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On Characterization of Metric Completeness

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Abstract

We give seven necessary and sufficient conditions for a metric space to be complete.

Key Words: Completeness, fixed point, stationary point, multimap, order.

1. Introduction

Metric completeness has been characterized in various ways; in particular, Kuratowski [3] proved that a metric space (X, d) is complete if and only if every sequence $\{F_n\}$ of closed subsets in X with $F_n \supset F_{n+1}$ and $\alpha(F_n) \to 0$ as $n \to \infty$ has a nonempty intersection. Using the existence of stationary points of a class of multimaps, Dancs, Hegedus and Medvegyev [2] obtained a necessary and sufficient condition for a metric space to be complete. For an ordered metric space Conserva and Rizzo [1] gave two characterizations of order completeness by using fixed point theorems and the convergence of Cauchy chains.

In this paper we introduce three classes of multimaps, one of which contains the maps used in [2], and characterize the metric completeness by virtue of the classes of multimaps. On the other hand we obtain connections between the metric completeness and the convergence of nondecreasing sequences in ordered metric spaces.

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2. Preliminaries

Throughout this paper, (X,d) is a metric space, $\leq$ an order (i.e, a reflexive, antisymmetric and transitive relation) on $X, CL(X)$ the family of all nonempty closed subsets of X. N denotes the set of all positive integers. For $A \subset X$, $\delta(A)$ and $\bar{A}$ denote the diameter and the closure of A, respectively; $\alpha(A) = inf \ \ \{\varepsilon > 0$: there exists a finite convering of A with sets having a diameter less than $\varepsilon\}$ if $\delta(A) < \infty$ and $\alpha(A) = \infty$ if $\delta(A) = \infty$. A sequence $\{x_n\}_{n\epsilon N}$ in X is said to be nondecreasing if $x_m \leq x_n$ for all m, n in N with $m \leq n$.

For each multimap f of X into $CL(X)$, let

(a) $fy \subset fx$ for each x in X and each y in fx;

(b) There exists a sequence $\{x_n\}_{n\epsilon N}$ in X such that $x_{n+1} \in fx_n$ for all n in N and that $\delta(fx_n) \rightarrow 0$ as $n \rightarrow \infty$;

(c) There exists a sequence $\{x_n\}_{n\epsilon N}$ in X such that $x_{n+1} \in fx_n$ for all n in N and that $\alpha(fx_n) \rightarrow 0$ as $n \rightarrow \infty$;

(d) $lim_{n\rightarrow\infty} d(x_n, x_{n+1}) = 0$ for each sequence $\{x_n\}_{n\epsilon N}$ in X with $x_{n+1} \in fx_n, n \in N$.

Motivated from Dancs *et al* [2], we introduce the following three classes of multimaps: $AB(X) = \{f : f$ satisfies (a) and (b)$\}$, $AC(X) = \{f : f$ satisfies (a) and (c)$\}$ and $AD(X) = \{f : f$ satisfies (a) and (d)$\}$.

In order to obtain our results, we need the following two lemmas.

Lemma 2.1. *If f is in AD(X), then there exists a point x in X such that fx is bounded.*

Proof. Suppose that fx is unbounded for each x in X. Then there exist a, b in fx satisfying

$$2 < d(a,b) \leq d(a,x) + d(b,x) \leq 2max\{d(a,x), d(b,x)\}$$

Therefore there is x_1 in fx such that $d(x, x_1) > 1$. Suppose that $x, x_1, \ldots\ldots, x_n$ were chosen so that $x_i \in f\ x_{i-1}$ and $d(x_i, x_{i-1}) > i$ for $i = 1, 2, \ldots, n$, where $x_0 = x$. Since fx_n is unbounded, there exist a, b in fx_n such that

$$2n + 2 < d(a,b) \leq 2max\{d(a,x_n), d(b,x_n)\}$$

This means that there exists x_{n+1} in fx_n with $d(x_n, x_{n+1}) > n + 1$. By induction, we obtain a sequence $\{x_n\}_{n\epsilon N}$ in X such that $x_{n+1} \in fx_n, d(x_n, x_{n+1}) > n + 1$ for all n in N. Consequently $d(x_n, x_{n+1}) \to \infty$ as $n \to \infty$. Since $f \in AD(X)$, it follows that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. This is a contradiction. □

Lemma 2.2. *AD(X) ⊂ AB(X) ⊂ AC(X).*

Proof. Lef f be in $AB(X)$. Note that $\alpha(fx_n) \leq \delta(fx_n)$. Then (b) implies (a) and hence $f \in AC(X)$, i.e, $AB(X) \subset AC(X)$.

Lef f be in $AD(X)$. In view of Lemma 2.1, there exists x_0 in X such that fx_0 is bounded. We can easily construct a sequence $\{x_n\}_{n\epsilon N}$ in X such that

$$d(x_n, x_{n-1}) \geq \delta(fx_{n-1})/2 - 1/2^{n-1}, x_n \in fx_{n-1} \text{ for all } n \text{ in } N$$

.

By (d) and the above inequality it follows that $\delta(fx_n) \to 0$ as $n \to \infty$. Hence $f \in AB(X)$; i. e., $AD(X) \subset AB(X)$. □

3. Main Results

Now we state our characterizations of the metric completeness.

Theorem 3.1 *For any metric space (X, d) the following statements are equivalent:*

(1) (X,d) is complete;

(2) Every f in $AC(X)$ has a fixed point;

(3) Every f in $AB(X)$ has a fixed point;

(4) Every f in $AD(X)$ has a fixed point.

Proof. Lemma 2.2 ensures that (2) ⇒ (3) ⇒ (4) hold.

(1) ⇒ (2) For each f in $AC(X)$, by (c), we obtain that $\alpha(\cap_{n\epsilon N} fx_n) \leq \alpha(fx_n) \to 0$ as $n \to \infty$. Consequently $\cap_{n\epsilon N} fx_n = F$ is compact. It follows from (1) that F is nonempty. For each x in F, (a) implies that $fx \subset F$, that is, F is invariant under f. By Zorn's lemma, there is a nonempty, minimal closed subset K of F which is invariant

under f. Let x be in K. It follows from (a) that fx is a nonempty closed subset of K which is invariant under f. By minimality of K, we conclude that $K = fx$. Therefore $x \in fx$.

(4) $\Rightarrow$ (2) Suppose that (X, d) is not complete. Then there exists a sequence $\{A_n\}_{n\geq 0}$ of nonempty closed subsets in X so that

$$X = A_0 \supset A_1 \supset \ldots \supset A_n \supset \ldots, \delta(A_n) \to 0 \text{ as } n \to \infty \text{ and } \cap_{n\epsilon N} A_n = \Phi.$$

Define a multimap f of X into $CL(X)$ by $fx = A_{i+1}$ if $x \in A_i$ and $x \notin A_{i+1}$. Clearly, f has no fixed point. It is easy to verify that f satisfies (a). Let $\{x_n\}_{n\epsilon N}$ be an arbitrary sequence in X with $x_{n+1} \in fx_n$ for each n in N. By the definition of f, there exists a sequence $\{i_n\}_{n\epsilon N}$ in N such that

$$x_{n+1} \in A_{i_{n+1}}, x_{n+1} \notin A_{i_{n+1}+1} \textit{ and } i_{n+1} \geq i_n + 1 \geq n \text{ for all } n \text{ in } N.$$

Consequently $d(x_n, x_{n+1}) \leq \delta(A_{i_n}) \to 0$ as $n \to \infty$; i.e, f satisfies (d). By (4), f has a fixed point. This is a contradiction. □

Theorem 3.2 *For each metric space (X, d), (1) is equivalent to each of the following;*

(5) Every f in $AB(X)$ has a stationary point w in X; i. e., $fw = \{w\}$;

(6) Every f in $AD(X)$ has a stationary point.

Proof. It follows from Lemma 2.2 that (5) implies (6).

(1) $\Rightarrow$ (5) Let f be in $AB(X)$. From (a), (b) and the completeness of X we easily conclude that $\cap_{n\epsilon N} fx_n = \{w\}$ for some w in X. Consequently $fw \subset \cap_{n\epsilon N} fx_n = \{w\}$ by (a). Hence $fw = \{w\}$.

(6) $\Rightarrow$ (1) Suppose that (X, d) is not complete. As in the proof of T heorem 3.1, we can construct a multimap f of X into $CL(X)$ such that f is in $AD(X)$ and that f has no fixed point. By (6), f has a stationary point w. Hence w is a fixed point of f. This is a contradiction. □

Theorem 3.3 *For any metric space (X, d), (1) is equivalent to each of the following:*

(7) If $\leq$ is an order on X and if $\{x_n\}_{n\epsilon N}$ in X is a nondecreasing sequence with $\alpha(\{x_n : n \in N\}) = 0$, then the sequence $\{x_n\}_{n\epsilon N}$ is convergent

(8) If $\leq$ is an order on X and if f is a selfmap on X satisfying $x \leq fx$ and $\alpha(\{f^n x : n \in N\}) = 0$ for all x in X, then the sequence $\{f^n x\}_{n\epsilon N}$ is convergent for each x in X.

Proof. (1) $\Rightarrow$ (7) In view of (1) and $\alpha(\{x_n : n \in N\}) = 0$ we conclude that the set $\overline{\{x_n : n \in N\}}$ is compact. This means that there exists a subsequence $\{x_{n_k}\}_{k\epsilon N}$ of $\{x_n\}_{n\epsilon N}$ such that $x_{n_k} \to w$ in X as $k \to \infty$. Since $\{x_n\}_{n\epsilon N}$ is nondecreasing, it follows that $x_n \to w$ as $n \to \infty$.

(7) $\Rightarrow$ (8) Note that $\{f^n x\}_{n\epsilon N}$ is nondecreasing for each x in X. Thus (8) follows from (7).

(8) $\Rightarrow$ (1) Suppose that (X, d) is not complete. Then there exists a sequence $X = F_0 \supset F_1 \supset \ldots \supset F_n \supset \ldots$ of nonempty closed subsets of X such that $\delta(F_n) \to 0$ as $n \to \infty$, but $\cap_{n\epsilon N} F_n = \Phi$. Define a mapping $i : X \to N \cup \{0\}$ by $i(x) = n$ if $x \in F_n$ and $x \notin F_{n+1}$. Define a relation $\leq$ on X as follows:

$$x \leq y \text{ in } X \text{ if and only if } y \in \{x\} \cup F_{i(x)+1}$$

We now prove that the relation $\leq$ is an order on X. Actually, reflexivity is clear. For antisymmetry, suppose that $x \leq y,\ y \leq x$ and $x \neq y$. Then $y \in F_{i(x)+1}$ and $x \in F_{i(y)+1}$, which implies that $i(y) \geq i(x)+1$ and $i(x) \geq i(y)+1$. Therefore $i(y) \geq i(x)+1 \geq i(y)+2$. This is a contradiction. For transitivity, suppose that $x \leq y$ and $y \leq z$. If either $x = y$ or $y = z$, then $x \leq z$; if $x \neq y$ and $y \neq z$, then $i(z) \geq i(y) + 1 \geq i(x) + 2 > i(x) + 1$; i. e., $z \in F_{i(z)} \subset F_{i(x)+1}$ and hence $x \leq z$.

Choose a choice function f on $\{F_{i(x)+1} : x \in X\}$. It is easy to verify that $x \leq fx$ and $a \neq fx$ for all x in X. For any x in X and n in N, let $x_n = f^n x$. By the definitions of f and i and $fx_n = x_{n+1}$ in $F_{i(x_n)+1}$ we conclude that $i(x_n) > i(x_n) + 1 \leq i(x_{n+1})$ for all n in N; i.e, $i(x_n) \to \infty$ as $n \to \infty$. From $F_{i(x_n)} \supset \{x_m : m \geq i(x_n)\}$ it follows that

$$\alpha(\{x_m : m \geq i(x_n)\}) \leq \delta(F_{i(x_n)}) \to 0 \text{ as } n \to \infty.$$

Therefore for each $\epsilon > 0$ there exists k in N such that $\alpha(\{x_m : m \geq i(x_k)\}) < \epsilon$. Thus we have

$$\alpha(\{x_n : n \in N\}) = max\{\alpha(\{x_1, x_2, \ldots, x_{i(x_k - 1)}\}), \alpha(\{x_m : m \geq i(x_k)\})\} < \epsilon$$

Letting ϵ tend to zero we obtain that $\alpha(\{x_n : n \in N\}) = 0$. Hence the sequence $\{x_n\}_{n \epsilon N}$ does not converge because $\cap_{n \epsilon N} F_{i(x_n)} = \Phi$ and $F_{i(x_n)} \supset \{x_n, x_{n+1}, \ldots\}$. But, (8) implies that the sequence $\{x_n\}_{n \epsilon N}$ is convergent. This is a contradiction.

□

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