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A Generalized Trapezoid Inequality for Functions of Bounded Variation

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Abstract

We establish a generalization of a recent trapezoid inequality for functions of bounded variation. A number of special cases are considered. Applications are made to quadrature formulæ, probability theory, special means and the estimation of the beta function.

Key Words: Trapezoid inequality, bounded variation, numerical integration, beta function.

1. Introduction

In [1], Dragomir proved the following trapezoid inequality for functions of bounded variation. Here and subsequently in the paper, if f is of bounded variation on [a,b], we denote its total variation on that interval by $\bigvee_a^b (f)$.

Theorem A. *Let $f : [a,b] \to \mathbf{R}$ be of bounded variation on $[a,b]$. Then*

$$\left| \int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) \right| \le \frac{1}{2}(b-a) \bigvee_a^b (f). \tag{1.1}$$

The constant $1/2$ is best possible.

We introduce the notation $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ for a division of the interval $[a,b]$, with $h_i := x_{i+1} - x_i$ $(0 \le i < n)$ and $\nu(h) := \max\{h_i \mid i = 0, ..., n-1\}$ for the norm of the division. Then we may deduce from Theorem A that

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$$\int_a^b f(t)\,dt = T(f, I_n) + R(f, I_n), \tag{1.2}$$

where

$$T(f, I_n) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_i) + f(x_{i+1})\right] h_i, \tag{1.3}$$

and that the remainder term satisfies

$$|R(f, I_n)| \le \frac{1}{2}\nu(h)\bigvee_a^b (f). \tag{1.4}$$

Here the constant $1/2$ is also best possible.

The main aim of this paper is to compare $\int_a^b f(t)\,dt$ with

$$f(a)(x-a) + f(b)(b-x),$$

where $x \in [a, b]$ is a free parameter. The choice $x = (a+b)/2$ gives the trapezoid estimate

$$\frac{f(a) + f(b)}{2}(b-a)$$

for mappings of bounded variation.

In Section 2 we derive our basic estimate, which provides an upper bound for the difference between $\int_a^b f(t)dt$ and the estimate proposed above for the case when f is a function of bounded variation. We examine the important special cases when f has a continuous derivative or is Lipschitz, monotone or convex. In Section 3 these results are applied to the estimation of the error term in some quadrature formulæ and in Section 4 to some estimates in probability theory, in particular, that of the mean $E(X)$ of a random variable X. Section 5 uses particular choices of f to obtain some apparently new inequalities subsisting amongst various well–known means of a pair of positive numbers. Finally a further special choice is taken in Section 6 to address the estimation of Euler's beta function.

For a compendious treatment of other inequalities of trapezoid type, see [2] and the references therein.

2. Some Integral Inequalities

We start with a basic integral inequality for mappings of bounded variation. For convenience we set

$$J(x) := \int_a^b f(t)\,dt - f(a)(x-a) - f(b)(b-x).$$

Theorem 1 *Let $f : [a,b] \to \mathbf{R}$ be a mapping of bounded variation. Then*

$$|J(x)| \leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{2.1}$$

for all $x \in [a,b]$. The constant $1/2$ is best possible.

Proof. By the integration by parts formula for a Riemann–Stieltjes integral, we have

$$\int_a^b (x-t)\,df(t) = (x-t)\,f(t)\big|_a^b + \int_a^b f(t)\,dt,$$

whence we derive the identity

$$\int_a^b f(t)\,dt = (b-x)\,f(b) + (x-a)\,f(a) + \int_a^b (x-t)\,df(t) \tag{2.2}$$

for all $x \in [a,b]$.

If $g, v : [a,b] \to \mathbf{R}$ are such that g is continuous and v of bounded variation on $[a,b]$, then $\int_a^b g(t)\,dv(t)$ exists and

$$\left|\int_a^b g(t)\,dv(t)\right| \leq \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (v).$$

Thus

$$\left|\int_a^b (x-t)\,df(t)\right| \leq \sup_{t\in[a,b]} |x-t| \bigvee_a^b (f). \tag{2.3}$$

As

$$\sup_{t\in[a,b]} |x-t| = \max\{x-a, b-x\} = \frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|,$$

(2.1) follows from (2.3) and (2.2).

Now suppose that (2.1) holds with a constant $c > 0$, that is,

$$|J(x)| \leq \left[c(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f)$$

for all $x \in [a,b]$. For $x = (a+b)/2$, we get

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a)\right| \leq c(b-a) \bigvee_a^b (f). \tag{2.4}$$

Define $f : [a,b] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x = a \\ 1 & \text{if } x \in (a,b) \\ 0 & \text{if } x = b. \end{cases}$$

Then f is of bounded variation on $[a,b]$ and

$$\int_a^b f(x)\,dx = b-a, \qquad \bigvee_a^b (f) = 2.$$

For this choice of f, (2.4) provides

$$b - a \leq 2c(b-a)$$

or $c \geq 1/2$, concluding the proof. □

Remark 1 *a) The choice $x = b$ supplies the "left rectangle" inequality*

$$\left| \int_a^b f(x)\,dx - f(a)(b-a) \right| \le (b-a) \bigvee_a^b (f).$$

b) Setting $x = a$ yields the "right rectangle" inequality

$$\left| \int_a^b f(x)\,dx - f(b)(b-a) \right| \le (b-a) \bigvee_a^b (f).$$

c) For $x = (a+b)/2$ we obtain the known "trapezoid" inequality (1.1). *This is the best possible inequality we can derive from* (2.1) *in the sense that the constant* $1/2$ *is best possible.*

Further standard assumptions about f lead to useful corollaries.

Corollary 1 *Suppose $f \in C^{(1)}[a,b]$. Then*

$$|J(x)| \le \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \|f'\|_1$$

for all $x \in [a,b]$. Here as subsequently $\|\cdot\|_1$ is the L_1-norm

$$\|f'\|_1 := \int_a^b |f'(t)|\,dt.$$

Corollary 2 *Let $f : [a,b] \to \mathbf{R}$ be a Lipschitzian mapping with the constant $L > 0$. Then*

$$|J(x)| \le \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] (b-a)\,L$$

for all $x \in [a,b]$.

Proof. As f is $L-$Lipschitzian on $[a,b]$, it is also of bounded variation. If $Div[a,b]$ denotes the family of divisions on $[a,b]$, then

$$\begin{aligned}\bigvee_a^b (f) &= \sup_{I_n \in Div[a,b]} \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \\ &\leq L \sup_{I_n \in Div[a,b]} |x_{i+1} - x_i| \\ &= (b-a)L,\end{aligned}$$

and the desired result is proved. □

Corollary 3 *Let $f : [a,b] \to \mathbf{R}$ be a monotone mapping on $[a,b]$. Then*

$$|J(x)| \leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] |f(b) - f(a)|$$

for all $x \in [a,b]$.

For $f : [a,b] \to \mathbf{R}$ convex on $[a,b]$, we have the Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

The above results enable us to place bounds on the difference between the two sides of the second inequality. Thus if f is convex and of bounded variation on $[a,b]$, (1.1) provides

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{1}{2} \bigvee_a^b (f).$$

If f is convex and Lipschitzian with the constant L on $[a,b]$, then Corollary 2.3 yields

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{1}{2}(b-a)L.$$

If f is convex and monotonic on $[a,b]$, then by Corollary 2.4

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2}\left|f(b)-f(a)\right|.$$

Finally, if $f \in C^{(1)}[a,b]$ and convex, then by Corollary 2.2

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2}\left\|f'\right\|_1.$$

3. Applications to Quadrature Formulæ

We now introduce the intermediate points $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ in the division I_n of $[a,b]$ and define

$$T_P(f, I_n, \xi) := \sum_{i=0}^{n-1}\left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})\right].$$

We have the following result concerning the approximation by T_P of the integral $\int_a^b f(x)\,dx$.

Theorem 2 *Let $f : [a,b] \to \mathbf{R}$ be of bounded variation on $[a,b]$. Then*

$$\int_a^b f(x)\,dx = T_P(f, I_n, \xi) + R_P(f, I_n, \xi), \tag{3.1}$$

with remainder term satisfying

$$\left|R_P(f, I_n, \xi)\right| \leq \left[\frac{1}{2}\nu(h) + \max_{0\leq i<n}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\bigvee_a^b(f) \leq \nu(h)\bigvee_a^b(f). \tag{3.2}$$

The constant $1/2$ is best possible.

Proof. Application of Theorem 1 to the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ gives

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \left[f(x_i)(\xi_i - x_i) + f(x_{i+1})(x_{i+1} - \xi_i)\right]\right|$$

$$\leq \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}} (f)$$

for all $i \in \{0, ..., n-1\}$.

By this and the generalized triangle inequality,

$$\begin{aligned} |R_P(f, I_n, \xi)| &\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(t)\, dt - [f(x_i)(\xi_i - x_i) + f(x_{i+1})(x_{i+1} - \xi_i)] \right| \\ &\leq \sum_{i=0}^{n-1} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}} (f) \\ &\leq \max_{0 \leq i < n} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) \\ &\leq \left[\frac{1}{2}\nu(h) + \max_{0 \leq i < n} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b (f) \end{aligned}$$

and the first inequality in (3.2) is proved.

For the second, we observe that

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2}h_i, \qquad i = 0, ..., n-1$$

so that

$$\max_{0 \leq i < n} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2}\nu(h),$$

proving the theorem. □

Remark 2 *a) Choosing* $\xi_i = x_{i+1} \quad (i = 0, ..., n-1)$ *provides*

$$\int_a^b f(x)\, dx = D_L(f, I_n) + R_L(f, I_n).$$

Here $D_L(f, I_n)$ *is constructed from the left rectangle rule*

$$D_L(f, I_n) = \sum_{i=0}^{n-1} f(x_i) h_i$$

and the remainder satisfies

$$|R_L(f, I_n)| \leq \nu(h) \bigvee_a^b (f).$$

b) Taking $\xi_i = x_i \quad (i = 0, ..., n-1)$ *gives*

$$\int_a^b f(x) \, dx = D_R(f, I_n) + R_R(f, I_n),$$

where $D_R(f, I_n)$ *is built from the right rectangle rule*

$$D_R(f, I_n) = \sum_{i=0}^{n-1} f(x_{i+1}) h_i$$

and the remainder term satifies

$$|R_R(f, I_n)| \leq \nu(h) \bigvee_a^b (f).$$

c) Finally, if we choose $\xi_i = (x_i + x_{i+1})/2$*, we get* (1.2) *with* (1.3) *and* (1.4).

Corollary 4 *Let* $f : [a, b] \to \mathbf{R}$ *be Lipschitzian with constant* $L > 0$*. Then we have* (3.1) *and the remainder satisfies*

$$|R_T(f, I_n, \xi)| \leq L \left[\frac{1}{2}\nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \leq L\nu(h).$$

Corollary 5 *Let* $f : [a, b] \to \mathbf{R}$ *be monotone on* $[a, b]$*. Then we have the quadrature formula* (3.1) *and the remainder satisfies*

$$\begin{aligned} |R_T(f, I_n, \xi)| &\leq \left[\frac{1}{2}\nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] |f(b) - f(a)| \\ &\leq \nu(h) |f(b) - f(a)|. \end{aligned}$$

4. Applications to Probability

Proposition 1 *Let $f : [a, b] \to \mathbf{R}$ be a probability density function of bounded variation on $[a, b]$ and $F : [a, b] \to \mathbf{R}$ the corresponding distribution function*

$$F(x) = \int_a^x f(t)\,dt, \qquad x \in [a, b].$$

Then

$$|F(x) - [f(a)(y-a) + f(x)(x-y)]| \leq \left[\frac{1}{2}(x-a) + \left|y - \frac{a+x}{2}\right|\right] \bigvee_a^x (f) \tag{4.1}$$

for all $a \leq y \leq x$. In particular, choosing $y = (a+x)/2$ gives

$$\left|F(x) - \frac{f(a) + f(x)}{2}(x-a)\right| \leq \frac{1}{2}(x-a) \bigvee_a^x (f) \tag{4.2}$$

for all $x \in [a, b]$. The constant $1/2$ in (4.1) and (4.2) is best possible.

Proof. The result is immediate from Theorem 1. □

The following approximation holds for the expectation of a random variable.

Proposition 2 *Let X be a random variable having distribution function F and expectation $E(X)$. Then*

$$\left|E(X) - \sum_{i=0}^{n-1} F(x_i)(\xi_{i+1} - \xi_i) - \xi_{n-1}\right| \leq \frac{1}{2}\nu(h) + \max_{0 \leq i < n}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|. \tag{4.3}$$

Proof. We apply Theorem 2 to F to get

$$\left|\int_a^b F(t)\,dt - \sum_{i=0}^{n-1} F(x_i)(\xi_i - x_i) - \sum_{i=0}^{n-1} F(x_{i+1})(x_{i+1} - \xi_i)\right| \tag{4.4}$$

$$\leq \left[\frac{1}{2}\nu(h) + \max_{0 \leq i < n}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b (F).$$

But

$$\bigvee_{a}^{b}(F) = F(b) - F(a) = 1$$

and

$$\int_{a}^{b} F(t)\,dt = tF(t)|_{a}^{b} - \int_{a}^{b} tf(t)\,dt = bF(b) - aF(a) - E(X) = b - E(X).$$

By (4.4),

$$\begin{aligned} &\Bigg| b - E(X) - F(a)(\xi_0 - a) - \sum_{i=1}^{n-1} F(x_i)(\xi_i - x_i) \\ &- \sum_{i=0}^{n-2} F(x_{i+1})(x_{i+1} - \xi_i) - F(b)(b - \xi_{n-1}) \Bigg| \\ &\leq \frac{1}{2}\nu(h) + \max_{0\leq i<n}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \end{aligned}$$

or

$$\left| -E(X) - \sum_{i=1}^{n-1} F(x_i)(\xi_i - \xi_{i-1}) + \xi_{n-1} \right| \leq \frac{1}{2}\nu(f) + \max_{0\leq i<n}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|$$

and the proposition is proved. □

Remark 3 *a) Suppose the division is reduced to the endpoints, that is,* $a = x_0 < x_1 = b$ *and* $\xi_1 = \xi \in [a, b]$*. Then by* (4.3)

$$|E(X) - \xi| \leq \frac{1}{2}(b - a) + \left|\xi - \frac{a + b}{2}\right|$$

for all $\xi \in [a, b]$.

b) Suppose $a = x_0 < x < x_2 = b$ *and* $\xi \in [a, x], \mu \in [x, b]$.
Then by (4.3)

$$\begin{aligned}
&|E(X) - F(x)(\xi - \mu) - \mu| \\
&\quad \leq \frac{1}{2}\max\{|x-a|, |b-x|\} + \max\left\{\left|\xi - \frac{a+x}{2}\right|, \left|\mu - \frac{x+b}{2}\right|\right\} \\
&\quad = \frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right| + \max\left\{\left|\xi - \frac{a+x}{2}\right|, \left|\mu - \frac{x-b}{2}\right|\right\}
\end{aligned}$$

for all $a \leq \xi \leq x \leq \mu \leq b$.

In particular, if $\xi = (a+x)/2$ *and* $\mu = (x+b)/2$*, then*

$$\left|E(X) - \frac{1}{2}F(x)(a-b) - \frac{x+b}{2}\right| \leq \frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|$$

for all $x \in [a, b]$.

5. Applications to Special Means

We now derive some results for various well–known means. For $a, b \geq 0$ we have the arithmetic mean

$$A = A(a, b) := (a+b)/2$$

and the geometric mean

$$G = G(a, b) := \sqrt{ab}.$$

For $a, b > 0$ we have the harmonic mean

$$H = H(a, b) := 2/\left(a^{-1} + b^{-1}\right),$$

the logarithmic mean

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \dfrac{b-a}{\ln b - \ln a} & \text{if } a \neq b, \end{cases}$$

the identric mean

$$I := I(a,b) = \begin{cases} a & \text{if } a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}$$

and for $p \in \mathbf{R} \backslash \{-1, 0\}$, the p–logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if } a \neq b; \\ a & \text{if } a = b. \end{cases}$$

It is well–known that with $L_{-1} := L$ and $L_0 := I$, the net (L_p) is monotone nondecreasing in $p \in \mathbf{R}$. In particular, we have the inequalities

$$H \leq G \leq L \leq I \leq A.$$

In what follows we establish some rather more involved inequalities for the above means by the use of (2.1), which we express in the equivalent form

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{bf(b) - af(a)}{b-a} + x \cdot \frac{f(b) - f(a)}{b-a} \right| \tag{5.1}$$

$$\leq \left[\frac{1}{2}(b-a) + |x - A|\right] \frac{1}{b-a} \bigvee_a^b (f).$$

Define $f : [a,b] \subset (0, \infty) \to \mathbf{R}$ by $f(x) = x^p$, $p \in \mathbf{R} \backslash \{-1, 0\}$. Then

$$\frac{1}{b-a} \int_a^b f(t)\, dt = L_p^p(a,b),$$

$$\frac{bf(b) - af(a)}{b-a} = (p+1) L_p^p(a,b),$$

$$\frac{f(b) - f(a)}{b-a} = pL_{p-1}^{p-1}(a,b),$$

$$\frac{1}{b-a}\bigvee_a^b (f) = \frac{1}{b-a}\int_a^b |f'(t)|\,dt = |p|\,L_{p-1}^{p-1}(a,b).$$

We deduce from (5.1) that

$$\left|L_p^p - (p+1)\,L_p^p + pxL_{p-1}^{p-1}\right| \le \left[\frac{1}{2}(b-a) + |x-A|\right] |p|\,L_{p-1}^{p-1},$$

which is equivalent to

$$\left|xL_{p-1}^{p-1} - L_p^p\right| \le \left[\frac{1}{2}(b-a) + |x-A|\right] L_{p-1}^{p-1}, \quad x \in [a,b].$$

The choice $x = A$ yields

$$\left|AL_{p-1}^{p-1} - L_p^p\right| \le \frac{1}{2}(b-a)\,L_{p-1}^{p-1}.$$

If instead we define $f : [a,b] \subset (0,\infty) \to \mathbf{R}$ by $f(x) = 1/x$, then

$$\frac{1}{b-a}\int_a^b f(t)\,dt = L^{-1}(a,b),$$

$$\frac{bf(b) - af(a)}{b-a} = 0,$$

$$\frac{f(b) - f(a)}{b-a} = -G^{-2}(a,b),$$

$$\frac{1}{b-a}\bigvee_a^b (f) = \frac{1}{b-a}\int_a^b |f'(t)|\,dt = G^{-2}(a,b).$$

From (5.1), we deduce that

$$\left|L^{-1} - xG^{-2}\right| \le \left[\frac{1}{2}(b-a) + |x-A|\right] G^{-2},$$

or equivalently

$$\left|xL - G^2\right| \leq \frac{1}{2}\left[(b-a) + |x-A|\right] L, \quad x \in [a,b].$$

Choosing $x = A$, we get

$$0 \leq AL - G^2 \leq \frac{1}{2}(b-a).$$

Finally, define $f : [a,b] \subset (0,\infty) \to \mathbf{R}$ by $f(x) = \ln x$, so that

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \ln I(a,b),$$

$$\frac{bf(b) - af(a)}{b-a} = \ln I(a,b) + 1,$$

$$\frac{f(b) - f(a)}{b-a} = L^{-1}(a,b),$$

$$\frac{1}{b-a}\int_a^b |f'(t)|\,dt = L^{-1}(a,b).$$

From (5.1), we deduce that

$$|x - L| \leq \frac{1}{2}(b-a) + |x-A|, \quad x \in [a,b].$$

With $x = A$, we get

$$0 \leq A - L \leq \frac{1}{2}(b-a).$$

6. Application to Euler's Beta Function

Let β be the Euler beta function given by

$$\beta(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \qquad p, q > 0.$$

Proposition 3 *If $p, q > 1$, then*

$$\beta(p,q) = T(p,q,I_n,\xi) + R(p,q,I_n,\xi),$$

where

$$T(p,q,I_n,\xi) = \sum_{i=0}^{n-1} \left[(\xi_i - x_i) x_i^{p-1} (1-x_i)^{q-1} + (x_{i+1} - \xi_i) x_{i+1}^{p-1} (1-x_{i+1})^{q-1} \right]$$

and the remainder $R(p,q,I_n,\xi)$ satisfies

$$|R| \leq \left[\frac{1}{2}\nu(h) + \max\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \max(p-1, q-1)\, \beta(p-1, q-1).$$

Proof. For $p, q > 1$ define $f_{p,q} : (0,1) \to \mathbf{R}$ by

$$f_{p,q}(t) = t^{p-1}(1-t)^{q-1}.$$

We have

$$f'_{p,q}(t) = [(p+q-2)t - q + 1]\, t^{p-2}(1-t)^{q-2},$$

so that

$$\begin{aligned} \bigvee_0^1 (f_{p,q}) &= \int_0^1 \left| f'_{p,q}(t) \right| dt \\ &\leq \int_0^1 |(p+q-2)t - q + 1|\, t^{p-2}(1-t)^{q-2}\, dt \\ &\leq \max(q-1, p-1) \int_0^1 t^{p-2}(1-t)^{q-2}\, dt \\ &= \max(q-1, p-1)\, \beta(p-1, q-1) \end{aligned}$$

and the proposition is proved. □

Remark 4 *The choice* $\xi_i = (x_i + x_{i+1})/2$ *yields*

$$\beta(p,q) = T(p,q,I_n) + R(p,q,I_n)$$

where

$$T(p,q,I_n) := \frac{1}{2}\sum_{i=0}^{n-1}\left[x_i^{p-1}(1-x_i)^{q-1} + x_{i+1}^{p-1}(1-x_{i+1})^{q-1}\right]h_i$$

corresponds to the trapezoid rule and the remainder satisfies

$$|R(p,q,I_n)| \leq \frac{1}{2}\nu(h)\max(p-1,q-1)\,\beta(p-1,q-1)$$

for all $p,q > 1$.

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