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On Non-Homogeneous Riemann Boundary Value Problem

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Abstract

In this paper we consider non-homogeneous Riemann boundary value problem with unbounded oscillating coefficients on a class of open rectifiable Jordan curve.

Key Words: Curve, Cauchy type integral, singular integral, Riemann problem.

In [1], homogeneous Riemann boundary value problem was studied when curve γ satisfies the condition $\theta(\delta) \approx \delta$ and G is an oscillating function at the end points of the curve. In this work we investigate the non-homogeneous Riemann problem in the same case and we will use terminology and notations introduced in [1].

We need the following class of functions for our future references:

$$H^{a_1}(\mu_1,\nu_1) + H^{a_2}(\mu_2,\nu_2) = \{g \in C_\gamma : g = g_1 + g_2, g_k \in C_\gamma, \Omega^{a_k}_{g_k}(\xi) = O(\xi^{-\nu_k}), \omega^{a_k}_{g_k}(\delta,\xi) = O(\delta^{\mu_k}\xi^{-\mu_k-\nu_k})\}$$

where $k = 1; 2, \mu_k \in (0,1], \nu_k \in [0,1), \delta, \xi \in (0,d], \delta \leq \xi,\ d = \operatorname{diam}\gamma$.

Lemma 1. [3] Suppose that γ satisfies $\theta(\delta) \approx \delta$, $G(t)=\exp(2\pi i f(t))$, $\Omega^{a_k}_f(\xi) = O\,(\ln\frac{1}{\xi}), \omega^{a_k}_f(\delta,\xi) = O\,(\frac{\delta}{\xi}), \delta \leq \xi$, $k = 1,2$, $g \in H^{a_1}(\mu_1,\nu_1) + H^{a_2}(\mu_2,\nu_2)$ and suppose

1991 *AMS subject classification:* Primary 30E20, 30E25; secondary 45E05 h is holomorphic in $\mathbb{C}\backslash\gamma$, continuously extendable to $\hat{\gamma}$ from both sides, h(z)≠0 for all z∈ $\mathbb{C}\backslash\gamma$, $h^{\pm}(t)\neq 0$ for all t∈ $\hat{\gamma}$, $h^{+}(t) = G(t)h^{-}(t)$, $g/h^{+} \in L\ (\gamma)$. Then the function

$$\Phi_0(z) = \frac{h(t)}{2\pi i} \int\limits_{\gamma} \frac{g(\tau)}{h^{+}(\tau)(\tau - z)} d\tau, z \notin \gamma \tag{1}$$

is holomorphic in $\mathbb{C}\backslash\gamma$, continuously extendable to $\hat{\gamma}$ from both sides and satisfies the homogeneous boundary conditions

We introduce the quantity

$$\overline{\Delta}_G^k = \overline{\lim_{z\to a_k}} \frac{1}{\ln \mid z - a_k \mid} Re \int\limits_{\gamma} \frac{f(\tau)}{\tau - z} d\tau, z \notin \gamma \tag{2}$$

and if $\overline{\Delta}_G^k$ is finite introduce

$$æ_k' = \begin{cases} \overline{\Delta}_G^k, & \text{if } \overline{\Delta}_G^k \in \mathrm{Z} \\ \left[\overline{\Delta}_G^k\right] + 1, & \text{if } \overline{\Delta}_G^k \notin Z \end{cases} \tag{3}$$

k=1,2.

Lemma 2. Suppose that γ satisfies $\theta(\delta) \approx \delta$, G and g are as in lemma1 and

$$\chi_0(z) = (z - a_1)^{-æ_1'} (z - a_2)^{-æ_2'} \exp \int\limits_{\gamma} \frac{f(\tau)}{\tau - z} d\tau, z \notin \gamma.$$

Then g/χ_0^+ is integrable on γ.

Proof. It is obvious that g/χ_0^+ is bounded on a compact subset of $\gamma\backslash\{a_1, a_2\}$ and measurable. Therefore it is integrable on compact subset of γ. Now we estimate g/χ_0^+ on γ_δ (a_1) for small enough δ. Since g∈ $H^{a_1}(\mu_1, \nu_1) + H^{a_2}(\mu_2, \nu_2)$ we have

$$\mid g(t) \mid \leq \Omega_{g_2}^{a_1}(\mid t - a_1 \mid) + \Omega_{g_2}^{a_2}(\mid t - a_2 \mid) \leq$$

$$\Omega_{g_1}^{a_1}(\mid t - a_1 \mid) + \Omega_{g_2}^{a_2}(\mid a_2 - a_1 \mid - \delta) \leq C \mid t - a_1 \mid^{-\nu_1} + C \leq C \mid t - a_1 \mid^{-\nu_1}.$$

From (2) we have

$$-Re\int\limits_{\gamma}\frac{f(\tau)}{\tau-z}d\tau\leq-(\overline{\Delta}_G^1+\varepsilon)\ln\mid z-a_1\mid,$$

where z$\notin\gamma$ is close enough to a_1. Hence

$$\left|\frac{1}{\chi_0^+(t)}\right|=\mid(t-a_1)^{æ_1'}(t-a_2)^{æ_2'}\mid\exp\left(-\mathrm{Re}\int\limits_{\gamma}\frac{f(\tau)}{\tau-z}d\tau\right)\leq$$

$$\leq C\mid(t-a_1)\mid^{æ_1'}\exp\left(-(\overline{\Delta}_G^1+\varepsilon)\ln\mid z-a_1\mid\right)=C\mid t-a_1\mid^{ae_1'-\overline{\Delta}_G^1-\varepsilon}.$$

Therefore

$$\left|\frac{g(t)}{\chi_0^+(t)}\right|\leq C\mid t-a_1\mid^{-\nu_1+æ_1'-\overline{\Delta}_G^1\varepsilon}.$$

For small enough ε, we have q=$\nu_1-æ_1'+\overline{\Delta}_G^1+\varepsilon>-1$, that is, in the vicinity of a_1

$$\left|\frac{g(t)}{\chi_0^+(t)}\right|\leq C\mid t-a_1\mid^q,q>-1.$$

Analoguosly, similar estimation exists in the vicinity of a_2. This yields $\frac{g(t)}{\chi_0^+(t)}$ as integrable.

Lemma 3. Suppose γ satisfies $\theta(\delta)\approx\delta$, G satisfies the conditions of lemma1, g$\in$ H$^{a_1}(\mu_1,\nu_1)+$H$^{a_2}(\mu_2,\nu_2)$ and

$$\Delta_G^k=\lim_{z\to a_k}\frac{1}{\ln\mid z-a_k\mid}\mathrm{Re}\int\limits_{\gamma}\frac{f(\tau)}{\tau-z}d\tau,z\notin\gamma \tag{4}$$

exists. Then the function in (1) is piecewise holomophic while h=χ.

Proof. It is obvious that according to condition (4) we may take $æ_1=æ_1',æ_2=æ_2'$ and $\chi_0=\chi$. Then for function (1) we only need to estimate in endpoints.

We shall investigate function (1) in the vicinity of a_1 (in the other end we may show the proof anologously). Take $2\eta=\mid z-a_1\mid,q=-æ_1+\Delta_G^1$. Since $q\in(-1,0],\nu_1\in[0,1)$.

Choose ε small enough such that $q+\nu_1+\varepsilon<1$. Let $\lambda>0$ such that for $z\in\{\xi\in\mathbb{C}:\ \mid \xi-u_1\mid<\lambda\}\backslash\gamma$,

$$(\Delta_G^1+\varepsilon)\ln\mid z-a_1\mid\leq \operatorname{Re}\int\frac{f(\tau)}{\tau-z}d\tau\leq(\Delta_G^1-\varepsilon)\ln\mid z-a_1\mid. \tag{5}$$

Let $t_z\in\{y\in\gamma:\mid z-y\mid=\operatorname{dist}(z,\gamma_\lambda(a_1)\backslash\gamma_\eta(z))\}$. We decompose (1) as follows:

$$\frac{\Phi_0(z)}{\chi(z)}=\frac{1}{2\pi i}\int\limits_\gamma\frac{g(\tau)}{\chi^+(\tau)(\tau-z)}d\tau=\frac{1}{2\pi i}\int\limits_{\gamma\backslash\gamma_\lambda(a_1)}\frac{g(\tau)}{\chi^+(\tau)(\tau-z)}d\tau+$$

$$\frac{1}{2\pi i}\int\limits_{\gamma_\lambda(a_1)\backslash\gamma_\eta(z)}\frac{g(\tau)}{\chi^+(\tau)(\tau-z)}d\tau+\frac{1}{2\pi i}\int\limits_{\gamma_\eta(z)}\frac{g(\tau)-g(t_z)}{\chi^+(\tau)(\tau-z)}d\tau+\frac{g(t_z)}{2\pi i}\int\limits_{\gamma_\eta(z)}\frac{d\tau}{\chi^+(\tau)(\tau-z)}$$

$$=A_1+A_2+A_3+A_4.$$

It is obvious that since A_1 does not depend on η it is bounded in the vicinity of a_1.

Let $\tau\in\gamma_\lambda(a_1)\backslash\gamma_\eta(z)$ therefore $\mid\tau-a_1\mid+\eta\leq\mid\tau-z\mid+3\eta\leq\mid\tau-z\mid$. From lemma 2 and [2] we have

$$\mid A_2\mid\leq\frac{1}{2\pi}\int\limits_{\gamma_\lambda(a_1)\backslash\gamma_\eta(z)}\frac{\Omega_{g_1}^{a_1}(\mid\tau-a_1\mid)+\Omega_{g_1}^{a_2}(\mid a_2-a_1\mid-\lambda)}{\mid\chi^+(\tau)\mid(\mid\tau-a_1\mid+\eta)}\mid d\tau\mid$$

$$\leq C\int\limits_{\gamma_\lambda(a_1)}\frac{\mid\tau-a_1\mid^{-\nu_1}\mid d\tau\mid}{\mid\tau-a_1\mid^{q+\varepsilon}(\mid\tau-a_1\mid+\eta)}d\tau\leq C\int\limits_0^\lambda\frac{x^{-\nu_1-q-\varepsilon}}{x+\eta}d\theta(x)$$

$$\leq C\int\limits_0^\lambda\frac{x^{-\nu_1-q-\varepsilon}}{x+\eta}dx\leq C\int\limits_0^\lambda\frac{x^{-\nu_1-q-\varepsilon}}{\eta+\varepsilon}dx$$

$$\leq C(\frac{1}{\eta}\int\limits_0^\lambda x^{-\nu_1-q-\varepsilon}dx+\int\limits_\eta^\lambda x^{-\nu_1-q-\varepsilon-1}dx)\leq C\eta^{-\nu_1-q-\varepsilon}.$$

If $\gamma_\eta(z)=\varnothing$, then $A_3=0$. Otherwise, since $\mid z-t_z\mid\leq\eta$ and $\gamma_\eta(z)\subset\gamma_{2\eta}(t_z)$ we get

$$
\begin{aligned}
| A_3 | &\leq \frac{1}{2\pi} \int\limits_{\gamma_\eta(z)} \frac{\omega_{g_1}^{a_1}(| \tau - t_z |, \eta/2) + \omega_{g_2}^{a_2}(| \tau - t_z |, | a_2 - a_1 | -\lambda)}{| \chi^+(\tau) || \tau - t_z |} | d\tau | \\
&\leq \frac{1}{2\pi} \int\limits_{\gamma_\eta(z)} \frac{| \tau - t_z |^{\mu_1} \eta^{-\mu_1-\nu_1} + | \tau - t_z |^{\mu_2}}{\eta^{q+\varepsilon} | \tau - t_z |} | d\tau | \\
&\leq C\eta^{-\eta-\varepsilon}(\eta^{-\mu_1-\nu_1} \int\limits_{\gamma_{2\eta}(t_z)} | \tau - t_z |^{\mu_1-1} | d\tau | \\
&+ \int\limits_{\gamma_{2\eta}(t_z)} | \tau - t_z |^{\mu_2-1} | d\tau |) \leq C\eta^{-\eta-\varepsilon}(\eta^{-\mu_1-\nu_1} \int\limits_0^{2\eta} \tau^{\mu_1-1} d\theta(\tau) \\
&+ \int\limits_0^{2\eta} \tau^{\mu_2-1} d\theta(\tau)) \leq C\eta^{-\eta-\varepsilon}(\eta^{-\mu_1-\nu_1} \int\limits_0^{2\eta} \tau^{\mu_1-1} d\tau \\
&+ \int\limits_0^{2\eta} \tau^{\mu_2-1} d\tau) \leq C\eta^{-q-\varepsilon}(C\eta^{-\nu_1} + C\eta^{\mu_2}) \leq C\eta^{-q-\varepsilon-\nu_1}.
\end{aligned}
$$

Now we investigate A_4. If $\gamma_\eta(z) = \varnothing$, then $A_4 = 0$. Otherwise since $| z - t_z |\leq \eta$ and $| \tau_z - a_1 |\geq \eta$ then

$$
\begin{aligned}
| g(t_z) | &\leq \Omega_{g_1}^{a_1}(| t_z - a_1 |) + \Omega_{g_2}^{a_2}(| t_z - a_2 |) \leq \Omega_{g_1}^{a_1}(| t_z - a_1 |) \\
&+ \Omega_{g_2}^{a_2}(| a_2 - a_1 | -\delta) \leq C | t_z - a_1 |^{-\nu_1} + C \leq C\eta^{-\nu_1}.
\end{aligned}
$$

Suppose that $a_2 \notin \gamma_\eta$(z) and γ_η(z) $= \Lambda \cup (\bigcup\limits_{k=1}^{p} \overset{\smile}{c_k d_k}), 1 \leq p \leq \infty, c_k, d_k \in \sum_\eta$ (z) $=$ $\{\xi \in \mathbb{C} :| \xi - z |= \eta\}$. Arcs $\overset{\smile}{c_k d_k}$ are connected components of γ_η(z) and $\Lambda \subset \sum_\eta$ (z). The number of $\overset{\smile}{c_k d_k}$'s may not be more than countable since arbitrary partition of interval [0,d], d=diam γ, is countable.

The points $c_k, d_k, c_k \neq d_k$ divide $\sum_\eta$ (z) into two arcs with enpoints c_k, d_k. Denote one of them by Λ_k oriented from c_k to d_k. Let D be the domain bounded by $\Lambda_k \cup \overset{\smile}{c_k d_k}$

and z$\notin$D. If meas $\Lambda_k \leq \pi\eta$ then meas $\Lambda_k \leq\mid c_k - d_k \mid \pi/2 \leq$(meas $\breve{c_k d_k})\pi/2$. If meas $\Lambda_k > \pi\eta$ then meas $\breve{c_k d_k} \geq 2\eta \geq$meas Λ_k/π. Therefore meas $\Lambda_k \leq$C(meas $\breve{c_k d_k}$), C=max$\{\pi, 2/\pi\} = \pi$. Meanwhile if $\tau \in \sum_\eta$(z) we have $\mid \tau - z \mid = \eta$. By means of Cauchy theorem we get

$$\left| \int\limits_{\gamma_\eta(z)} \frac{d\tau}{\tau - z} \right| = \left| (\int\limits_{\Lambda} + \sum_{k=1}^{p} \int\limits_{\breve{c_k d_k}}) \frac{d\tau}{\tau - z} \right| \leq \left| (\int\limits_{\Lambda} - \sum_{k=1}^{p} \int\limits_{\Lambda_k}) \frac{d\tau}{\tau - z} \right|$$

$$\leq \frac{1}{\eta}(\text{meas}\,\Lambda + \sum_{k=1}^{p}(\text{meas}\,\Lambda_k)$$

$$\leq \frac{\pi}{\eta}(\text{meas}\,\Lambda + \sum_{k=1}^{p} \text{meas } \breve{c_k d_k}) = \frac{\pi}{\eta}(\text{meas}\,\Lambda + \text{meas} \bigcup_{k=1}^{p} \breve{c_k d_k}) = \frac{\pi}{\eta}\text{meas}\,\gamma_\eta(\text{z})$$

$$\leq \frac{\pi}{\eta}\text{meas}\,\gamma_{2\eta}(t_z) = \frac{\pi}{\eta}\theta_{t_z}(2\eta) \leq \frac{\pi}{\eta}\theta(2\eta) \leq \frac{\pi}{\eta}C\eta = \pi C.$$

Therefore $\mid A_4 \mid\leq \frac{C\eta^{-\nu_1}}{2\pi}\eta^{-q-\varepsilon}C\pi = C\eta^{-q-\nu_1-\varepsilon}$. If we round up the result obtained we have $\mid \Phi_0(\text{z}) \mid = \left| \frac{\chi(\text{z})}{2\pi i} \int\limits_{\gamma} \frac{g(\tau)}{\chi^+(\tau)(\tau - z)} d\tau \right| \leq C\eta^{-q-\nu_1-\varepsilon} \mid \chi(\text{z}) \mid$. From (5) $\mid \chi(\text{z}) \mid\leq C\eta^{q-\varepsilon}$, therefore $\mid \Phi_0(z) \mid\leq C\eta^{-q-\nu_1-\varepsilon} \mid \chi(\text{z}) \mid\leq C\eta^{-\nu_1-2_\varepsilon}$.

Since ε is small enough we may assume that $\nu_1 + 2\varepsilon < 1$. This proves the lemma.

In [1] the solution of the homogeneous boundary value problem was given as $\chi(\text{z})\text{P}_{æ-1}(\text{z})$ where $\text{P}_{æ-1}(\text{z})$ is a polynomial whose degree is not greater than æ $-$ 1. If æ $=$ 0 $\text{P}_{æ-1}(\text{z}) \equiv 0$. For æ $<$ 0 the coefficients of z^{-1}, z^{-2},... $\text{z}^{-æ}$ in the expantion

$$\frac{1}{2\pi i} \int\limits_{\gamma} \frac{g(\tau)}{\chi^+(\tau)(\tau - z)} d\tau = -\frac{z^{-1}}{2\pi i} \int\limits_{\gamma} \frac{g(\tau)}{\chi^+(\tau)} d\tau - \frac{z^{-2}}{2\pi i} \int\limits_{\gamma} \frac{g(\tau)}{\chi^+(\tau)} \tau d\tau - \frac{z^{-3}}{2\pi i} \int\limits_{\gamma} \frac{g(\tau)}{\chi^+(\tau)} \tau^2 d\tau - \ldots$$

must be zero. Thus the following theorem is obtained.

Theorem. Suppose the conditions of lemma1 are satisfied and limit in (4) exists.

i) If æ ≥ 0, the Riemann boundary value problem is soluable in K(γ) unconditionally, the solution is given by

$$\Phi(\mathrm{z}) = \frac{\chi(\mathrm{z})}{2\pi i} \int\limits_{\gamma} \frac{g(t)}{\chi^{+}(t)(t-z)} dt + \chi(\mathrm{z})\ \mathrm{P}_{æ-1}(\mathrm{z}),$$

where $\mathrm{P}_{æ-1}(\mathrm{z})$ is arbitrary polynomial of degree not greater than $æ - 1(\mathrm{P}_{æ-1}(\mathrm{z}) \equiv 0$ for $æ = 0)$. ii) If $æ < 0$, then the Riemann boundary value problem is solvable in K(γ) if and only if the conditions

$$\int\limits_{\gamma} \frac{g(t)}{\chi^{+}(t)} t^{j} dt = 0, j = 0, 1, \ldots, -æ - 1$$

are satisfied. Under these conditions the solution is unique and is given by

$$\Phi(\mathrm{z}) = \frac{\chi(\mathrm{z})}{2\pi i} \int\limits_{\gamma} \frac{g(t)}{\chi^{+}(t)(t-z)} dt.$$

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