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Efficient Presentations for Some Direct Products of Groups

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Abstract

In this paper we give efficient presentations for $A_4 \times D_n$, where n is odd number, or n is even number and (n,3)=1. We also give efficient presentations for $A_5 \times D_n$ where n is an even or odd number.

1. Introduction

Let G be finite group with a presentation on n generators and r relations. The deficiency of the presentation is $n - r$. A group H of maximal order with the properties that there is a subgroup A with $A \leq Z(H) \cap H'$ and $H/A \cong G$ is called a covering group of G. In general, H is not unique but A is unique and is called the Schur multiplier $M(G)$ of G. For details see [1, 7, 15].

Schur [9] showed that any presentation for G with n generators requires at least n+rank($M(G)$) relations. If G has a presentation with n generators and precisely n+rank($M(G)$) relations we say that G is efficient. Not all groups are efficient and examples of soluble groups with trivial multipliers which are not efficient were given by Swan [10] and inefficient groups have been found by Wotherspoon [16]. Further details of such groups are given in [1], [14], [15].

For the finite field $GF(p)$, for a prime p, let $SL(2,p)$ denote the group of 2×2 matrices of determinant 1 over the field $GF(p)$. Define $PSL(2,p) = SL(2,p)/\{\pm I\}$, where I is the 2×2 identity matrix.

For any group G we shall use G' and $Z(G)$ to denote the derived group of G and the center of G, respectively. We also use the notation A_4 and A_5 to denote, respectively, the alternating groups of degree four and five. Let D_n denote the dihedral group of order 2n.

Questions concerning the efficiency of direct products have been of considerable interest for a number of years. The first questions concerning the efficiency of direct products were posed by Wiegold in [15]. In particular his questions were whether $PSL(2,5) \times PSL(2,5)$ and $SL(2,5) \times SL(2,5)$ are efficient.

The Schur-Künneth formula [6] gives the Schur multiplier of a direct product:

$$M(G \times H) = M(G) \times M(H) \times (G \otimes H).$$

The first of these questions was answered by Kenne in [8]. He showed that $PSL(2,5) \times PSL(2,5)$ is efficient. The second question was answered by Campbell et al. [2]. In [4] C. M. Campbell, E. F. Robertson and P. D. Williams have obtained efficient presentations for certain direct products involving field of the same characteristic. Some work on direct product of groups $PSL(2, p^{n_i})$ for a fixed prime p and different n_i' s is done and also some efficient presentations for $PSL(2, q_1) \times PSL(2, q_2)$, q_1, q_2 prime power, are given by Vatansever in [11]. In [12] the efficiency of the group $PSL(2, Z_n) \times PSL(2, Z_m)$, for certain n, m is given. In [5] D. M. Gill has obtained efficient presentations of direct products of familiar groups. In [13] the efficiency of the group $PSL(2,7) \times PSL(2,3^2)$ is given.

In this paper we consider the problem of giving efficient presentations for the direct products $A_4 \times D_n$ and $A_5 \times D_n$.

From the Schur-Künneth formula we have:

i) If n is odd then the rank of $M(A_k \times D_n)$ is 1 where $k = 4, 5$.

ii) If n is even then the rank of $M(A_k \times D_n)$ is 2 where $k = 4, 5$.

2. The direct products $A_4 \times D_n$ and $A_5 \times D_n$.

Theorem 1. *When n is odd, a presentation of* $(2, 3, 2r+1) \times D_n$ *is*

$$\langle x, y \mid (xy)^{2r+1}.\ x^{n+1}y^3,\ x^2yx^2y^5,\ y^3 = xy^3x\rangle.$$

Proof. Take $(2, 3, 2r+1)$ as $\langle a, b \mid a^2,\ b^3,\ (ab)^{2r+1}\rangle$ and D_n as $\langle c, d \mid c^2,\ d^n,\ (cd)^2\rangle$. Define $x = ad$, and $y = bc$, so $x^n = a$, $x^{n+1} = d$, $y^3 = c$, and $y^{-2} = b$. Therefore the direct product of these two groups has presentation

$$\langle x, y \mid x^{2n},\ y^6,\ (x^ny^{-2})^{2r+1},\ (y^3x^{n+1})^2,\ x^ny^3 = y^3x^n,\ x^2y^2 = y^2x^2\rangle.$$

Firstly $(y^3x^{n+1})^2$	$= y^5x^{n+1}yx^{n+1}$	(using $x^2y^2 = y^2x^2$), for which
we have	$y = x^{n+1}yx^{n+1}$	(using $y^6 = 1$)
and	$y = x^{2n+2}yx^{2n+2} = x^2yx^2$	(using $x^{2n} = 1$).

So add $y = x^2yx^2$ and note that $x^2y^2 = y^2x^2$ is then redundant.

Also,

$$(y^3x^{n+1})^2 = y^3xy^3x^{2n+1} \quad \text{(using } x^ny^3 = y^3x^n\text{) we have}$$
$$y^3 = xy^3x. \quad \text{(using } x^{2n} = 1, y^6 = 1\text{)}$$

So replace $(y^3x^{n+1})^2 = 1$ by $y^3 = xy^3x$. Then replace $(x^ny^{-2})^{2r+1} = 1$ by $(x^ny)^{2r+1}y^3 = 1$. (Therefore $x^ny^3 = y^3x^n$ is redundant).

So we now have:

$$\langle x,\, y \mid x^{2n},\, y^6,\, (x^ny)^{2r+1}y^3,\, y = x^2yx^2,\, y^3 = xy^3x\rangle.$$

But $1 = (x^ny)^{2r+1}y^3 = (x^nyx^ny)^rx^ny^4$

$$= (xyxy)^rx^ny^4 \quad \text{(using } y = x^2yx^2\text{)}$$
$$= (xy)^{2r}xx^{n-1}yy^3$$
$$= (xy)^{2r}xyx^{1-n}y^3 \quad \text{(using } y = x^2yx^2\text{)}$$
$$= (xy)^{2r+1}x^{1+n}y^3.$$

Replace $(x^ny)^{2r+1}y^3 = 1$ by this.

But now $x^{2n} = 1$ is redundant as from this new relation we see that

$$x^{-2n-2} = y^3(xy)^{2r+1}y^3(xy)^{2r+1}$$
$$= y^6(x^{-1}y)^{2r+1}(xy)^{2r+1} \quad \text{(using } xy^3 = y^3x^{-1}\text{)}$$
$$= x^{-1}y(x^{-1}yx^{-1}y)^r(xy)^{2r+1}$$
$$= x^{-1}y(xyxy)^r(xy)^{2r+1} \quad \text{(using } x^{-1}y = xyx^2\text{)}$$
$$= x^{-2}(xy)^{4r+2}$$
$$= x^{-2}y^{-3}x^{-1-n}y^{-3}x^{-1-n} \quad \text{(using } (xy)^{2r+1}x^{1+n}y^3 = 1\text{)}$$
$$= x^{-2}y^{-6} = x^{-2} \quad \text{(using } y^3 = xy^3x\text{).}$$

Consider

$$H = \langle x,\, y \mid (xy)^{2r+1}x^{1+n}y^3,\, x^2yx^2y^5,\, y^3 = xy^3x\rangle.$$

Now $(1 = x^2yx^2y^3y^2 = x^2y^4x^{-2}y^2)$ we have $x^2 = y^2x^2y^4$.

Similarly $(1 = y^3x^2yx^2y^2 = x^{-2}y^4x^2y^2)$ we have $x^2 = y^4x^2y^2$. Therefore $y^2x^2 = y^4x^2y^4 = x^2y^2$ and so $y^6 = 1$. Therefore H is a presentation of the direct product.

Corollary 2. *When n is odd, an efficient presentation of $A_4 \times D_n$ is*

$$\langle x,\, y \mid (xy)^3 x^{n+1} y^3,\, x^2 y x^2 y^5,\, y^3 = xy^3 x\rangle.$$

Corollary 3. *When n is odd, an efficient presentation of $A_5 \times D_n$ is*

$$\langle x,\, y \mid (xy)^5 x^{n+1} y^3,\, x^2 y x^2 y^5,\, y^3 = xy^3 x\rangle.$$

Theorem 4. *When n is even, a presentation of $(2,3,4r+1) \times D_n$ is*

$$\langle x,\, y \mid x^3 y^2 x^3 = y^2,\, (xy^{4r})^2,\, y^{4r+1} x^2 y^{4r+1} = x^2,\, (xy^{4r+1})^n x^{2n}\rangle.$$

Proof. Take $(2,3,4r+1)$ as $\langle a,\, b \mid a^2,\, b^3,\, (ab)^{4r+1}\rangle$ and D_n as $\langle c,\, d \mid c^2,\, d^n,\, (cd)^2\rangle$. Let $x = bcd$ and $y = abcd^2$, so $x^3 = cd$, $x^4 = b$, $y^{4r+1} = (cd^2)^{4r+1} = (cd.dcd.d)^{2r}.cd^2 = (cd.c.d)^{2r} cd^2 = cd^2$ and $y^{4r+2} = ab$.

Therefore $d = x^{-3} y^{4r+1}$, $c = x^3 y^{-4r-1} x^3$ and $a = y^{4r+2} x^{-4}$. Hence a presentation of the direct product is:

$$\langle x,\, y \mid (y^{4r+2} x^{-4})^2,\, x^6,\, y^{2(4r+1)},\, (x^{-3} y^{4r+1})^n,\, x^2 y^{4r+1} = y^{4r+1} x^2,\, x^3 y^2 = y^2 x^3\rangle.$$

The first relation is $1 = (y^{4r+2} x^{-4})^{-2}$

$= x^{-6} (y^{4r+2} x^{-1})^{-2}$ (using $x^3 y^2 = y^2 x^3$)

$= (xy^{4r})^2$ (by $x^6 = 1$, $y^{2(4r+1)} = 1$)

and the fourth is $1 = (x^{-3} y^{4r+1})^{-n}$

$= (x^{-1} y^{4r+1})^{-n} x^{2n}$ (by $x^2 y^{4r+1} = y^{4r+1} x^2$)

$= (y^{4r+1} x)^n x^{2n}$. (using $y^{2(4r+1)} = 1$)

So we have:

$$\langle x,\, y \mid (xy^{4r})^2,\, x^6,\, y^{2(4r+1)},\, (xy^{4r+1})^n x^{2n},\, x^2 y^{4r+1} = y^{4r+1} x^2,\, x^3 y^2 = y^2 x^3\rangle.$$

Now consider:

$$\langle x,\, y \mid x^3 y^2 x^3 = y^2,\, (xy^{4r})^2,\, y^{4r+1} x^2 y^{4r+1} = x^2,\, (xy^{4r+1})^n x^{2n}\rangle.$$

First note that $x^3 y^2 x^3 = y^2$ implies $x^3 y^4 = y^4 x^3$.

Now for $(xy^{4r})^2 = 1$ we have $xy^{4r} x = y^{-4r}$ so

$x^3 y^{4r} x^3 = x^2 y^{-4r} x^2$

we have $x^6 y^{4r} = x^2 y^{-4r} x^2$. (using $x^3 y^4 = y^4 x^3$)

This is $x^6 = x^2y^{-4r}x^2y^{-4r}$
$= x^2y.y^{-4r-1}x^2y^{-4r-1}.y$
$= x^2yx^2y$. (using $y^{4r+1}x^2y^{4r+1} = x^2$).
Also, $x^4y^{4r}x^4 = x^3y^{-4r}x^3 = x^6y^{-4r}$, but
$x^4y^{4r}x^4 = x^4y^{-1}y^{4r+1}x^4$
$= x^4y^{-1}x^4y^{4r+1}$. (using $y^{4r+1}x^2y^{4r+1} = x^2$).
Equating these gives

$$yx^2 = x^4y^{8r+1}. \tag{0.1}$$

We know $x^6 = x^2yx^2y$ so $x^6 = x^2.x^4.y^{8r+1}.y$, i.e. $y^{8r+2} = 1$ and hence $y^{4r+1}x^2y^{4r+1} = x^2$ shows that $x^2y^{4r+1} = y^{4r+1}x^2$.

As $\gcd(4, 8r+2) = 2$, $x^3y^4 = y^4x^3$ we have $x^3y^2 = y^2x^3$ and therefore from $x^3y^2x^3 = y^2$, we see that $x^6 = 1$ and we have a presentation of the direct product. □

Corollary 5. *An efficient presentation of $A_5 \times D_n$, where n is even, is*
$$\langle x, y \mid x^3y^2x^3 = y^2, (xy^4)^2, y^5x^2y^5 = x^2, (xy^5)^n x^{2n} \rangle.$$

Theorem 6. *An efficient presentation of $A_4 \times D_n$, when n is even and $(n,3) = 1$, is*
$$\langle x, y \mid x^6, y^{\varepsilon n-1} = x^3yx^3, (xy)^2, x = y^3xy^3 \rangle$$
where $\varepsilon \equiv \frac{n}{2}(mod 3)$ and $\varepsilon \epsilon \{-1, 1\}$.

Proof. $M(A_4 \times D_n) = C_2 \times C_2$. Let $D_n = \langle a, b \mid a^2, b^n, (ab)^2 \rangle$ and $A_4 = \langle c, d \mid c^3, d^3, (cd)^2 \rangle$. First take $n = 6r + 2$. Let $x = ac$ and $y = bd$, hence $x^3 = a$, $x^{-2} = c$, $y^{-n} = d$ and using ($n = 6r + 2$ and $(n,3) = 1$) we obtain $y^{n+1} = b$. So the direct product is

$$\langle x, y \mid x^6, y^{3n}, y^n = (x^3y)^2, y^{n-2} = (x^2y^{-1})^2, x^3y^n = y^nx^3, x^2y^3 = y^3x^2 \rangle.$$

Note that $x^3y^n = y^nx^3$ is redundant. (using $y^n = (x^3y)^2$)
Also we see that $1 = x^2y^{-1}x^2y^{-1}y^{2-n}$
$= x^2y^{-1}x^2y.y^{-n}$
$= x^2y^{-1}x^2y(x^3y)^{-2}$ (using $y^n = (x^3y)^2$)
we have $(xy)^2 = 1$.
Therefore replace $y^{n-2} = (x^2y^{-1})^2$ by $(xy)^2 = 1$.

Now $(y^{n-1})^3 = (x^3yx^3)^3$ (using $y^n = (x^3y)^2$)
$= x^3y^3x^3$ (using $x^6 = 1$)
we have $x^3y^3x^3y^3 = 1$ (using $y^{3n} = 1$)
we obtain $x = y^3xy^3$. (using $x^2y^3 = y^3x^2$, $x^6 = 1$).

Add the relation $x = y^3xy^3$. However $x = y^3xy^3$ we have $x^2y^3 = y^3x^2$, so the later is redundant. So we see that the direct product can be presented by

$$\langle x,\, y \mid x^6,\, y^{3n},\, y^n = (x^3y)^2,\, (xy)^2,\, x = y^3xy^3\rangle.$$

Consider

$$\langle x,\, y \mid x^6,\, y^{n-1} = x^3yx^3,\, (xy)^2,\, x = y^3xy^3\rangle.$$

As we just mentioned, $x = y^3xy^3$ we have $x^2y^3 = y^3x^2$.
And $(y^{n-1})^3 = (x^3yx^3)^3$
$= x^3y^3x^3$. (using $x^6 = 1$)

Hence

$$y^{3n} = x^3y^3x^3y^3. \tag{0.2}$$

But, $x = y^3xy^3$ we have $x^3 = y^3x^3y^3$ (using $x^2y^3 = y^3x^2$)
we obtain $x^3y^3x^3y^3 = 1$. (using $x^6 = 1$)
Therefore, from (0.2), we see that $y^{3n} = 1$.
Hence a presentation is one of $A_4 \times D_n$ when $n \equiv 2 (\mathrm{mod}\ 6)$ is

$$\langle x, y \mid x^6, y^{n-1} = x^3yx^3, (xy)^2, x = y^3xy^3\rangle\ . \tag{0.3}$$

When $n \equiv -2(\mathrm{mod} 6)$, consider $A_4 \times D_{-n}$. Note that $-n \equiv 2(\mathrm{mod} 6)$ and so using (0.3) we see that

$$A_4 \times D_{-n} \cong \langle x,\, y \mid x^6,\, y^{(-n)-1} = x^3yx^3,\, x = y^3xy^3,\, (xy)^2\rangle.$$

However as $D_n \cong \langle a,\, b \mid a^2,\, b^n,\, (ab)^2\rangle = \langle a,\, b \mid a^2,\, b^{-n},\, (ab)^2\rangle \cong D_{-n}$ we see that $A_4 \times D_{-n} \cong A_4 \times D_n$ hence, when $n \equiv -2(\mathrm{mod} 6)$,

$$A_4 \times D_n \cong \langle x,\, y \mid x^6,\, y^{-n-1} = x^3yx^3,\, x = y^3xy^3,\, (xy)^2\rangle,$$

and the theorem is proved. □

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