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Representing Systems of Exponentials and Projection on Initial Data in the Cauchy Problem*

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Abstract

The Cauchy problem for the equation

$$Mw \equiv \sum_{j=0}^{m}\sum_{s=0}^{l_j} a_{s,j}\frac{\partial^{s+j}w(z_1,z_2)}{\partial z_1^s \partial z_2^j} = 0 \tag{1}$$

$$\frac{\partial^n w(z_1,z_2)}{\partial z_2^n} \mid_{z_2=0} = \varphi_n(z_1), n = 0,1,\ldots,m-1 \tag{2}$$

is investigated under the condition $l_j \le l_m, j = 0,1,\ldots,m-1$. It is shown that the operator of projection of solution of (1) on its initial data (2) in a definite situation has a linear continuous right inverse which can be determined effectively with the help of representing systems of exponentials in the space of initial data.

Introduction

The Cauchy problem (C.p.) for the equation

$$Mw \equiv \sum_{j=0}^{m}\sum_{s=0}^{l_j} a_{s,j}\frac{\partial^{s+j}w(z_1,z_2)}{\partial z_1^s \partial z_2^j} = 0 \tag{3}$$

with $a_{s,j} \in \mathcal{C}$ was investigated in a number of works and in particular in the paper [1]. We need in what follows the contents of $\oint\oint$ 1-4 and $\oint$1 of this paper. For reader's

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convenience let us remind some definitions and results from [1]. Let $F_1 (l = 1, 2)$ be a dense in itself subset of C and let $C^\infty(F_1)$ be a space of functions $y(z_1) : F_1 \to \mathcal{C}$ infinitely differentiable at each point of F_1. The sequence $\{y_n(z_1)\}_{n=1}^\infty$ tends to $y(z_1)$ in $C^\infty(F_1)$ if $\forall s \geq 0 y_n^{(s)}(z_1) \to y^{(s)}(z_1)$ uniformly on each compact of F_1. Let $E_1(F_1)$ be a complete separable locally convex space (CSLCS) satisfying the following conditions:

1) $E_1(F_1) \hookrightarrow C^\infty(F_1)$;

2) the operator $Dy \equiv y'$ is continuous in $E_1(F_1)$;

3) there exists an absolutely representing system (ARS) of exponentials $E_{\Lambda,1} = \{exp\lambda_k z_1\}_{k=1}^\infty$ such that for each $k \geq 1 E_{\Lambda,k} = \{exp\lambda_j z_1\}_{j=k}^\infty$ is also an ARS in $E_1(F_1)$ and $lim_{k\to\infty} \mid \lambda_k \mid = \infty$.

It is worth reminding that he system $\{x_k\}_{k=1}^\infty$ of elements x_k of a complete locally convex space H is said to be an ARS in H (see e.g. [1], p. 556) if each element x of H can be represented in the form of the series $x = \sum_{k=1}^\infty a_k x_k$, absolutely converging in H.

Suppose that a standard decomposition of the polynomial $Q(\lambda, \mu) := \sum_{j=0}^m \mu^j$ $\sum_{s=0}^{l_j} a_{s,j}\lambda^s$ contains no irreducible polynomials depending only on one variable λ or μ. Then in some neighborhood of infinity the equation $Q(\lambda, \mu) = 0$ generates N different branches $\mu_j(\lambda)$ with multiplicity $p_j : \sum_{j=1}^N p_j = m$.

The symbol $(E_1(F_1); E_2(F_2))$ denotes the set of functions $u(z_1, z_2)$ such that $\forall z_l \in F_l\, u(z_1, z_2) \in E_{3-l}(F_{3-l})$ as a function of $z_{3-1}, l = 1, 2$.

Assuming that $0 \in F_2$ throughout the paper we look for the function $w(z_1, z_2)$ from $(E_1(F_1); E_2(F_2))$ satisfying the equation (1) and initial conditions with respect to z_2:

$$\frac{\partial^n w(z_1, z_2)}{\partial z_2^n} \mid_{z_2=0} = \varphi_n(z_1), n = 0, 1, \dots, m-1 \tag{4}$$

where $\varphi_n \in E_1(F_1)$, $0 \leq n \leq m-1$. To do this we expand at first the functions φ_n into series with respect to ARS $E_{\Lambda,1}$

$$\varphi_n(z_1) = \sum_{k=1}^\infty b_{k,n} exp\lambda_k z_1, n = 0, 1, \dots, m-1 \tag{5}$$

All the series (3) converge absolutely in $E_1(F_1)$. If $k \geq 1$ is fixed we always can find numbers $a_{j,s}^{(k)}$ from the system

$$b_{k,n} = \sum_{j=1}^{N} \sum_{s=0}^{q_j(n)} a_{j,s}^{(k)} s! C_n^s \mu_{j,k}^{n-s}, n = 0, 1, \ldots, m-1 \tag{6}$$

where $\mu_{j,k} = \mu_j(\lambda_k), q_j(n) = min(n, p_j - 1)$. After that we form the series

$$w_0(z_1, z_2) = \sum_{j=1}^{N} \sum_{s=0}^{p_j-1} \sum_{k=1}^{\infty} a_{j,s}^{(k)} (z_2)^s exp(\lambda_k z_1 + \mu_{j,k} z_2) \tag{7}$$

It was proved in [1] under definite suppositions that w_0 belongs to $(E_1(F_1); E_2(F_2))$ and satisfies (1), (2). We cite one result from [1] in this direction. Suppose that

$$l_j \leq l_m \text{ if } j \leq m-1 \text{ and if } a_{l_j,j} \neq 0 \tag{8}$$

Then [1] $\exists R_0 > 0 : v := sup\{| \mu_j(\lambda) | : | \lambda | \geq R_0, 1 \leq j \leq N\} < \infty$.

We put $F_2 = \mathcal{C}, \beta := max\{(p_j - 1) : 1 \leq j \leq N\}$ and introduce the Banach space $E_2(\mathcal{C}) = E(v, \beta)$ of entire functions $y(z_2)$ such that $\| y \|_2 := sup_{z_2 \in C} \frac{|y(z_2)| exp(-v|z_2|)}{|z_2|^\beta + 1} < \infty$.

Let $P = \{p\}$ be the set of seminorms defining the topology in CSLCS $E_1(F_1)$ with properties 1)-3). Denote by $\{E_1(F_1); E(v, \beta)\}$ the subspace of $(E_1(F_1); E(v, \beta))$ containing all functions $y(z_1, z_2)$ for which $\forall p \in P$

$$(p) y(z_1, z_2) := sup_{z_2 \in C} \frac{p(y(z_1, z_2)) exp(-v \mid z_2 \mid)}{\mid z_2 \mid^\beta + 1} < \infty.$$

$\{E_1(F_1); E(v, \beta)\}$ is a CSLCS with topology defined by the set $(P) := \{(p)\}_{p \in P}$ of seminorms (p). According to theorem 1 ([1], p. 559-560) the series (5) described above converges absolutely in $\{E_1(F_1); E(v, \beta)\}$ and its sum $w_0(z_1, z_2)$ is a solution of the Cauchy problem (1)(2) for arbitrary chosen φ_n from $E_1(F_1)$. Let us introduce the CSLCS $(E_1(F_1))^m$ with the standard set of seminorms $P_{(m)} = \{p_m(\varphi) = \sum_{k=0}^{m-1} p(\varphi_k), p \in P, \varphi = (\varphi_0, \ldots, \varphi_{m-1})\}$ and the operator T_M of projection to initial data: for each $w(z_1, z_2) \in A := M^{-1}(0) \cap \{E_1(F_1); E(v, \beta)\}$

$$T_M w = \left(w(z_1, 0), \ldots, \frac{\partial^{m-1} w(z_1, z_2)}{\partial z_2^{m-1}} |_{z_2=0} \right).$$

It is easy to see that T_M is a continuous operator from A (with the topology induced from $\{E_1(F_1); E(v,\beta)\}$ into $(E_1(F_1))^m$). Under the relation (6) and conditions 1)-3) T_M is an epimorphism of A onto $(E_1(F_1))^m$. We shall indicate further the conditions under which the operator T_M has a linear continuous right inverse (LCRI). In order to formulate the main result we need to introduce the CSLCS

$$A_2^k = A_2(E_{\Lambda,k}, E_1(F_1)) = \{c = (c_s)_{s=k}^{\infty} : q_p^k(c) := \sum_{s=k}^{\infty} \mid c_s \mid p(exp\lambda_s z_1) < \infty, \forall p \in P\}$$

with the set of seminorms $Q_P^k = \{q_p^k\}_{p\in P}$ and the representation operator (RO) L_k:

$$\forall c = (c_s)_{s=k}^{\infty} \in A_2^k \to L_k c = \sum_{s=k}^{\infty} c_s exp\lambda_s z_1 \in E_1(F_1).$$

It is evident that L_k acts continuously from A_2^k into $E_1(F_1)$.

Theorem 1. *Let the relations (6) be valid and let the CSLCS $E_1(F_1)$ satisfy the conditions 1)-3). Suppose that $\forall k \geq 1$ the operator L_k has a LCRI B_k. Then the projection operator T_M has a LCRI which can be determined effectively.*

Proof. Let us fix $R_0 < \infty$ so that the above described branches $\mu_j(\lambda)$ are determined in the set $\mid \lambda \mid \geq R_0$. Let us also fix $k \geq 1$ so that $\mid \lambda_j \mid \geq R_0, \forall j \geq k$. If $\varphi = (\varphi_n)_{n=0}^{m-1} \in (E_1(F_1))^m$ then $\forall z_1 \in F_1 \varphi_n(z_1) = \sum_{s=k}^{\infty} (B_k\varphi_n)_s exp\lambda_s z_1, n = 0, 1, \ldots, m-1$. Moreover $\forall p_1 \in P \; \exists d < \infty \; \exists p_0 \in P$

$$\forall y \in E_1(F_1) \sum_{s=k}^{\infty} \mid (B_k y)_s \mid p_1(exp\lambda_s z_1) \leq dp_0(y). \tag{9}$$

Following [1], $\oint 3$ and $\oint 11$ we form the series (5)

$$w_k(z_1, z_2) = \sum_{j=1}^{N} \sum_{s=0}^{p_j-1} \sum_{r=k}^{\infty} a_{j,s}^{(r)}(z_2)^s exp(\lambda_r z_1 + \mu_{j,r} z_2) = \sum_{r=k}^{\infty} u_r(z_1, z_2)$$

where $\mu_{j,r} = \mu_j(\lambda_r)$ and the coefficients $a_{j,s}^{(r)} (r \geq k)$ are determined from the system $(B_k\varphi_n)_r = \sum_{j=l}^{N} \sum_{s=0}^{q_j(n)} a_{j,s}^{(r)} s! C_n^s \mu_{j,r}^{n-s}, n = 0, 1, \ldots, m-1$. According to inequality (9) from [1]

$$\exists D<\infty\exists H<\infty:\forall r\geq k\forall j\leq N\forall s\leq q_j(n)\mid a_{j,s}^{(r)}|\leq D(\lambda_r\mid^H\sum_{n=0}^{m-1}\mid(B_k\varphi_n)_r\mid.$$

The constants D,H do not depend on j,s and r (when k is fixed). We have for all $p\in P$ and $r\geq k$

$$(p)u_r\leq sup_{z_2\in C}\sum_{j=1}^{N}\sum_{s=0}^{p_j-1}\mid a_{j,s}^{(r)}\mid\frac{exp(-v\mid z_2\mid)}{\mid z_2\mid^\beta+1}\mid z_2\mid^s p(exp\lambda_r z_1)exp\mid\mu_{j,r}\mid\mid z_2\mid$$

$$\leq D\mid\lambda_r\mid^H\sum_{n=0}^{m-1}\mid(B_k\varphi_n)\mid p(exp\lambda_r z_1)\leq D_2\sum_{n=0}^{m-1}\mid(B_k\varphi_n)_r\mid p_1(exp\lambda_r z_1).$$

Taking into account (7) we find that $\forall p\in P\exists p_0\in P$:

$$\sum_{r=k}^{\infty}(p)u_k\leq D_2d\sum_{n=0}^{m-1}p_0(\varphi_n)=D_3(p_0)_m(\varphi).$$

Hence $w_k(z_1,z_2)=Q_k\varphi$ where Q_k is a linear continuous operator from $(E_1(F_1))^m$ into $\{E_1(F_1);E(v,\beta)\}$. Besides, $w_k\in M^{-1}(0)\cap\{E_1(F_1),E(v,\beta)\}$ and $T_m w_k=\varphi$, i.e. $T_M Q_k\varphi,\forall\varphi\in(E_1(F_1))^m$. □

In conclusion we mention some examples of the spaces $E_1(F_1)$ of initial data and some classes of equations (1) satisfying the suppositions of theorem I.

I. 1. Let G be a bounded convex domain in C and let $H(G)$ be the Frechet space of all functions analytic in G with the standard open-compact topology. It is proved in [2] that if the function $\Psi(z)$ maps conformly the disc $\mid z\mid<1$ onto G and satisfies the condition $sup\{\mid\Psi'(z)\mid:\mid z\mid<1\}<\infty$, then there exists an ARS $(exp\lambda_k z)_{k=1}^\infty$ in $H(G)$ such that $lim_{k\to\infty}sup\frac{k}{|\lambda_k|}<\infty,\forall k\geq 1(exp\lambda_r z)_{r=k}^\infty$ is an ARS in $H(G)$ and the corresponding RO L_k has a LCRO B_k. So we can put in theorem I $F_1=G,E_1(F_1)=H(G)$.

2. For any $R \in (0, +\infty)$ denote by $C^{\infty}[-R, R]$ the Frechet space of all complex-valued functions infinitely differentiable on $[-R, R]$, with the set of norms $\| y \|_n = max\{| y(j)(x) |: x \in [-R, R], 0 \leq j \leq n\}, n = 0, 1, \dots$ According to [3], $\oint 5$, for each $\theta < 1$ and $k \geq 0 E^k_{\theta,R} := \{exp\frac{ij\theta\pi x}{R}\}_{|j| \geq k}$ is an ARS in $C^{\infty}[-R, R]$ and the RO L_k has a LCRI. So theorem I is valid if $F_1 = [-R, R], 0 < R < \infty, E_1(F_1) = C^{\infty}[-R, R]$.

3. Assume that $M_0 = 1, M_l \uparrow +\infty, l \geq 1, R \in (0, +\infty)$. Denote by $E_{(M_l}[-R, R]$ the Beurling space of all functions $y(x)$ from $C^{\infty}[-R, R]$ such that

$$\forall h > 0 \mid y \mid_{R,h} := sup \left\{ \frac{| y^{(l)}(x) |}{h^l M_l} : l \geq 0, x \in [-R, R] \right\} < \infty.$$

The topology in $E_{(M_l)}[-R, R]$ is defined by the set of norms $| y |_{R,1/n}, n = 1, 2, \dots$ Suppose that (M_l) satisfies the following conditions:

$$\forall \varepsilon < 0 \exists \delta > 0 \exists d < \infty : \forall l \geq 0 \sum_{j=0}^{l} M_j \delta^j C^j_l \leq d\varepsilon^l M_l, \tag{10}$$

$$sup\frac{m_p}{p} \sum_{j \geq p} \frac{1}{m_j} < \infty, \tag{11}$$

$$sup(m_p)^{1/p} < \infty, \tag{12}$$

where $m_0 = 1, M_p = m_p M_{p-1}, p \geq 1$. It is proved in [3] ($\oint 5$, theorem 5.3) with the help of the results of [4] that for each $\theta < 1$ and $k \geq 0 E^k_{\theta,R}$ is an ARS in $E_{(M_l)}[-R, R]$ and the RO L_k has a LCRI B_k. So under conditions (8)-(10) theorem I works in the case $F_1 = [-R, R], E_1(F_1) = E_{(M_l)}[-R, R]$. In particular, we can put $M_l = (l!)^{\alpha}, \alpha > 1, l \geq 1$. In this case $E_1(F_1)$ coincides with the well-known Gevrey class of minimal type:

$$E_1(F_1) = \{y(x) \in C^{\infty}[-R, R] : \forall h > 0 sup[| y^{(l)}(x) | (l!)^{-\alpha} h^{-l} : l \geq 1, x \in [-R, R]] < \infty\}.$$

4. As the last example we regard the Roumieu space

$$E_{\{M_l\}}[-R, R] = \{y \in C^{\infty}[-R, R] : \exists h > 0 :| y |_{R,h} < \infty\}.$$

If $M_0 = 1, M_l \uparrow +\infty$, if the conditions (8), (10) are fulfilled and if

$$\exists l > 1 \lim_{p \to \infty} \frac{m_p}{p} \sum_{j=lp}^{\infty} \frac{1}{m_j} = 0 \tag{13}$$

then according to $\oint 5$ of [3] for each $\theta < 1$ and $k \geq 0 E^k_{\theta,R}$ is an ARS in $E_{\{M_l\}}[-R, R]$. Moreover under the conditions (8), (10), (11) the operator L_k has a LCRI, and theorem I is again applicable. In particular, we can take for $E_1(F_1)$ the Gevrey space of maximal type:

$$E_1(F_1) = \{y(x) \in C^\infty[-R, R] : \exists h > 0 sup[| y^{(l)}(x) | (l!)^{-\alpha} h^{-1} : l \geq 1, x \in [-R, R]] < \infty\}.$$

II. The characteristical polynomial of the equation (1) can be written in the following form

$$Q(\lambda, \mu) = \sum_{j=0}^{m} \sum_{s=0}^{l_j} a_{s,j} \mu^j \lambda^s = \sum_{k=0}^{m} \mu^k R_k(\lambda) = T_\lambda(\mu).$$

It is well known that discriminant of $T_\lambda(\mu)$ (as a polynomial with respect to μ) is a polynomial $v(\lambda)$ in λ. Suppose that $v(\lambda)$ is not identically zero. Then $\exists R_1 \in (0, +\infty)$: $v(\lambda) \neq 0$, if $| \lambda | \geq R_1$. If the space $E_1(F_1)$ satisfies the conditions 1)- 3), then (see $\oint$ 12 of [1]) the representation (5) of the solution w_0 can be simplified:

$$w_0(z_1, z_2) = \sum_{j=1}^{m} \sum_{k=1}^{\infty} a_j^{(k)} exp(\lambda_k z_1 + \mu_j(\lambda_k) z_2)$$

where each branch $\mu_j(\lambda)$ is a simple one (i.e. $p_j = 1, j = 1, 2, \ldots, m$). If the condition (6) holds, then Theorem 1 is applicable, and the magnitude β is equal to zero : $E(v, \beta) = E(v, 0)$. In particular, the polynomial $v(\lambda)$ is not identically zero, if $Q(\lambda, \mu)$ is an irreducible polynomial.

Let us consider as an example the Cauchy problem for the Sobolev-Galpern equation

$$\sum_{k=0}^{l_l} a_k \frac{\partial^{k+1} w(z_1, z_2)}{\partial z_1^k \partial z_2} = \sum_{s=0}^{l_0} b_s \frac{\partial^s w(z_1, z_2)}{\partial z_1^s}; w(z_1, 0) = f(z_1). \tag{14}$$

We have for this equation $m = 1$,

$$Q(\lambda, \mu) = \mu \sum_{k=0}^{l_1} a_k \lambda^k + \sum_{s=0}^{l_0} b_s \lambda^s = \mu P_1(\lambda) + P_2(\lambda).$$

If $l_0 \leq l_1$, the conditions 1)-3) for $E_1(F_1)$ are satisfied and if $\forall k \geq 1$ the operator L_k has a LCRJ B_k, then the projection operator T_μ has a LCRJ Q_k. The operator Q_k can be expressed in the following form:

$$w_k(z_1, z_2) = \sum_{r=k}^{\infty} a_1^{(r)} exp(\lambda_r z_1 + \mu_1(\lambda_r) z_2).$$

Here $\mu_1(\lambda_r) = -\frac{P_2(\lambda_r)}{P_1(\lambda_r)}, a_1^{(r)} = (B_k f)_r; \beta = 0; v = \varepsilon + \frac{|b_{l_0}|}{|a_{l_0}|}$ for the case $l_0 = l_1$, and $v = \varepsilon$ if $l_0 < l_1$. The positive number ε can be fixed arbitrarily small.

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