Turk J Math
24 (2000) , 165 – 172.

Some Commutativity Results for S-unital Rings

Moharram A. Khan

Abstract

In the present paper, it is shown that if R is a left (resp. right) s-unital ring satisfying $[f(y^m x^r y^s) \pm x^t y, x] = 0$ (resp. $[f(y^m x^r y^s) \pm yx^t, x] = 0$), where m, r, s, t are fixed non-negative integers and $f(\lambda)$ is a polynomial in $\lambda^2\mathbf{Z}[\lambda]$, then R is commutative. Commutativity of R has also been investigated under different sets of constraints on integral exponents.

Key Words and phrases: Automorphisms, commutativity theorems, nilpotent elements, polynomial constraints, s-unital rings.

1. Introduction

Throughout this paper, R will denote an associative ring (may be without unity 1), $N(R)$ the set of nilpotent elements of R, $U(R)$ the group of units of R and $\mathbf{Z}[X]$ the totality of polynomials in X with coefficients in $\mathbf{Z}$, the ring of integers. As usual, $[x, y]$ will denote the commutator $xy - yx$.

Following [3], a ring R is said to be a left (resp. right) s-unital ring if $x \in Rx$ (resp. $x \in xR$) for each $x \in R$. Further R is called s-unital if it is left as well as right s-unital.

Now, we consider the following ring properties:

(C) Let m, r, s and t be fixed non-negative integers. For each $x, y \in R$, there exists a polynomial $f(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$ such that

$$[f(y^m x^r y^s) \pm x^t y, x] = 0.$$

(C^*) For each $x, y \in R$, there exist a polynomial $f(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$ and non-negative integers m, r, s, t such that

$$[f(y^m x^r y^s) \pm x^t y, x] = 0.$$

Mathematics Subject Classification: Primary 16U80; Secondary 16U99.

(C_1) Let m, r, s and t be fixed non-negative integers. For each $x, y \in R$, there exists a polynomial $f(\lambda)$ in $\lambda^2\mathbf{Z}[\lambda]$ such that

$$[f(y^m x^r y^s) \pm yx^t, x] = 0.$$

(C_1^*) For each $x, y \in R$, there exist a polynomial $f(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$ and non-negative integers m, r, s, t such that

$$[f(y^m x^r y^s) \pm yx^t, x] = 0.$$

(C_2) For each $y \in R$, there exist polynomials $f(\lambda), p(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$ such that

$$p(y)[x, f(y)]q(x) = \pm x^t[x^m, y] \quad \text{and} \quad p(y)[x, f(y)]q(x) = \pm x^t[x^n, y]$$

for all $x \in R$, where $q(\lambda) \in \mathbf{Z}[\lambda]$ is a fixed polynomial with $q(1) = \pm 1$, and m, n, t are fixed positive integers such that $(m, n) = 1$.

(C_2^*) For every $x, y \in R$, there exist polynomials $f(\lambda), p(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$ and non-negative integers $m \geq 1, n \geq 1$ and t with $(m, n) = 1$ such that

$$p(y)[x, f(y)]q(x) = \pm x^t[x^m, y] \text{ and } p(y)[x, f(y)]q(x) = \pm x^t[x^n, y]$$

where $q(\lambda) \in \mathbf{Z}[\lambda]$ is a fixed polynomial.

(C_3) For each $y \in R$, there exist polynomials $f(\lambda), p(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$ such that

$$p(y)[x, f(y)]q(x) = \pm[x^m, y]x^t \quad \text{and} \quad p(y)[x, f(y)]q(x) = \pm[x^n, y]x^t$$

for all $x \in R$, where $q(\lambda) \in \mathbf{Z}[\lambda]$ is a fixed polynomial with $q(1) = \pm 1$, and m, n, t are fixed positive integers such that $(m, n) = 1$.

(C_3^*) For every $x, y \in R$, there exist polynomials $f(\lambda), p(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$ and non-negative integers $m \geq 1, n \geq 1$ and t with $(m, n) = 1$, such that

$$p(y)[x, f(y)]q(x) = \pm[x^m, y]x^t \quad \text{and} \quad p(y)[x, f(y)]q(x) = \pm[x^n, y]x^t$$

where $q(\lambda) \in \mathbf{Z}[\lambda]$ is a fixed polynomial.

(CH) For every $x, y \in R$, there exist $f(\lambda), h(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$ such that $[x - f(x), y - h(y)] = 0$.

A well-known theorem of Herstein [2] asserts that if for each $x, y \in R$, there exists a polynomial $f(t) \in t^2\mathbf{Z}[t]$ such that $[x - f(x), y] = 0$, then R is commutative. Further, the author jointly with Bell and Quadri [1], established the commutativity of R with identity 1 satisfying the polynomial identity $[xy - f(xy), x] = 0$, where $f(t) \in t^2\mathbf{Z}[t]$. More recently, several commutativity theorems have been found when the underlying polynomials $f(\lambda), p(\lambda), \in \lambda^2\mathbf{Z}[\lambda]$, and $q(\lambda) \in \mathbf{Z}[\lambda]$ in (C), (C_1), (C_2) and (C_3) are particularly assumed to be monomials [3, 5, 6, 7, 10]. In the present paper, our objective is to extend these results to the rings satisfying the above properties. Moreover, commutativity theorems for one-sided s-unital rings are obtained under different sets of conditions. Finally, commutativity of rings satisfying Chacron's criterion (CH) together with any one of the properties $(C^*), (C_1^*), (C_2^*)$ and (C_3^*) has been studied. In fact, our results generalise many well-known commutativity theorems namely; [1, Theorems 2 and 3], [5, Theorem 2], [6, Theorems 1-3], [7, Theorem], [8, Theorem] and [10, Theorem].

2. Preliminary Results

Consider the following types of rings.

$(i)_l$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

$(i)_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(i) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(ii) $M_\sigma(F) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \middle| \ a,\ b \in F \right\}$, where F is a finite field with a non-trivial automorphism σ.

(iii) A non-commutative ring with no non-zero divisors of zero.

(iv) $S =< 1 > +T, T$ is non-commutative subring of S such that $T[T, T] = [T, T]T = 0$.

In a recent paper [11], Streb classified non-commutative rings, which have been used effectively to establish several commutativity theorems [5, 6, 7, 8, 9]. One can easily observe, from the proof of [9, Corollary 1], that if R is a non-commutative s-unital ring, then there exists a factor subring S of R which is of type $(i)_l$, (ii), (iii) or (iv). This gives the following result which plays a vital role in our subsequent discussion [9, Meta theorem].

Lemma 2.1. Let P be a ring property which is inherited by factor subrings. If no ring of type $(i)_l$, (ii), (iii) or (iv) satisfies (P), then every left s-unital ring satisfying P is commutative.

Remark 2.1. We pause to remark that the dual of the above lemma holds; if P is a ring property which is inherited by factorsubrings, and if no ring of type $(i)_r$, (ii), (iii) or (iv) satisfies (P), then every right s-unital ring satisfying P is commutative.

3. Main Results

The main results of the present paper are as follows.

Theorem 3.1. Let R be a left (resp. right) s-unital ring satisfying (C) (resp. (C_1)). Then R is commutative.

Theorem 3.2. Let R be a left (resp. right) s-unital ring satisfying (C_2) (resp. (C_3)). Then R is commutative.

We need the following known results.

Lemma 3.1 [5]. Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, ..., x_n$ with relatively prime integer coefficients. Then the following statements are equivalent :

(a) For any ring R satisfying $f = 0$, the commutator ideal of R is nil ideal.

(b) For every prime p, the ring $(GF(p))_2$ fails to satisfy $f = 0$.

Lemma 3.2 [8]. Let R be a left (resp. right) s-unital ring which is not right (resp. left) s-unital. Then R has a factor subring of type $(i)_l$ (resp. $(i)_r$).

Lemma 3.3 [9]. Let R be a ring with unity 1 satisfying (CH). If R is non-commutative, then there exists a factorsubring of R which is of type (i) or (ii).

Proof of Theorem 3.1. Let S be any ring of type $(i)_l$, and let $f(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$. Then

$$[f(e_{12}^m e_{11}^r e_{12}^s) \pm e_{11}^t e_{12}, e_{11}] = \pm e_{12} \neq 0$$

hence S does not satisfy (C). It follows by Lemma 3.2 that if R is any left s-unital ring satisfying (C), then R is right s-unital as well. Thus, in view of Proposition 1 of [3], we may assume that R has unity 1.

Suppose that $R = M_\sigma(F)$, is the ring of type (ii). Taking $x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix} (\sigma(a) \neq a), y = e_{12}$ in (C) we get

$$[f(y^m x^r y^s) \pm x^t y, x] = \pm a^t(a - \sigma(a))e_{12} \neq 0,$$

for every $f(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$ and then R does not satisfy (C).

Let R be a ring of type (iii). Since $x = e_{22}$ and $y = e_{21}$ do not satisfy (C), by Lemma 3.1, we see that the commutator ideal of R is nil and hence no ring of type (iii) satisfies (C).

Let R be a ring of type (iv) and let $a, b \in T$ such that $[a, b] \neq 0$. Then by hypothesis, we have

$$(1 + a)^t[a, b] = \pm[1 + a, f(1 + a)^m b^r(1 + a)^s)] = 0.$$

This implies that $[a, b] = 0$, which gives a contradiction.

Hence we have seen that no ring of type $(i)_l$, (ii), (iii) or (iv) satisfies (C) and by Lemma 2.1, R is commutative.

Using the similar arguments as above we see that no ring of type $(i)_r$, (ii), (iii), or (iv) satisfies the property (C_1) (see also Remark 2.1).

Proof of Theorem 3.2. Let S be of type $(i)_l$ and let $f(\lambda) \in \lambda^2\mathbf{Z}[\lambda], g(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$ and $h(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$. Taking $x = e_{11} + e_{12}, y = e_{12}$ in (C_2), we get

$$x^t[x^m, y] = \pm g(y)[x, f(y)]h(x) = e_{12} \neq 0,$$

because $x^t[x^m, y] = e_{12} \neq 0$ and $\pm g(y)[x, f(y)]h(x) = 0$. Hence, R does not satisfy (C_2). It follows by Lemma 3.2 that if R is any left s-unital ring satisfy (C_2), then R is right s-unital and hence, s-unital. In view of Proposition 1 of [3], we may assume that the ring R has unity 1.

Consider the ring $R = M_\sigma(F)$, a ring of type (ii). Notice that $N(R) = Fe_{12}$. Hence for $b \in N(R)$ and arbitrary unit $u \in U(R)$, we obtain that there exists a polynomial $f(\lambda) \in \lambda^2\mathbf{Z}[\lambda]$ such that

$$u^t[u^m, b] = \pm g(b)[u, f(b)]h(u) = 0,$$

and

$$u^t[u^n, b] = \pm g(b)[u, f(b)]h(u) = 0.$$

Since $b^2 = 0$ and u is a unit of R, the last two equations yield $[u^m, b] = 0$ and $[u^n, b] = 0$. This implies that $[u, b] = 0$. Now, particularly for non-central element $b = e_{12}$, $[u, e_{12}] = 0$. This gives that e_{12} is central which is a contradiction.

Let R be a ring of type (iii). By hypothesis we have

$$p(y)[x, f(y)]q(x) = \pm x^t[x^m, y]. \tag{1}$$

Replacing x by $x + 1$ in (1), we get

$$p(y)[x, f(y)]q(x + 1) = \pm(x + 1)^t[(x + 1)^m, y]. \tag{2}$$

Multiply (1) (resp. (2)) by $q(x+1)$(resp. $q(x)$) on the right and compare the equations so obtained to get

$$(x+1)^t[(x+1)^m, y]q(x) = x^t[x^m, y]q(x+1).$$

This is a polynomial identity, and $x = e_{12} - e_{22}$ and $y = e_{12}$ in $(GF(p))_2$ fail to satisfy this equality. Hence, by Lemma 3.1, the commutator ideal of R is nil, yields a contradiction.

Finally, let R be a ring of type (iv) and let $[a, b] \neq 0$, where $a, b \in T$. There exists $f(\lambda)$ in $\lambda^2\mathbf{Z}[\lambda]$ such that

$$m[a, b] = (1+a)^t)[(1+a)^m, b] = \pm p(b)[a, f(b)]q(1+a) = 0,$$

and

$$n[a, b] = (1+a)^t[(1+a)^n, b] = \pm p(b)[a, f(b)]q(1+a) = 0.$$

Since $(m, n) = 1$, we get $[a, b] = 0$, and this gives a contradiction.

Hence, no ring of type $(i)_l$, (ii), (iii) or (iv) satisfies (C_2) and by Lemma 2.1, R is commutative.

We remark that the same conclusion holds; if R satisfies (C_3), then trivially, we see that no ring of type $(i)_r$, (ii), (iii) or (iv) satisfies $(C)_3$.

From the previous proofs of Theorems 3.1 and 3.2, we see that no ring of type $(i)_l$ satisfies (C^*) or (C_2^*), and no ring of type $(i)_r$ satisfies (C_1^*) or (C_3^*).

Combining this fact with Lemma 3.2, we obtain the following:

Theorem 3.3 Let R satisfy (CH). Then the following are equivalent:

(I) R is commutative.

(II) R is left (resp. right) s-unital ring satisfying (C^*) (resp. (C_1^*)).

(III) R is left (resp. right) s-unital ring satisfying (C_2^*) (resp. (C_3^*)).

Remark 3.1 The following example shows that in the hypotheses of Theorem 3.2, the existence of both conditions in (C_2) are not superfluous (even if R has unity 1).

Example 3.1. Let

$$R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \middle| \alpha, \beta, \gamma, \delta \in GF(2) \right\}.$$

Then R is a non-commutative ring with unity satisfying the condition $x^t[x^4, y] = y^s[x, y^4]$, where s and t are fixed non-negative integers.

Remark 3.2. The following example demonstrates that there are non-commutative left (resp. right) s-unital rings satisfying (C_1) (resp.(C)).

Example 3.2. Let

$$R_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

$$(\text{resp.} R_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\})$$

be subring of 2×2 matrices over $GF(2)$. Then for any fixed positive integers m, n, r, s, t larger than 1, R_1 (resp. R_2) satisfies $[(y^m x^r y^s)^n \pm yx^t, x] = 0$ (resp. $[(y^m x^r y^s)^n \pm x^t y, x] = 0$). However, R_1 (resp. R_2) is a non-commutative left (resp. right) s-unital ring.

References

[1] H. E. Bell, M. A. Quadri and M. A. Khan, Two commutativity theorems for rings, Rad. Mat. 3 (1987), 255-260.

[2] I. N. Herstein, Two remarks on the commutativity of rings, Canad. J. Math. 7 (1955), 411 - 412.

[3] Y. Hirano, Y. Kobayashi and H. Tominaga, Some polynomial identities and commutativity of s-unital rings, Math. J. Okayama Univ. 24 (1982), 7-13.

[4] T. P. Kezlan, A note on commutativity of semiprime PI-rings, Math. Japon. 27 (1982), 267 - 268.

[5] M. A. Khan, Commutativity of right s-unital rings with polynomial constraints, Jour. Inst. Math & Comp. Sci. 12 (1999), 47 - 51.

[6] M. A. Khan, Commutativity theorems through a Streb's classification, Proc. Irish Math. Acad. Sci. No. 2 (2000) (to appear).

[7] H. Komatsu, A commutativity theorem for rings, Math. J. Okayama Univ. 26 (1984), 109 - 111.

[8] H. Komatsu, T. Nishinaka and H. Tominaga, On commutativity of rings, Rad. Mat. 6 (1990), 303 - 311.

[9] H. Komatsu and H. Tominaga, Chacron's condition and commutativity theorems, Math. J. Okayama Univ. 31 (1989), 101 - 120.

[10] E. Psomopoulos, A commutativity theorem for rings, Math. Japon. 29 (1984), 371 - 373.

[11] W. Streb, Zur struktur nichtkommtativer ringe, Math. J. Okayama Univ. 31 (1989), 135 - 140.

Moharram A. KHAN
Department of Mathematics
Faculty of Science
King Abdulaziz University
P.O.Box 30356
Jeddah - 21477-SUADI ARABIA
e-mail nassb@hotmail.com

Received 25.04.2000