) we get

$$\begin{gathered}g((\nabla_X\phi_a)Y, Z) = \eta_a(Y)\alpha(X, Z) + \eta_b(\nabla_X\xi_a)g(\phi_b Y, Z) \\ +\alpha(X, \xi_c)g(\phi_b Y, Z) + (\eta_c(\nabla_X\xi_a) - \alpha(X, \xi_b))g(\phi_c Y, Z) \\ -\alpha(X, Y)\eta_a(Z).\end{gathered} \tag{III.13}$$

Thus if $h(X,Y)$ has no components in $\Gamma\left(v^{\perp}\right)$ and $D_a, a = 1,2,3$ are parallel in M, then from (III.12) and (III.13), we get $(\nabla_X \phi_a) Y = 0$ and $(\nabla_X \eta_a)(Y) = 0$ that is M is a manifold with cosymplectic 3-structure.

As immediate consequence of theorem we have the following;

Corollary 3.4 *Let M be real hypersurface of quaternion Kaehler manifold $\overline{M}$. If M is totally geodesic and ξ_a are parallel in M then M is a manifold with cosymplectic 3-structure.*

Corollary 3.5 *Let M be real hypersurface in $IR^{4m}(m>1)$. Then M is totally geodesic if and only if M is a manifold with cosymplectic 3-structure.*

Proof. Since $\overline{M} = IR^{4m}(m>1)$ we have $\overline{\nabla}_X J_a = 0$. Thus from (III.13) we get

$$g((\nabla_X \phi_a) Y, Z) = \eta_a(Y)\alpha(X,Z) - \alpha(X,Y)\eta_a(Z) \tag{III.14}$$

for any $X, Y, Z \in \Gamma(TM)$. Let M be a totally geodesic real hypersurface of $IR^{4m}(m>1)$ then from (III.14) we have

$$g((\nabla_X \phi_a) Y, Z) = 0$$

and from (III.8) we obtain

$$(\nabla_X \eta_a)(Y) = 0$$

for any $X, Y \in \Gamma(TM)$. Thus M is a manifold with cosymplectic 3-structure.

Conversely, if M is a manifold with cosymplectic 3-structure, from (III.14) we get

$$\eta_a(Y) A_N X = \alpha(X,Y)\xi_a$$

or

$$\eta_1(Y) A_N X = \alpha(X,Y)\xi_1$$
$$\eta_2(Y) A_N X = \alpha(X,Y)\xi_2$$
$$\eta_3(Y) A_N X = \alpha(X,Y)\xi_3$$

since ξ_1, ξ_2, ξ_3 are linearly independent we get $\alpha(X,Y) = 0$. Thus proof is complete.

From (III.14) we have the following corollary.

Corollary 3.6 *Let M be real hypersurface in $IR^{4m}(m>1)$. Then M is a manifold with Sasakian 3-structure if and only if $\alpha(X,Y)=g(X,Y)$ for any $X,Y\in\Gamma(TM)$.*

Corollary 3.7 *Let M be totally umbilical real hypersurface in $IR^{4m}(m>1)$. Then M is a manifold with nearly Sasakian 3-structure.*

Proof. For any $X,Y\in\Gamma(TM)$, from (III.7) we get

$$\begin{aligned} g((\nabla_X\phi_a)X,Y) &= \eta_a(X)g(h(X,Y),N)-\alpha(X,X)\eta_a(Y) \\ &= \eta_a(X)g(X,Y)-g(X,X)g((Y,\xi_a). \end{aligned}$$

Thus M is a manifold with nearly Sasakian 3-structure.

Theorem 3.8 *Let M be a QR-submanifold of quaternion Kaehler manifold such that $(\bigtriangledown_X\phi_a)Y=0$,$X,Y\in\Gamma(D)$. Then the quaternion distribution is involutive.*

Proof. By using (III.6), we obtain

$$\begin{aligned} g([X,Y],\xi_a) &= g(\bigtriangledown_XY,\xi_a)-g(\bigtriangledown_YX,\xi_a) \\ &= Xg(Y,\xi_a)-g(Y,\bigtriangledown_X\xi_a)-Yg(X,\xi_a)+g(X,\bigtriangledown_Y\xi_a) \\ &= g(X,\bigtriangledown_Y\xi_a)-g(Y,\bigtriangledown_X\xi_a) \\ &= -2g(Y,\bigtriangledown_X\xi_a) \\ &= 2g(\bigtriangledown_XY,\xi_a) \end{aligned}$$

for any $X,Y\in\Gamma(D)$ and $\xi_a\in\Gamma(D^{\perp})$. Since D is invariant under ϕ_a there exists nonzero vector field $Z\in\Gamma(D)$ such that $Y=\phi_aZ$. Thus we have

$$\begin{aligned} g([X,Y],\xi_a) &= 2g(\bigtriangledown_X\phi_aZ,\xi_a) \\ &= 2g((\bigtriangledown_X\phi_a)Z+\phi_a(\bigtriangledown_XZ),\xi_a) \\ &= 2g(\phi_a(\bigtriangledown_XZ),\xi_a) \end{aligned}$$

here, by using (II.3) we obtain $g([X,Y],\xi_a)=0$ i.e. $[X,Y]\in\Gamma(D)$.

Corollary 3.9 *Let M be a QR-submanifold of quaternionn Kaehler manifold. If D is involutive, then each leaf of D is totally geodesic in M.*

Proof. The proof is obvious from proof of the Theorem 3.7.

Theorem 3.10 *Let M be a QR-submanifold of quaternionn Kaehler manifold. Then we have*

$$PA_NX = -X \Longleftrightarrow P \bigtriangledown_X \xi_a = -\phi_a X$$

for any $X \in \Gamma(D)$. Where P is the projection morphism of TM to the quaternion distribution D.

Proof. From (II.2) we have

$$-\overline{\bigtriangledown}_X J_aN - h(X, J_aN) = \bigtriangledown_X \xi_a$$
$$-(\overline{\bigtriangledown}_X J_a)N - J_a(\overline{\bigtriangledown}_X N) - h(X, J_aN) = \bigtriangledown_X \xi_a.$$

By using (II.2) and (II.15) we obtain

$$-Q_{ab}(X)J_bN - Q_{ac}(X)J_cN - J_a(A_NX + \bigtriangledown^{\perp}{}_X N - h(X, \xi_a) = \bigtriangledown_X \xi_a.$$

Here,(II.1) and [1] we have

$$\begin{aligned}
&-Q_{ab}(X)J_bN - Q_{ac}(X)J_cN + J_aPA_NX \\
&+\eta_b(A_nX)\xi_c - \eta_c(A_nX)\xi_b - \eta_a(A_nX)N \\
&\qquad -B_a\bigtriangledown^{\perp}{}_X N - C_a\bigtriangledown^{\perp}{}_X N \\
&\qquad\qquad -h(X, \xi_a) = P \bigtriangledown_X \xi_a + \eta_a(\bigtriangledown_X \xi_a)\xi_a \\
&\qquad\qquad\qquad +\eta_b(\bigtriangledown_X \xi_a)\xi_b + \eta_c(\bigtriangledown_X \xi_a)\xi_c
\end{aligned}$$

thus we obtain $J_aPA_NX = P \bigtriangledown_X \xi_a$. Hence we get assertion of theorem.

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