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On the Efficiency of Finite Simple Semigroups

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Abstract

Let S be a finite simple semigroup, given as a Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ over a group G.

We prove that the second homology of S is $H_2(S) = H_2(G) \times \mathbb{Z}^{(|I|-1)(|\Lambda|-1)}$.

It is known that for any finite presentation $\langle A \mid R \rangle$ of S we have $|R| - |A| \geq \operatorname{rank}(H_2(S))$; we say that S is efficient if equality is attained for some presentation. Given a presentation $\langle A_1 \mid R_1 \rangle$ for G, we find a presentation $\langle A \mid R \rangle$ for S such that $|R| - |A| = |R_1| - |A_1| + (|I| - 1)(|\Lambda| - 1) + 1$. Further, if R_1 contains a relation of a special form, we show that $|R| - |A|$ can be reduced by one. We use this result to prove that S is efficient whenever G is finite abelian or dihedral of even degree.

1. Introduction

The purpose of this paper is to investigate the efficiency of finite simple semigroups.

It is well known that a finite semigroup S is simple if and only if it is isomorphic to a finite Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$. Here G is a group, I and Λ are non-empty sets, and $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix with entries from G. Then the *Rees matrix semigroup* $\mathcal{M}[G; I, \Lambda; P]$ is the set $I \times G \times \Lambda = \{ (i, a, \lambda) \mid i \in I,\ a \in G,\ \lambda \in \Lambda \}$ with the multiplication

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu).$$

It is known that the matrix P can be chosen to be *normal*, that is $p_{\lambda 1} = p_{1i} = 1_G$ for all

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$\lambda \in \Lambda$, $i \in I$, where 1_G is the identity of G; see for example [7] or [4].

Let A be an alphabet and let A^+ denote the *free semigroup* on A. A *presentation* is an ordered pair $\langle A \mid R \rangle$, where $R \subseteq A^+ \times A^+$. A semigroup S is said to be *defined by* $\langle A \mid R \rangle$ if $S \cong A^+/\rho$ where ρ is the congruence generated by R. If both A and R are finite sets then $\langle A \mid R \rangle$ is said to be a *finite presentation* and S is said to be *finitely presented.* The *deficiency of a finite presentation* $\mathcal{P} = \langle A \mid R \rangle$ is defined to be $|R| - |A|$ and is denoted by $\operatorname{def}(\mathcal{P})$. The *deficiency of a finitely presented semigroup* S is defined by

$$\operatorname{def}(S) = \min\{ \operatorname{def}(\mathcal{P}) \mid \mathcal{P} \text{ is a finite presentation for } S \}.$$

For a semigroup S, let S^1 denote the monoid S with an identity adjoined to it. For a finite semigroup S, it is well-known that $\operatorname{def}(S) \geq 0$. Recently it has been shown by S. J. Pride (unpublished) that there exists a better lower bound for the deficiency of finite semigroups, namely

$$\operatorname{def}(S) \geq \operatorname{rank}(H_2(S)),$$

where $H_2(S)$ is the second integral homology of S^1.

We call a finite semigroup S *efficient* if S has a presentation $\mathcal{P} = \langle A \mid R \rangle$ such that $\operatorname{def}(\mathcal{P}) = \operatorname{rank}(H_2(S))$ and *inefficient* otherwise. Examples of both efficient semigroups and of inefficient semigroups are given in [1], where it is also shown that finite rectangular bands are efficient. Of course rectangular bands are simple. In this paper we first compute the second integral homology of a general finite simple semigroup $S = \mathcal{M}[G; I, \Lambda; P]$. If G is efficient, then we find a presentation $\mathcal{P}$ for S with $\operatorname{def}(\mathcal{P}) = \operatorname{rank}(H_2(S)) + 1$. We are able to modify this presentation to reduce the deficiency by one and hence show that S is efficient when G is a finite abelian group or a dihedral group D_{2n} with even n. It is not known whether this can be done for an arbitrary finite group, or whether there exists a finite group G such that $\operatorname{def}(S) = \operatorname{rank}(H_2(S)) + 1$. Finally, we show that there exist non-simple efficient semigroups which have non-trivial second homology.

2. A rewriting system for Rees matrix semigroups

In [1] the bar resolution was used to compute the second homology of rectangular bands $R_{m,n}$ to be $\mathbb{Z}^{(m-1)(n-1)}$, and the nth ($n \geq 1$) homology of semigroups with a left or a right zero to be trivial. Here we use another resolution which is described by Squier in [8]. Since this resolution is defined by using a presentation in which the set of relations is a uniquely terminating rewriting system, we first find a presentation for a Rees matrix semigroup in which the set of relations forms such a system. We begin by introducing some elementary concepts about rewriting systems.

Let A be a set and let A^* be the free monoid on A. A *rewriting system* R on A is a subset of $A^* \times A^*$. For $w_1, w_2 \in A^*$, we write $w_1 \equiv w_2$ if they are identical words. We say that w_1 *rewrites to* w_2 if there exist $b, c \in A^*$ and $(u, v) \in R$ such that $w_1 \equiv buc$ and $w_2 \equiv bvc$ and we write $w_1 \to w_2$. We denote by $\xrightarrow{*}$ the reflexive transitive closure of $\to$ and by $\sim$ the equivalence relation generated by $\to$.

For a word w we say that w is *reducible* if there is a word z such that $w \to z$; otherwise we call w *irreducible*. If $w \xrightarrow{*} y$ and y is irreducible, then we say that y is an *irreducible form* of w. A rewriting system R is said to be *terminating* if there is no infinite sequence (w_n) such that $w_n \to w_{n+1}$ for all $n \geq 1$. We denote by $|w|$ the length of the word w. We call R *length-reducing* if $|u| > |v|$ for all $(u, v) \in R$. It is clear that if R is a length-reducing rewriting system, then R is a terminating rewriting system.

We say that R is *confluent* if, for any $x, y, z \in A^*$ such that $x \xrightarrow{*} y$, $x \xrightarrow{*} z$, there exists $w \in A^*$ such that $y \xrightarrow{*} w$, $z \xrightarrow{*} w$. A rewriting system R is *complete* if it is both terminating and confluent. For a given R, define $R_1 \subseteq A^*$ to consist of all $r \in A^*$ such that there exists $(r, s) \in R$ for some $s \in A^*$. The system R is said to be *reduced* provided that, for each $(r, s) \in R$, we have $R_1 \cap A^* r A^* = \{r\}$ and s is R-irreducible. A reduced complete rewriting system $R \subseteq A^* \times A^*$ is called a *uniquely terminating rewriting system.*

Lemma 2.1 *Let R be a terminating rewriting system. Then the following are equivalent:*

(i) *R is confluent (and hence complete);*

(ii) *for any $(r_1 r_2, s_{12})$, $(r_2 r_3, s_{23}) \in R$, where r_2 is non-empty, there exists a word $w \in A^*$ such that $s_{12} r_3 \xrightarrow{*} w$, $r_1 s_{23} \xrightarrow{*} w$; for any $(r_1 r_2 r_3, s_{12})$, $(r_2, s_{23}) \in R$, there exists a word $w \in A^*$ such that $s_{12} \xrightarrow{*} w$, $r_1 s_{23} r_3 \xrightarrow{*} w$;*

(iii) *any word $w \in A^*$ has exactly one irreducible form. Moreover $w \sim w'$ if and only if w and w' have the same irreducible form.* □

For a proof see [3] or [8].

We define the *overlaps* to be the ordered pairs of the form $[(r_1 r_2, s_{12}), (r_2 r_3, s_{23})]$

and $[(r_4r_5r_6, s_{45}), (r_5, s_{56})]$ where (r_1r_2, s_{12}), (r_2r_3, s_{23}), $(r_4r_5r_6, s_{45})$, $(r_5, s_{56}) \in R$, and r_2 and r_5 are non-empty.

First, we give a presentation for a Rees matrix semigroup with a normal matrix. For ease of notation we assume that I and Λ contain a distinguished element denoted by 1.

Theorem 2.2 *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a Rees matrix semigroup, where G is a group and $P = (p_{\lambda i})$ is a normal $\Lambda \times I$ matrix with entries from G. Let $\langle X \mid R \rangle$ be a semigroup presentation for G, let $e \in X^+$ be a non-empty word representing the identity of G, and let $Y = X \cup \{ y_i \mid i \in I - \{1\} \} \cup \{ z_\lambda \mid \lambda \in \Lambda - \{1\} \}$. Then the presentation*

$$\langle Y| \;\; R,\;\; y_ie = y_i,\;\; ey_i = e,\;\; z_\lambda e = e,\;\; ez_\lambda = z_\lambda,\;\; z_\lambda y_i = p_{\lambda i}$$
$$(i \in I - \{1\},\; \lambda \in \Lambda - \{1\}) \,\rangle$$

defines S in terms of the generating set $\{ (1, x, 1) \mid x \in X \} \cup \{ (i, e, 1) \mid i \in I - \{1\} \} \cup \{ (1, e, \lambda) \mid \lambda \in \Lambda - \{1\} \}$.

Proof. The result is a special case of Theorem 6.2 in [5]. □

In the previous presentation, there are some overlaps, for example $[y_ie = y_i, ey_i = e]$, which show that the set of the relations is not a uniquely terminating rewriting system. Now we construct a new presentation with a uniquely terminating rewriting system of relations. We can take the presentation $\langle X \mid R \rangle$ to be the Cayley table, that is $X = G$ and $R = \{ (x_1x_2, x_3) \mid x_1, x_2, x_3 \in X,\; x_1x_2 = x_3 \text{ in } G \}$. It is clear that R is a uniquely terminating rewriting system on X. Let $x_0 \in X$ represent the identity of G. Then, taking $e \equiv x_0$ and adding the new relations $xy_i = x$, $z_\lambda x = x$, $y_iy_{i'} = y_i$ and $z_\lambda z_{\lambda'} = z_{\lambda'}$ $(x \in X - \{x_0\};\, i, i' \in I - \{1\};\, \lambda, \lambda' \in \Lambda - \{1\})$, which are easily seen to hold in S, yields the presentation

$$\langle Y| \;\; R,\;\; y_ix_0 = y_i,\;\; xy_i = x,\;\; y_iy_{i'} = y_i,\;\; z_\lambda x = x,\;\; x_0z_\lambda = z_\lambda,\;\; z_\lambda z_{\lambda'} = z_{\lambda'},$$
$$z_\lambda y_i = p_{\lambda i}\;\; (i, i' \in I - \{1\},\;\; \lambda, \lambda' \in \Lambda - \{1\},\;\; x \in X) \,\rangle$$

which defines $S = \mathcal{M}[G; I, \Lambda; P]$.

For ease of notation, we assume that G is finite and $X = \{x_0, x_1, \ldots, x_m\}$. We further assume that the entries $p_{\lambda i}$ of the matrix P are represented by the words of length one.

Theorem 2.3 *Let $\langle\, X \mid R \,\rangle$ be the Cayley table of the finite group G and let $x_0 \in X$ be the representative of the identity. With the above notation, the presentation*

$$\mathcal{P} = \langle\, Y \mid R,\ \ y_i x_0 = y_i,\ \ x_k y_i = x_k,\ \ y_i y_{i'} = y_i,\ \ z_\lambda x_k = x_k,\ \ x_0 z_\lambda = z_\lambda,$$
$$z_\lambda z_{\lambda'} = z_{\lambda'},\ \ z_\lambda y_i = p_{\lambda i}\ (0 \le k \le m,\ \ i, i' \in I - \{1\},\ \ \lambda, \lambda^{'} \in \Lambda - \{1\})\,\rangle,$$

which defines $S = \mathcal{M}[G; I, \Lambda; P]$, has a uniquely terminating rewriting system of relations on Y.

Proof. Let Q denote the set of relations of $\mathcal{P}$. Recall that all rewriting rules in R have the form (x_1x_2, x_3) $(x_1, x_2, x_3 \in X)$ so that all the rewriting rules in Q are length-reducing. Therefore Q is terminating. It is clear that Q is reduced. To prove that Q is confluent, we list the overlaps:

$U_{1,k,k',k''} = [(x_k x_{k'}, x_l),\ (x_{k'} x_{k''}, x_{l'})]$, $U_{2,k,k',i} = [(x_{k'} x_k, x_l),\ (x_k y_i, x_k)]$,
$U_{3,k,\lambda} = [(x_k x_0, x_k),\ (x_0 z_\lambda, z_\lambda)]$, $U_{4,k,i} = [(y_i x_0, y_i),\ (x_0 x_k, x_k)]$,
$U_{5,i,i'} = [(y_i x_0, y_i),\ (x_0 y_{i'}, x_0)]$, $U_{6,i,\lambda} = [(y_i x_0, y_i),\ (x_0 z_\lambda, z_\lambda)]$,
$U_{7,k,i} = [(x_k y_i, x_k),\ (y_i x_0, y_i)]$, $U_{8,k,i,i'} = [(x_k y_i, x_k),\ (y_i y_{i'}, y_i)]$,
$U_{9,i,i'} = [(y_i y_{i'}, y_i),\ (y_{i'} x_0, y_{i'})]$, $U_{10,i,i',i''} = [(y_i y_{i'}, y_i),\ (y_{i'} y_{i''}, y_{i'})]$,
$U_{11,k,k',\lambda} = [(z_\lambda x_k, x_k),\ (x_k x_{k'}, x_l)]$, $U_{12,k,i,\lambda} = [(z_\lambda x_k, x_k),\ (x_k y_i, x_k)]$,
$U_{13,\lambda,\lambda'} = [(z_\lambda x_0, x_0),\ (x_0 z_{\lambda'}, z_{\lambda'})]$, $U_{14,k,\lambda} = [(x_0 z_\lambda, z_\lambda),\ (z_\lambda x_k, x_k)]$,
$U_{15,\lambda,\lambda'} = [(x_0 z_\lambda, z_\lambda),\ (z_\lambda z_{\lambda'}, z_{\lambda'})]$, $U_{16,i,\lambda} = [(x_0 z_\lambda, z_\lambda),\ (z_\lambda y_i, p_{\lambda i})]$,
$U_{17,k,\lambda,\lambda'} = [(z_\lambda z_{\lambda'}, z_{\lambda'}),\ (z_{\lambda'} x_k, x_k)]$, $U_{18,\lambda,\lambda',\lambda''} = [(z_\lambda z_{\lambda'}, z_{\lambda'}),\ (z_{\lambda'} z_{\lambda''}, z_{\lambda''})]$,
$U_{19,i,\lambda,\lambda'} = [(z_\lambda z_{\lambda'}, z_{\lambda'}),\ (z_{\lambda'} y_i, p_{\lambda' i})]$, $U_{20,i,\lambda} = [(z_\lambda y_i, p_{\lambda i}),\ (y_i x_0, y_i)]$,
$U_{21,i,i',\lambda} = [(z_\lambda y_i, p_{\lambda i}),\ (y_i y_{i'}, y_i)]$,

$(i, i', i'' \in I - \{1\};\ \lambda, \lambda^{'}, \lambda^{''} \in \Lambda - \{1\};\ 1 \le k, k', k'' \le m)$, and then apply Lemma 2.1(ii), which is straightforward. □

3. The second homology of Rees matrix semigroups

Now we describe the resolution of $\mathbb{Z}$ given by Squier in [8], which we use to compute the second homology of a finite Rees matrix semigroup.

Let S be a monoid and let $\langle\, A \mid R \,\rangle$ be a presentation for S in which R is a uniquely terminating rewriting system. Then Squier defined the free resolution of $\mathbb{Z}$ as follows:

$$P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where P_0 is the free $\mathbb{Z}S$-module on a single formal symbol $[\,]$, the augmentation map $\varepsilon : P_0 \longrightarrow \mathbb{Z}$ is defined by $\varepsilon([\,]) = 1$, P_1 is the free $\mathbb{Z}S$-module on the set of formal symbols $[x]$ for all $x \in A$ and $\partial_1 : P_1 \longrightarrow P_0$ is defined by

$$\partial_1([x]) = (x-1)[\,]$$

where $x \in A$. Further P_2 is the free $\mathbb{Z}S$-module on the set of formal symbols $[r,s]$, one for each $(r,s) \in R$. For $x \in A$, we define a function $\partial/\partial_x : A^* \longrightarrow \mathbb{Z}A^*$ inductively by

$$\begin{aligned} \partial/\partial_x(1) &= 0 \\ \partial/\partial_x(wx) &= \partial/\partial_x(w) + w \quad (w \in A^*) \\ \partial/\partial_x(wy) &= \partial/\partial_x(w) \qquad\quad (w \in A^* \text{ and } y \neq x). \end{aligned}$$

This function is called a *derivation*.

Now we define $\partial_2 : P_2 \longrightarrow P_1$ by

$$\partial_2([r,s]) = \sum_{x \in A} \phi(\partial/\partial_x(r) - \partial/\partial_x(s))[x]$$

where $\phi : \mathbb{Z}A^* \longrightarrow \mathbb{Z}S$ is induced by the natural homomorphism from A^* to S.

Next, P_3 is the free $\mathbb{Z}S$-module on the set of overlaps $[(r_1r_2, s_{12}), (r_2r_3, s_{23})]$ from R. Let w be in A^* and let u be the irreducible form of w. Then we have a sequence $w \equiv b_1r_1c_1,\ b_1s_1c_1 \equiv b_2r_2c_2, \ldots, b_qs_qc_q \equiv u$ where $b_i, c_i \in A^*$ and $(r_i, s_i) \in R$ for all $i = 1, \ldots, q$. Define $\Phi : A^* \longrightarrow P_2$ by

$$\Phi(w) = \sum_{i=1}^{q} \phi(b_i)[r_i, s_i].$$

Now we define $\partial_3 : P_3 \longrightarrow P_2$ by

$$\partial_3\Big([(r_1r_2, s_{12}), (r_2r_3, s_{23})]\Big) = r_1[r_2r_3, s_{23}] - [r_1r_2, s_{12}] + \Phi(r_1s_{23}) - \Phi(s_{12}r_3).$$

Squier [8] showed that $P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$ is an exact sequence when R is a uniquely terminating rewriting system.

We now use this resolution to compute the second homology of a finite Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$.

Theorem 3.4 *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a finite Rees matrix semigroup. Then the second integral homology of S is*

$$H_2(S) = H_2(G) \times \mathbb{Z}^{(|I|-1)(|\Lambda|-1)}.$$

Proof. Without loss of generality we may assume that P is normal. We consider the uniquely terminating rewriting system Q on Y given in Theorem 2.3 and the resolution of $\mathbb{Z}$ arising from it. By applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}S^1} -$ to this resolution, we obtain the chain complex of abelian groups

$$\mathbb{Z} \otimes P_3 \xrightarrow{1\otimes\partial_3} \mathbb{Z} \otimes P_2 \xrightarrow{1\otimes\partial_2} \mathbb{Z} \otimes P_1 \xrightarrow{1\otimes\partial_1} \mathbb{Z} \otimes P_0 \xrightarrow{1\otimes\varepsilon} \mathbb{Z} \otimes \mathbb{Z} \longrightarrow 0,$$

or simply

$$\bar{P}_3 \xrightarrow{\bar{\partial}_3} \bar{P}_2 \xrightarrow{\bar{\partial}_2} \bar{P}_1 \xrightarrow{\bar{\partial}_1} \mathbb{Z} \longrightarrow 0,$$

where $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ are the free abelian groups on the sets of formal symbols $[x]$ $(x \in Y)$, $[r, s]$ $((r, s) \in Q)$ and $[(r_1r_2, s_{12}), (r_2r_3, s_{23})]$, one for each overlap from Q, respectively. The mappings $\bar{\partial}_2 : \bar{P}_2 \to \bar{P}_1$ and $\bar{\partial}_3 : \bar{P}_3 \to \bar{P}_2$ are defined respectively by

$$\bar{\partial}_2([r, s]) = \sum_{x\in Y}((\text{the number of } x\text{'s in } r) - (\text{the number of } x\text{'s in } s))[x]$$

and

$$\bar{\partial}_3([(r_1r_2, s_{12}), (r_2r_3, s_{23})]) = [r_2r_3, s_{23}] - [r_1r_2, s_{12}] + \bar{\Phi}(r_1s_{23}) - \bar{\Phi}(s_{12}r_3),$$

where $\bar{\Phi}$ is defined by

$$\bar{\Phi}(w) = \sum_{i=1}^{q}[r_i, s_i]$$

if $\Phi(w) = \sum_{i=1}^{q} \phi(b_i)[r_i, s_i]$.

Before we compute the second homology of S, $H_2(S) \cong \ker \bar{\partial}_2 / \mathrm{im}\, \bar{\partial}_3$, we assume that $H_2(G) \cong \ker \bar{\partial}_2^G / \mathrm{im}\, \bar{\partial}_3^G$ where $\ker \bar{\partial}_2^G$ is the free abelian group on $\{ W_j \mid j \in J \}$ and $\mathrm{im}\, \bar{\partial}_3^G$ is the free abelian group on $\{ V_l \mid l \in L \}$ which are found by using the Squier resolution on the Cayley table of G. Notice that since G is a finite group, $H_2(G)$ is finite, and so $|J| = |L|$. Moreover, since

$$\bar{\partial}_2^G([x^2, u_1] + [u_1 x, u_2] + \cdots + [u_{n_x - 1} x, x]) = n_x [x] \tag{1}$$

where $x \in X$, $u_i = x^{i+1}$ and n_x is the order of x, we have $\mathrm{rank}\,(\mathrm{im}\, \bar{\partial}_2^G) = |X| = |G|$, and so $|J| = |L| = |G|^2 - |G|$.

Now we find a generating set for $\mathrm{im}\, \bar{\partial}_3$ by using the overlaps from the proof of Theorem 2.3. First observe that $\bar{\partial}_3(U_{1,k,k',k''})$ gives a generating set which may be reduced to the basis $\{ V_l \mid l \in L \}$ for $\mathrm{im}\, \bar{\partial}_3^G$. Next we have

$$\begin{aligned} \bar{\partial}_3(U_{2,k,k',i}) &= [x_k y_i, x_k] - [x_{k'} x_k, x_l] + \bar{\Phi}(x_{k'} x_k) - \bar{\Phi}(x_l y_i) \\ &= [x_k y_i, x_k] - [x_l y_i, x_l] \end{aligned}$$

since $\bar{\Phi}(x_{k'} x_k) = [x_{k'} x_k, x_l]$ and $\bar{\Phi}(x_l y_i) = [x_l y_i, x_l]$. Similarly, we compute that

$$\begin{aligned} \bar{\partial}_3(U_{3,k,\lambda}) &= [x_0 z_\lambda, z_\lambda] - [x_k x_0, x_k], \\ \bar{\partial}_3(U_{4,k,i}) &= [x_0 x_k, x_k] - [y_i x_0, y_i], \\ \bar{\partial}_3(U_{5,i,i'}) &= [x_0 y_{i'}, x_0] - [y_i y_{i'}, y_i], \\ \bar{\partial}_3(U_{6,i,\lambda}) &= [x_0 z_\lambda, z_\lambda] - [y_i x_0, y_i], \\ \bar{\partial}_3(U_{7,k,i}) &= [y_i x_0, y_i] - [x_k x_0, x_k], \\ \bar{\partial}_3(U_{8,k,i,i'}) &= [y_i y_{i'}, y_i] - [x_k y_{i'}, x_k], \\ \bar{\partial}_3(U_{9,i,i'}) &= [y_{i'} x_0, y_{i'}] - [y_i x_0, y_i], \\ \bar{\partial}_3(U_{10,i,i',i''} &= [y_{i'} y_{i''}, y_{i'}] - [y_i y_{i''}, y_i], \\ \bar{\partial}_3(U_{11,k,k',\lambda}) &= [z_\lambda x_l, x_l] - [z_\lambda x_k, x_k], \\ \bar{\partial}_3(U_{12,k,i,\lambda}) &= 0, \\ \bar{\partial}_3(U_{13,\lambda,\lambda'}) &= [z_\lambda z_{\lambda'}, z_{\lambda'}] - [z_\lambda x_0, x_0], \end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{14,k,\lambda}) &= -[x_0 z_\lambda, z_\lambda] + [x_0 x_k, x_k],\\
\bar{\partial}_3(U_{15,\lambda,\lambda'}) &= [x_0 z_{\lambda'}, z_{\lambda'}] - [x_0 z_\lambda, z_\lambda],\\
\bar{\partial}_3(U_{16,i,\lambda}) &= -[x_0 z_\lambda, z_\lambda] + [x_0 p_{\lambda i}, p_{\lambda i}],\\
\bar{\partial}_3(U_{17,k,\lambda,\lambda'}) &= [z_\lambda x_k, x_k] - [z_\lambda z_{\lambda'}, z_{\lambda'}],\\
\bar{\partial}_3(U_{18,\lambda,\lambda',\lambda''}) &= [z_\lambda z_{\lambda''}, z_{\lambda''}] - [z_\lambda z_{\lambda'}, z_{\lambda'}],\\
\bar{\partial}_3(U_{19,i,\lambda,\lambda'}) &= -[z_\lambda z_{\lambda'}, z_{\lambda'}] + [z_\lambda p_{\lambda' i}, p_{\lambda' i}],\\
\bar{\partial}_3(U_{20,i,\lambda}) &= [y_i x_0, y_i] - [p_{\lambda i} x_0, p_{\lambda i}],\\
\bar{\partial}_3(U_{21,i,i',\lambda}) &= [y_i y_{i'}, y_i] - [p_{\lambda i} y_{i'}, p_{\lambda i}].
\end{aligned}$$

It is easy to see that we have a smaller generating set for $\operatorname{im}\bar{\partial}_3$: the generating set $\{\, V_l \mid l \in L \,\}$ for $\operatorname{im}\bar{\partial}_3^G$ together with

$$V_{k,i} = [y_i x_0, y_i] - [x_k x_0, x_k], \quad V_{k,i,i'} = [y_i y_{i'}, y_i] - [x_k y_{i'}, x_k],$$

$$V_{k,\lambda} = [x_0 z_\lambda, z_\lambda] - [x_0 x_k, x_k], \quad V_{k,\lambda,\lambda'} = [z_\lambda z_{\lambda'}, z_{\lambda'}] - [z_\lambda x_k, x_k]$$

($0 \le k \le m$; $i, i' \in I - \{1\}$; $\lambda, \lambda' \in \Lambda - \{1\}$). For example, observe that $\bar{\partial}_3(U_{2,k,k',i}) = V_{l,i',i} - V_{k,i',i}$ and $\bar{\partial}_3(U_{3,k,\lambda}) = V_{k,\lambda} - ([x_k x_0, x_k] - [x_0 x_k, x_k])$ where, of course, $([x_k x_0, x_k] - [x_0 x_k, x_k]) \in \operatorname{im}\bar{\partial}_3^G$. The remaining proofs are similar. Therefore

$$B = \{\, V_l,\ V_{k,i},\ V_{k,i,i'},\ V_{k,\lambda},\ V_{k,\lambda,\lambda'} \mid l \in L;\ 0 \le k \le m;\ i, i' \in I - \{1\};\ \lambda, \lambda' \in \Lambda - \{1\} \,\}$$

generates $\operatorname{im}\bar{\partial}_3$.

Next we find a basis for $\ker\bar{\partial}_2$. First notice that since $\bar{\partial}_2([y_{i'} y_i, y_{i'}]) = [y_i]$ and $\bar{\partial}_2([z_\lambda z_{\lambda'}, z_{\lambda'}]) = [z_\lambda]$, it follows from (1) that

$$\operatorname{rank}(\operatorname{im}\bar{\partial}_2) = \operatorname{rank}(\bar{P}_1) = |G| + (|\Lambda| - 1) + (|I| - 1).$$

Therefore

$$\begin{aligned}
\operatorname{rank}(\ker\bar{\partial}_2) = \operatorname{rank}(\bar{P}_2) - \operatorname{rank}(\bar{P}_1) = (|G|^2 - |G|) + |G|\Big((|\Lambda| - 1)\\
+ (|I| - 1)\Big) + (|\Lambda| - 1)^2 + (|I| - 1)^2 + (|\Lambda| - 1)(|I| - 1).
\end{aligned}$$

Since each $\alpha \in \bar{P}_2$ has the form

$$\alpha = \sum_{k,k'=0}^{m} \alpha_{x_k,x_{k'}}[x_k x_{k'}, x_l] + \sum_{i\in I-\{1\}} \Bigg(\alpha_{1,i}[y_i x_0, y_i] + \sum_{i'\in I-\{1\}} \alpha_{2,i,i'}[y_{i'} y_i, y_{i'}]$$

$$+ \sum_{k=0}^{m} \alpha_{3,k,i}[x_k y_i, x_k] \Bigg) + \sum_{\lambda\in\Lambda-\{1\}} \Bigg(\beta_{1,\lambda}[x_0 z_\lambda, z_\lambda] + \sum_{\lambda'\in\Lambda-\{1\}} \beta_{2,\lambda,\lambda'}[z_\lambda z_{\lambda'}, z_{\lambda'}]$$

$$+ \sum_{k=0}^{m} \beta_{3,k,\lambda}[z_\lambda x_k, x_k] + \sum_{i\in I-\{1\}} \gamma_{\lambda,i}[z_\lambda y_i, p_{\lambda i}] \Bigg)$$

where all the coefficients are integers, $\alpha \in \ker \bar{\partial}_2$ if and only if

$$0 = \bar{\partial}_2(\alpha) = \sum_{k,k'=0}^{m} \alpha_{x_k,x_{k'}}([x_k] + [x_{k'}] - [x_l])$$

$$+ \sum_{i\in I-\{1\}} \Bigg(\alpha_{1,i}[x_0] + \sum_{i'\in I-\{1\}} \alpha_{2,i,i'}[y_i] + \sum_{k=0}^{m} \alpha_{3,k,i}[y_i] \Bigg)$$

$$+ \sum_{\lambda\in\Lambda-\{1\}} \Bigg(\beta_{1,\lambda}[x_0] + \sum_{\lambda'\in\Lambda-\{1\}} \beta_{2,\lambda,\lambda'}[z_\lambda] + \sum_{k=0}^{m} \beta_{3,k,\lambda}[z_\lambda]$$

$$+ \sum_{i\in I-\{1\}} \gamma_{\lambda,i}([z_\lambda] + [y_i] - [p_{\lambda i}]) \Bigg).$$

Equivalently, $\alpha \in \ker \bar{\partial}_2$ if and only if

$$\alpha_{x_0,x_0} = -\sum_{k=1}^{m} (\alpha_{x_k,x_0} + \alpha_{x_0,x_k} - \alpha_{x_k,x_k^{-1}}) - \sum_{i\in I-\{1\}} \alpha_{1,i} - \sum_{\lambda\in\Lambda-\{1\}} \beta_{1,\lambda} \qquad (2)$$

$$+ \sum_{\substack{\lambda\in\Lambda-\{1\},\, i\in I-\{1\} \\ p_{\lambda i}\equiv x_0}} \gamma_{\lambda,i},$$

$$0 = 2\alpha_{x_k,x_k} + \sum_{\substack{k'=1\\k'\neq k}}^{m} (\alpha_{x_k,x_{k'}} + \alpha_{x_{k'},x_k} - \alpha_{x_{k'},x_{k'}^{-1}x_k}) \tag{3}$$

$$- \sum_{\substack{\lambda\in\Lambda-\{1\},\ i\in I-\{1\}\\ p_{\lambda i}\equiv x_k}} \gamma_{\lambda,i} \qquad (1 \leq k \leq m),$$

$$\alpha_{2,i,2} = -\Big(\sum_{i'\in I-\{1,2\}} \alpha_{2,i,i'} + \sum_{k=0}^{m} \alpha_{3,k,i} + \sum_{\lambda\in\Lambda-\{1\}} \gamma_{\lambda,i} \Big) \ \ (i \in I - \{1\}), \tag{4}$$

$$\beta_{2,\lambda,2} = -\Big(\sum_{\lambda'\in\Lambda-\{1,2\}} \beta_{2,\lambda,\lambda'} + \sum_{k=0}^{m} \beta_{3,k,\lambda} + \sum_{i\in I-\{1\}} \gamma_{\lambda,i} \Big) \ \ (\lambda \in \Lambda - \{1\}). \tag{5}$$

We have assumed that $|I|$, $|\Lambda| \geq 2$ and that 2 is a common element. The cases $|I| = 1$ or $|\Lambda| = 1$ are treated similarly. By using the system of equations above, we find a basis for $\ker \bar{\partial}_2$. First, if we take all $\alpha_{1,i}$, $\alpha_{2,i,i'}$, $\alpha_{3,k,i}$, $\beta_{1,\lambda}$, $\beta_{2,\lambda,\lambda'}$, $\beta_{3,k,\lambda}$ and $\gamma_{\lambda,i}$ to be zero, we have

$$\sum_{x_k,x_{k'}\in X} \alpha_{x_k,x_{k'}}([x_k] + [x_{k'}] - [x_l]) = 0,$$

which gives the basis $\{ W_j \mid j \in J \}$ of $\ker \bar{\partial}_2^G$ where $H_2(G) = \ker \bar{\partial}_2^G / \mathrm{im}\, \bar{\partial}_3^G$.

Now if we fix $\alpha_{1,i} = 1$ and all the other variables on the right-hand side in (2)–(5) to be zero, then we obtain $\alpha_{x_0,x_0} = -1$. Therefore we obtain the following generators:

$$W_i = [y_i x_0, y_i] - [x_0^2, x_0] \ \ (i \in I - \{1\}).$$

By using similar arguments, we obtain certain other generators:

$$\begin{array}{lcll}
W_\lambda & = & [x_0 z_\lambda, z_\lambda] - [x_0^2, x_0] & (\lambda \in \Lambda - \{1\}),\\
W_{i,k} & = & [y_2 y_i, y_2] - [x_k y_i, x_k] & (0 \leq k \leq m,\ i \in I - \{1\}),\\
W_{\lambda,k} & = & [z_\lambda z_2, z_2] - [z_\lambda x_k, x_k] & (0 \leq k \leq m,\ \lambda \in \Lambda - \{1\}),\\
W_{i,i'} & = & [y_{i'} y_i, y_{i'}] - [y_2 y_i, y_2] & (i, i' \in I - \{1\},\ i' \neq 2),\\
W_{\lambda,\lambda'} & = & [z_\lambda z_{\lambda'}, z_{\lambda'}] - [z_\lambda z_2, z_2] & (\lambda, \lambda' \in \Lambda - \{1\},\ \lambda' \neq 2).
\end{array}$$

We note that to construct a basis for $\ker \bar{\partial}_2$ we need a further $(|\Lambda| - 1)(|I| - 1)$ independent elements. We will see that we do not need to identify these remaining elements $W_{\lambda,i}$ $(\lambda \in \Lambda - \{1\};\ i \in I - \{1\})$ of the basis:

$$Z = \{\ W_j,\ W_i,\ W_\lambda,\ W_{i,k},\ W_{\lambda,k},\ W_{i,i'}, W_{\lambda,\lambda'},\ W_{\lambda,i},\ |\ j \in J;\ 0 \le k \le m;$$

$$i, i' \in I - \{1\}\ (i' \neq 2);\ \lambda, \lambda' \in \Lambda - \{1\}\ (\lambda' \neq 2)\ \}.$$

Now we express the V's in B in terms of the W's in Z. First, for each $l \in L$, write $V_l(W)$ for the expression of V_l in terms of the W_j $(j \in J)$ as in the calculation of $H_2(G)$. Now observe that

$$\begin{array}{lll} V_{0,i} = W_i, & V_{k,i} = W_i + \bar{\partial}_3([(x_k x_0, x_k), (x_0 x_0, x_0)] & (k \neq 0), \\ V_{0,\lambda} = W_\lambda, & V_{k,\lambda} = W_\lambda - \bar{\partial}_3([(x_0 x_0, x_0), (x_0 x_k, x_k)] & (k \neq 0), \\ V_{k,2,i} = W_{i,k}, & V_{k,i',i} = W_{i,i'} + W_{i,k} & (i' \neq 2), \\ V_{k,\lambda,2} = W_{\lambda,k}, & V_{k,\lambda,\lambda'} = W_{\lambda,\lambda'} + W_{\lambda,k} & (\lambda' \neq 2). \end{array}$$

We obtain the following abelian group presentation for $H_2(S)$:

$$\begin{aligned} \langle\ Z|\ \ & V_l(W) = 0,\ W_i = 0,\ W_i + V_k(W) = 0\ (k \neq 0),\ W_\lambda = 0, \\ & W_\lambda + V'_k(W) = 0\ (k \neq 0),\ W_{i,k} = 0,\ W_{i,i'} + W_{i,k} = 0, \\ & W_{\lambda,k} = 0,\ W_{\lambda,\lambda'} + W_{\lambda,k} = 0\ \ (l \in L;\ 0 \le k \le m; \\ & \lambda \in \Lambda - \{1\};\ \lambda' \in \Lambda - \{1,2\};\ i \in I - \{1\};\ i' \in I - \{1,2\})\ \rangle \end{aligned}$$

where $V_k(W)$ expresses $\bar{\partial}_3([(x_k x_0, x_k), (x_0 x_0, x_0)]$ in terms of the W_j, and similarly for $V'_k(W)$. It is clear that some of the generators in the above presentation are redundant. By eliminating these redundant generators, we obtain the abelian group presentation:

$$\langle\ V_j,\ W_{\lambda,i}\ (j \in J;\ \lambda \in \Lambda - \{1\};\ i \in I - \{1\})\ |\ V_l(W) = 0\ (l \in L)\ \rangle$$

which defines the abelian group

$$H_2(G) \times \mathbb{Z}^{(|I|-1)(|\Lambda|-1)},$$

as required. □

4. A small presentation for Rees matrix semigroups

Consider the presentation for $S = \mathcal{M}[G; I, \Lambda; P]$, a Rees matrix semigroup with P normal, which is given in Theorem 2.2 by

$$\begin{aligned} \mathcal{P}_1 = \langle\, Y \mid R, & y_i e = y_i, \quad e y_i = e \ \ (2 \le i \le m), & (6)\\ & z_\lambda e = e, \quad e z_\lambda = z_\lambda \ \ (2 \le \lambda \le n), & (7)\\ & z_\lambda y_i = p_{\lambda i} \ \ (2 \le i \le m,\ 2 \le \lambda \le n) \,\rangle \end{aligned}$$

where e is a non-empty representative of the identity of G, and where $I = \{1, \ldots, m\}$ and $\Lambda = \{1, \ldots, n\}$. From now on, we write $S = \mathcal{M}[G; m, n; P]$ instead of $S = \mathcal{M}[G; I, \Lambda; P]$.

The deficiency of $\mathcal{P}_1$ is $\operatorname{def}(\mathcal{P}_1) = \operatorname{def}(\mathcal{P}_G) + (m-1)(n-1) + (m-1) + (n-1)$, where $\mathcal{P}_G = \langle\, X \mid R \,\rangle$ is a semigroup presentation for G. With the above notation, we give a presentation for S with deficiency $\operatorname{def}(\mathcal{P}_G) + (m-1)(n-1) + 1$, which is one higher than the rank of $H_2(S)$ (see Theorem 3.4), provided that $\mathcal{P}_G$ is an efficient presentation for G.

Proposition 4.5 *The presentation*

$$\begin{aligned} \mathcal{P}_2 = \langle\, Y \mid R, & e y_2 = e, \quad y_i y_{i+1} = y_i \ (2 \le i \le m-1), & (8)\\ & e z_2 = z_2, \quad z_\lambda z_{\lambda+1} = z_{\lambda+1} \ (2 \le \lambda \le n-1), & (9)\\ & y_m z_n e = y_m, & (10)\\ & z_\lambda y_i = p_{\lambda i} \ \ (2 \le i \le m,\ 2 \le \lambda \le n) \,\rangle \end{aligned}$$

defines the Rees matrix semigroup $S = \mathcal{M}[G; m, n; P]$ with $m, n > 1$.

Proof. From (6), we have

$$y_i y_{i+1} = (y_i e) y_{i+1} \equiv y_i (e y_{i+1}) = y_i e = y_i \ \ (2 \le i \le m-1).$$

Similarly, from (7), we have

$$z_\lambda z_{\lambda+1} = z_{\lambda+1} \ \ (2 \le \lambda \le n-1).$$

Moreover, from (7) and (6), we have

$$y_m z_n e = y_m e = y_m.$$

Therefore, every relation in $\mathcal{P}_2$ holds in S. Now we show that every relation in $\mathcal{P}_1$ is a consequence of the relations in $\mathcal{P}_2$.

By induction, it follows from (8) that $y_i y_{i'} = y_i$ $(2 \leq i < i' \leq m)$. In particular,

$$y_i y_m = y_i \text{ and } y_2 y_i = y_2 \ (2 \leq i \leq m). \tag{11}$$

Similarly, from (9),

$$z_\lambda z_n = z_n \text{ and } z_2 z_\lambda = z_\lambda \ (2 \leq \lambda \leq n). \tag{12}$$

Since G is finite and e is a representative of the identity of G, there exists $k \in \mathbb{N}$ such that the relation $p_{nm}^k = e$ holds in G, and so $(z_n y_m)^k = e$ is a consequence of the relations from $R \cup \{ z_n y_m = p_{nm} \}$. It follows from (12), (9) and (10) that

$$z_n e = (z_2 z_n)e = (e z_2) z_n e = e z_n e = (z_n y_m)^{k-1} z_n (y_m z_n e) = (z_n y_m)^k = e. \tag{13}$$

Moreover, since $e^2 = e$ is a consequence of the relations from R, it follows from (10) that

$$y_m e = (y_m z_n e)e = y_m z_n e = y_m. \tag{14}$$

Next, from (11) and (14), we have

$$y_i e = (y_i y_m)e \equiv y_i(y_m e) = y_i y_m = y_i \ (2 \leq i \leq m-1)$$

and, from (8) and (11), we have

$$e y_i = (e y_2) y_i \equiv e(y_2 y_i) = e y_2 = e \ (3 \leq i \leq m).$$

Similarly, from (9), (12) and (13), we have $e z_\lambda = z_\lambda$ $(3 \leq \lambda \leq n)$ and $z_\lambda e = e$ $(2 \leq \lambda \leq n-1)$, as required. □

5. Efficiency of Rees matrix semigroups

The presentation $\mathcal{P}_2$ is not efficient, but it proves useful in the following results. In [1] we proved that finite abelian groups and dihedral groups D_{2r} with r even are efficient (as semigroups). In particular, we found efficient semigroup presentations of the form $\langle X \mid R_1, \ x^{k+1} = x \rangle$ with identity x^k. In the following theorem we use semigroup presentations for groups of a similar form in $\mathcal{P}_2$ to obtain efficient semigroup presentations for Rees matrix semigroups.

Theorem 5.6 *Let $S = \mathcal{M}[G; m, n; P]$ be a finite Rees matrix semigroup with P normal. If G has a semigroup presentation of the form $\mathcal{P}_G = \langle\, X \mid R_1,\ xux = x \,\rangle$ with identity xu ($x \in X,\ u \in X^+$), then S has a semigroup presentation whose deficiency is* $\operatorname{def}(\mathcal{P}_G) + (m-1)(n-1)$.

Proof. First assume that $m, n > 1$ and consider the presentation $\mathcal{P}_2$ for S. Take $e \equiv xu$. Since, from (8) and (13), the relations $xuy_2 = xu$, $z_n x = x$ and $xux = x$ hold in S, we have

$$xuy_2z_nx \equiv (xuy_2)z_nx = xu(z_nx) = xux = x.$$

Therefore, S is a homomorphic image of the semigroup T defined by the presentation obtained from $\mathcal{P}_2$ by adding the relation $xuy_2z_nx = x$ and removing the relations $xuy_2 = xu$ and $xux = x$:

$$\begin{aligned} \mathcal{P}_3 = \langle\, Y \mid R_1,\ & xuy_2z_nx = x, && (15)\\ & y_iy_{i+1} = y_i \ \ (2 \le i \le m-1), && (16)\\ & xuz_2 = z_2, && (17)\\ & z_\lambda z_{\lambda+1} = z_{\lambda+1} \ (2 \le \lambda \le n-1), && (18)\\ & y_mz_nxu = y_m, && (19)\\ & z_\lambda y_i = p_{\lambda i} \ \ (2 \le i \le m,\ 2 \le \lambda \le n) \,\rangle. \end{aligned}$$

Note that if $m = 2$, then (16) is absent and if $n = 2$, then (18) is absent. Now we show that the relations $xuy_2 = xu$ and $xux = x$ hold in T so that $S \cong T$.

As before, from (16), (18) and (17), we have

$$y_2y_m = y_2, \quad z_2z_n = z_n \text{ and } xuz_n = z_n. \tag{20}$$

It follows from (20), (19) and (15) that

$$xuy_2 = xu(y_2y_m) = xuy_2(y_mz_nxu) = (xuy_2z_nx)u = xu \tag{21}$$

and also that

$$z_nx = (z_2z_n)x = (xuz_2)z_nx = xuz_nx = (xuy_2)z_nx = x. \tag{22}$$

Therefore, from (15), (21), (20) and (22), we have

$$xux = xu(xuy_2z_nx) = xu(xuz_n)x = (xuz_n)x = z_nx = x$$

and hence S is efficient, as required.

Similarly, it may be shown that if $m = 1$, then

$$\mathcal{P}_3' = \langle\, X, z_2, ..., z_n \mid R_1,\ \ uxz_2 = z_2,\ \ z_\lambda z_{\lambda+1} = z_{\lambda+1}\ (2 \le \lambda \le n-1),\ \ xz_nux = x\,\rangle$$

is an efficient presentation for S. Similarly, if $n = 1$, then

$$\mathcal{P}_3'' = \langle\, X, y_2, ..., y_m \mid R_1,\ \ xuy_2x = x,\ \ y_iy_{i+1} = y_i\ (2 \le i \le m-1),\ \ y_mxu = y_m\,\rangle$$

is an efficient presentation for S. The proof is now complete. □

As we mentioned at the beginning of this section, finite abelian groups and dihedral groups D_{2r} with r even, have efficient semigroup presentations of the required form (see [1]). (For further examples of groups which are efficient as semigroups, see [2].) Therefore we have the following result.

Corollary 5.7 *Finite Rees matrix semigroups over finite abelian groups or dihedral groups with even degree are efficient.* □

6. Efficient non-simple semigroups

All the efficient semigroups in [1] and in this paper so far are simple. In this section, we give a family of efficient non-simple semigroups which have non-trivial second homology. Consider the following presentation:

$$\langle\, a_1, ..., a_r \mid a_i^{n_i+1} = a_i\ \ (1 \le i \le r),\ \ a_ja_i = a_ia_j\ \ (1 \le i < j \le r)\ \rangle$$

where $n_1 > 1$ and n_i divides n_{i+1} for $i = 1, ..., r-1$.

This semigroup presentation is related to the standard group presentation of the abelian group $C_{n_1} \times \cdots \times C_{n_r}$, where C_{n_i} is the cyclic group of order n_i. For $r \ge 2$, it is clear that this semigroup presentation defines a commutative semigroup S which is not a group. For $r \ge 2$, the subset $I = \{\, a_1^{m_1} \cdots a_r^{m_r} \mid 1 \le m_i \le n_i \text{ for } i = 1, ..., r\}$ is a proper (minimal) ideal of S, so that S is not simple.

Theorem 6.8 *Let S be the semigroup defined by the presentation*

$$\langle\ a_1, \ldots, a_r \mid a_i^{n_i+1} = a_i \ \ (1 \leq i \leq r),\ \ a_j a_i = a_i a_j \ \ (1 \leq i < j \leq r)\ \rangle$$

where $n_1 > 1$ and n_i divides n_{i+1} for $i = 1, ..., r-1$. Then the second homology of S is

$$H_2(S) = C_{n_1}^{(r-1)} \times C_{n_2}^{(r-2)} \times \cdots \times C_{n_{r-1}}.$$

In particular, S is an efficient semigroup.

Proof. Since the system of relations is uniquely terminating, use the Squier resolution [8] to determine the second homology. □

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