Turk J Math
24 (2000) , 109 – 120.

A Local Zero-Two Law and S ome Applications

R. Zaharopol

Abstract

In the paper we obtain a local zero-two law for positive contractions of L^1-spaces, which we use in order to offer new proofs of a theorem of Orey concerning Markov chains, and of the strong asymptotic stability of certain Markov operators that have appeared in the study of the Tjon-Wu equation and in connection with the Hannsgen and Tyson model of the cell cycle.

1991 Mathematics Subject Classification Numbers:
Primary: 47A35
Secondary: 47B38, 47D07, 47N20, 47N60, 60J10

1. Introduction

Let (X, Σ, μ) be a measure space (the measure μ is not necessarily σ-finite).

A linear bounded operator $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is called a positive contraction if T is positive (that is, $Tf \geq 0$ whenever $f \in L^1(X, \Sigma, \mu), f \geq 0$) and T is a contraction (i.e., $\parallel T \parallel \leq 1$).

The starting point for the results of this paper is a beautiful theorem of Ornstein and Sucheston (Theorem 1.1 of [10]) known as the zero-two law for positive contractions of L^1-spaces. For our purposes the theorem can be stated as follows:

Theorem 1.1. *Let T: $L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ be a positive contraction. If there exists $\epsilon \in \mathbb{R}, \epsilon > 0$ such that*

$$lim_{n\to\infty}\frac{\parallel T^{n+1}u - T^n u \parallel}{\parallel u \parallel} \leq 2(1-\epsilon)$$

for every $u \in L^1(X, \Sigma, \mu), u \neq 0$*, then*

$$lim_{n\to\infty} \parallel T^{n+1}u - T^n u \parallel = 0$$

for every $u \in L^1(X, \Sigma, \mu)$.

A linear bounded operator $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is called a Markov operator if T is a positive operator and $\parallel Tf \parallel = \parallel f \parallel$ for every $f \in L^1(X, \Sigma, \mu), f \geq 0$. (Clearly, a Markov operator is a positive contraction.)

Now, let $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ be a Markov operator.

Given $A \in \Sigma$, set $T_A = 1_A T 1_A$ and

$$\delta_A = sup\left\{lim_{n\to\infty}\frac{\parallel T^{n+1}u - T^n u \parallel}{\parallel u \parallel} \mid u \in L^1(X, \Sigma, \mu), u1_A = u, u \neq 0\right\}.$$

We say that A is a strong zero set (for T) if the sequence of positive contractions $(T_A^n)_{n\in\mathbb{N}}$ converges strongly to zero.

Given $\rho \in R, 0 \leq \rho \leq 1$, we say that T is of 2ρ type for the zero-two law on A (to shorten, we write that T is 2ρ; 0-2 on A) if $\delta_A \leq 2\rho$.

Note that for any measurable subset A of X it follows that T is 2; 0-2 on A.

The main result of the paper is the following theorem:

Theorem 1.2. *Let* $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ *be a Markov operator. If there exists* $A \in \Sigma$ *such that* $X \setminus A$ *is a strong zero set, and such that* T *is* 2η *; 0-2 on* A *for some* $\eta \in R, 0 \leq \eta < 1$*, then* $\lim_{n\to\infty} \parallel T^{n+1}u - T^n u \parallel = 0$ *for every* $u \in L^1(X, \Sigma, \mu)$.

We call Theorem 1.2 a "local" zero-two law because the theorem has the following consequence: if $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is a Markov operator and if $A \in \Sigma$ is such that $X \setminus A$ is a strong zero set for T, then

$$sup\left\{lim_{n\to\infty}\frac{\parallel T^{n+1}u - T^n u \parallel}{\parallel u \parallel} \mid u \in L^1(X, \Sigma, \mu), u1_A = u, u \neq 0\right\} = 0 \ \ or \ \ 2,$$

and because one may think of Theorem 1.1 as a “global” zero-two law for positive contractions of L^1-spaces.

We will use the main result in order to obtain a new proof of a theorem of Orey [9] (see also Ornstein and Sucheston [10]) and new proofs of the strong asymptotic stability of the following two Markov operators:

- and operator that has appeared in the study of a special case of the linear Boltzmann equation known as the linear Tjon-Wu equation (the operator is discussed in detail in the book by Lasota and Mackey [6]; see also Lasota, Li, and Yorke [5] and Malczak [8]);

- an operator that was defined in connection with a model of the cell cycle in biology created by Hannsgen and Tyson [3] (see also Gacki and Lasota [2], Komorowski and Tyrcha [4], Lasota and Mackey [6], Malczak [8], and Tyrcha [11]).

The paper is organized as follows: in the next secton (Section 2) we prove the main result of the paper (Theorem 1.2) and discuss several consequences; the results of Section 2 are used to obtain a new proof of a theorem of Orey [9] in Section 3, and to obtain (in Section 4) new proofs of the asymptotic stability of the Markov operators defined by stochastic kernels that we mentioned earlier (an operator that stems from the study of the linear Tjon-Wu equation, and another one which appeared in the study of the cell cycle).

Throughout the paper we will use results and terminology from Foguel [1], Lasota and Mackey [6], Lin [7], Ornstein and Sucheston [10].

2. A Zero-Two Law for Complements of Strong Zero Sets

Let (X, Σ, μ) be a measure space (not necessarily σ-finite) and let $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ be a Markov operator.

The next lemma is a reformulation of Lemma 2.2 of Lin [7].

Lemma 2.1 *Let $A \in \Sigma$, and let $u \in L^1(X, \Sigma, \mu)$ be such that $u \geq 0, u1_{X\setminus A} = u$. Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of positive elements of $L^1(X, \Sigma, \mu)$ such that:*

(a) $g_n 1_A = g_n \quad \forall n \in \mathbb{N}$;

(b) $\Sigma_{k=1}^{n} T^{n-k} g_k + T^n_{X\setminus A} u = T^n u$ for every $n \in \mathbb{N}$;

(c) $\| \Sigma_{k=1}^{n} g_k \| = \| T^n u - T^n_{X\setminus A} u \|$ for every $n \in \mathbb{N}$.

Proof. Set $g_n = 1_A T T_{X\setminus A}^{n-1} u$ for every $n \in \mathbb{N}$. Clearly, $g_n \geq 0$ and $g_n 1_A = g_n$ for every $n \in \mathbb{N}$.

Since $g_1 + T_{X\setminus A} u = Tu$, and since $\Sigma_{k=1}^{n} T^{n-k} g_k + T_{X\setminus A}^{n} u = \Sigma_{k=1}^{n-1} T^{n-k} g_k + 1_A T T_{X\setminus A}^{n-1} u + T_{X\setminus A}^{n} u = \Sigma_{k=1}^{n-1} T^{n-k} g_k + T T_{X\setminus A}^{n-1} u = T(\Sigma_{k=1}^{n-1} T^{n-1-k} g_k + T_{X\setminus A}^{n-1} u)$, by an induction argument it follows that (b) holds true for every $n \in \mathbb{N}$.

Finally, taking into consideration that (b) is true it follows that

$$\| \Sigma_{k=1}^{n} g_k \| = \Sigma_{k=1}^{n} \| T^{n-k} g_k \| = \| \Sigma_{k=1}^{n} T^{n-k} g_k \| = \| T^n u - T_{X\setminus A}^{n} u \| \text{ for every } n \in \mathbb{N}.$$

□

Proof of Theorem 1.2.

We first note that the theorem is trivially true if $\mu(A) = 0$ (if $\mu(A) = 0$, then X is a strong zero set); thus, we may and will assume that $\mu(A) > 0$.

Next, note that if $\mu(X \setminus A) = 0$, then the conclusion of the theorem follows from the zero-two law for positive contractions of L^1-spaces of Ornstein and Sucheston (Theorem 1.1); therefore, we may and do assume that $\mu(X \setminus A) > 0$.

Let $\rho \in R, 0 < \rho < 1 - \eta$. Since T is 2η; 0-2 on A, it follows that $lim_{n\to\infty} \| T^{n+1}u - T^n u \| < 2(1-\rho) \| u \|$ for every $u \in L^1(X, \Sigma, \mu), u1_A = u, u \neq 0$.

Using again Theorem 1.1, we infer that in order to prove that $lim_{n\to\infty} \| T^{n+1}u - T^n u \| = 0$ for every $u \in L^1(X, \Sigma, \mu)$, it is enough to prove that for every $v \in L^1(X, \Sigma, \mu), v \geq 0, v \neq 0$ there exists $n_0 \in \mathbb{N}$ such that $\| T^{n_0+1}v - T^{n_0}v \| \leq 2(1 - \frac{\rho}{2}) \| v \|$.

To this end, let $v \in L^1(X, \Sigma, \mu), v \geq 0, v \neq 0$. Set $v_A = v1_A$ and $v_{X\setminus A} = v1_{X\setminus A}$.

It follows that there exists $n_1 \in \mathbb{N}$ such that $\| T^{n+1}v_A - T^n v_A \| \leq 2(1-\rho) \| v_A \|$ for every $n \geq n_1$.

Since $X \setminus A$ is a strong zero set, it follows that there exists $n_2 \in \mathbb{N}, n_2 \geq n_1$ such that $\| T_{X\setminus A}^{n} v_{X\setminus A} \| \leq \frac{\rho}{8} \| v_{X\setminus A} \|$ for every $n \geq n_2$.

Set $g_k = 1_A T T_{X\setminus A}^{k-1} v_{X\setminus A}$ for every $k \in N$.

Since $g_k 1_A = g_k$ for every $k \in \mathbb{N}$, it follows that there exists $n_3 \in \mathbb{N}, n_3 \geq n_2$ such that $\| T^{n_3+1}g_k - T^{n_3}g_k \| \leq 2(1-\rho) \| g_k \|$ for every $k = 1, 2, \ldots, n_2$.

Set $n_0 = n_2 + n_3$. Using (b) and (c) of Lemma 2.1, we obtain that

$$\| T^{n_0+1}v - T^{n_0}v \| \leq \| T^{n_0+1}v_A - T^{n0}v_A \|$$
$$+ \| (T^{n_0+1} - T^{n_0})v_{X\setminus A} \| \leq 2(1-\rho) \| v_A \|$$
$$+ \| (T^{n_3+1} - T^{n_3})(\Sigma_{k=1}^{n_2} T^{n_2-k} g_k + T^{n_2}_{X\setminus A} v_{X\setminus A}) \|$$
$$\leq 2(1-\rho) \| v_A \| + (\Sigma_{k=1}^{n_2} \| T^{n_3+1}g_k - T^{n_3}g_k \|$$
$$+2 \| T^{n_2}_{X\setminus A} v_{X\setminus A} \| \leq 2(1-\rho) \| v_A \| + 2(1-\rho)\Sigma_{k=1}^{n_2} \| g_k \|$$
$$+\frac{\rho}{4} \| v_{X\setminus A} \| = 2(1-\rho) \| v_A \|$$
$$+2(1-\rho) \| T^{n_2}v_{X\setminus A} - T^{n_2}_{X\setminus A} v_{X\setminus A} \| + \frac{\rho}{4} \| v_{X\setminus A} \|$$
$$\leq 2(1-\rho) \| v_A \| + 2(1-\rho) \| v_{X\setminus A} \| + \frac{\rho}{4} \| v_{X\setminus A} \|$$
$$\leq 2(1-\frac{\rho}{2}) \| v_A \| + 2(1-\frac{\rho}{2}) \| v_{X\setminus A} \| = 2(1-\frac{\rho}{2}) \| v \| .$$

□

From now on, throughout this section, we will deal only with σ-finite measure spaces. Thus, we assume given a σ-finite measure space (X, Σ, μ) (and a Markov operator $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$).

Following [10] we say that T is ergodic and conservative if $\Sigma_{n=0}^{\infty} T^n f = +\infty$ on X for every $f \in L^1(X, \Sigma, \mu), f \geq 0, f \neq 0$.

Lemma 2.2 *Assume that $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is ergodic and conservative. Then any measurable subset $A \in \Sigma$ such that $\mu(A) > 0$ and $\mu(X \setminus A) > 0$ is a strong zero set.*

Proof. Let $T' : L^\infty(X, \Sigma, \mu) \to L^\infty(X, \Sigma, \mu)$ be the dual of T, and set $\Sigma_i = \{B \in \Sigma \mid T'1_B = 1_B\}$. It is the custom to say that Σ_i is trivial if Σ_i contains only negligible and complements of negligible sets. It is known (and not hard to prove) that T is ergodic and conservative if and only if T is conservative (that is, $X = C$) and Σ_i is trivial. In other words, the notions of "ergodic and conservative" of Ornstein and Sucheston's paper [10] and of Foguel's book [1] agree. □

Now, let $A \in \Sigma$ be such that $\mu(A) > 0$ and $\mu(X \setminus A) > 0$. Using the discussion on p. 67 of [1] we obtain that $((1_A T' 1_A)^n 1_X)_{n \in \mathbb{N} \cup \{0\}}$ is a decreasing sequence of elements of $L^\infty(X, \Sigma, \mu)$, and $\inf_{n \in \mathbb{N} \cup \{0\}} (1_A T' 1_A)^n 1_X = 0$.

By the Lebesgue dominated convergence theorem it follows that

$$lim_{n \to \infty} \parallel T_A^n u \parallel = lim_{n \to \infty} \int u((1_A T' 1_A)^n 1_X) d\mu$$

$$= \int lim_{n \to \infty} (u((1_A T' 1_A)^n 1_X)) d\mu = 0$$

for every $u \in L^1(X, \Sigma, \mu), u \geq 0$.

It follows that $(T_A^n)_{n \in \mathbb{N} \cup \{0\}}$ converges strongly to zero.

The operator $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is called completely mixing if $lim_{n \to \infty} \parallel T^n f \parallel = 0$ for every $f \in L^1(X, \Sigma, \mu), \int f d\mu = 0$.

Theorem 1.2 has the following consequence:

Corollary 2.3. *Assume that T is ergodic and conservative. If there exists $A \in \Sigma, \mu(A) > 0$, and $\eta \in R$, $0 \leq \eta < 1$ such that T is 2η; 0-2 on A, then T is completely mixing.*

Proof. Lemma 2.2 implies that $X \setminus A$ is a strong zero set. Thus, by Theorem 1.2 $lim_{n \to \infty} \parallel T^{n+1} u - T^n u \parallel = 0$ for every $u \in L^1(X, \Sigma, \mu)$. Using Corollary 1.3 of [10] we conclude that T is completely mixing. □

The operator $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is called (strongly) asymptotically stable if there exists $g \in L^1(X, \Sigma, \mu), g \geq 0, \parallel g \parallel = 1$ such that the sequence $(T^n f)_{n \in \mathbb{N}}$ converges in the norm topology of $L^1(X, \Sigma, \mu)$ to $\left(\int f d\mu\right) \cdot g$ for every $f \in L^1(X, \Sigma, \mu)$. Clearly, T is asymptotically stable if and only if T is completely mixing and there exists $u \in L^1(X, \Sigma, \mu), u \geq 0, u \neq 0$ such that $Tu = u$.

We conclude this section with two consequences of Theorem 1.1 which can be used in studying the asymptotic stability of Markov operators.

Corollary 2.4. *Assume that T satisfies the following conditions:*

(a) There exists $u \in L^1(X, \Sigma, \mu), u > 0$ on X such that $Tu = u$.

(b) $\bigcup_{n\in\mathbb{N}\cup\{0\}}\{T^n f > 0\} = X$ whenever $f \in L^1(X, \Sigma, \mu), f \geq 0, f \neq 0$.

(c) There exists $A \in \Sigma$, $\mu(A) > 0$, and $\eta \in R, 0 \leq \eta < 1$ such that T is 2η; 0-2 on A.

Then T is asymptotically stable.

Proof. If (a) is satisfied, then T is conservative (that is, $X = C$); if both (a) and (b) are satisfied, then T is ergodic and conservative. By Corollary 2.3 it follows that T is completely mixing whenever (a), (b), and (c) are satisfied.

Using again (a) it follows that T is asymptotically stable. □

Corollary 2.5. *Assume that T satisfies (a) and (b) of Corollary 2.4, and that there exists $A \in \Sigma$, $\mu(A) > 0$, and $\eta \in R$, $\eta > 0$ such that $\| (T^2 g) \wedge (Tg) \| \geq \eta$ for every $g \in L^1(X, \Sigma, \mu), g \geq 0, \| g \| = 1$, $g1_A = g$. Then T is asymptotically stable.*

Proof. Since T is a positive contraction, it follows that $\lim_{n\to\infty} \| T^{n+1}g - T^n g \| \leq \| T^2 g - Tg \| = \| | T^2 g - Tg | \| = \| T^2 g + Tg - 2((T^2 g) \wedge (Tg)) \| 2 - 2 \| (T^2 g) \wedge (Tg) \| \leq 2(1-\eta)$ for every $g \in L^1(X, \Sigma, \mu), g \geq 0, \| g \| = 1, g1_A = g$.

Thus, T is $2(1-\eta)$; 0-2 on A. By Corollary 2.4, it follows that T is asymptotically stable. □

3. An Application to Markov Chains

Recall that a real valued matrix $P = (p_{ij})_{i,j\in\mathbb{N}}$ is called a Markovian matrix (or a column stochastic matrix) if $p_{ij} \geq 0 \ \forall i, j$ and if $\Sigma_i p_{ij} = 1$ for every j. Such a matrix induces an operator on $l^1 = \{(a_n)_{n\in\mathbb{N}} \mid a_n \in R \ \ \forall n, \Sigma_n \mid a_n \mid < +\infty\}$ denoted again by P. Thus, $P : l^1 \to l^1, P((a_n)_n) = \left(\Sigma_{j=1}^{\infty} p_{ij} a_j\right)_{i\in\mathbb{N}}$ for every $(a_n)_n \in l^1$.

We will use the notation $P^n = (p_{ij}^{(n)})_{i,j}$. The values taken by the indices i and j are called states in Markov chain terminology.

Theorem 3.1. *(Orey [9]). Assume that the operator $P : l^1 \to l^1$ defined by the Markovian matrix $P = (p_{ij})_{i,j\in\mathbb{N}}$ is ergodic and conservative. If $p_{k_0 i_0}^{(n_0+1)} p_{k_0 i_0}^{(n_0)} \neq 0$ for* *some $i_0, k_0, n_0 \in \mathbb{N}$, then P is completely mixing.*

Proof. Since

$$\begin{aligned} \parallel (P^{n_0+1} - P^{n_0})1_{\{i_0\}} \parallel &= \sum_{k=1}^{\infty} \mid p_{ki_0}^{(n_0+1)} - p_{ki_0}^{(n_0)} \mid \\ &\leq \sum_{k=1}^{\infty} (p_{ki_0}^{(n_0+1)} + p_{ki_0}^{(n_0)} - 2(p_{k_0 i_0}^{(n_0+1)} \wedge (p_{k_0 i_0}^{(n_0+1)} \wedge p_{k_0 i_0}^{(n_0)}) \\ &= 2 - 2(p_{k_0 i_0}^{(n_0+1)} \wedge p_{k_0 i_0}^{(n_0)} < 2 \end{aligned}$$

it follows that T is 2η; 0-2 on $\{i_0\}$ for some $\eta \in \mathbb{R}$, $0 \leq \eta < 1$. By Corollary 2.3, T is completely mixing. □

4. Applications to Operators Defined by Stochastic Kernels

Our goal in this section is to discuss the use of the results of Section 2 in the study of the asymptotic stability of a certain type of Markov operators defined by stochastic kernels. Thus, we will offer new proofs of the asymptotic stability of a Markov operator that has appeared in the study of the linear Tjon-Wu equation and of another Markov operator that was created in connection with a model of the cell cycle.

Let (X, Σ, μ) be a σ-finite measure space, let $K : X \times X \to R$ be a function which is measurable with respect to the product σ-algebra $\Sigma \otimes \Sigma$ such that $K(x, y) \geq 0$ for every $x, y \in X$ and $\int_X K(x, y) d\mu(x) = 1$ for every $y \in X$. By Fubini's theorem the operator $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$, $(Tf)(x) = \int K(x, y) f(y) dy$ for every $f \in L^1(X, \Sigma, \mu)$ and $x \in X$ is well-defined (that is, $Tf \in L^1(X, \Sigma, \mu)$ for every $f \in L^1(X, \Sigma, \mu)$) and is a stochastic operator. The function K is called a stochastic kernel.

Theorem 4.1. *Assume that (X, Σ, μ) is a σ-finite measure space, that $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is defined by a stochastic kernel $K : X \times X \to R$, and that the following three conditions are satisfied.*

I. There exists $A \in \Sigma$, $\mu(A) > 0$ such that $inf_{x,y \in A} K(x, y) > 0$.

II. There exists $u \in L^1(X, \Sigma, \mu), u > 0$ on X such that $Tu = u$.

III. $\bigcup_{n \in \mathbb{N} \cup \{0\}} \{T^n f > 0\} = X$ for every $f \in L^1(X, \Sigma, \mu), f \geq 0, f \neq 0$.

Then T is (strongly) asymptotically stable.

Proof. Set $\eta = inf_{x,y\in A}K(x,y)$.
It follows that $K_2(x,y) = (K\star K)(x,y) = \int_X K(x,z)K(z,y)d\mu(z) \geq \int_A K(x,z)K(z,y)d\mu(z) \geq \eta^2\mu(A)$ for every $x, y \in A$.

Set $\rho = \eta^2\mu(A)$.

It follows that

$$\| (T^2w) \wedge (Tw) \| = \int_X ((\int_X K_2(x,y)w(y)dy) \wedge (\int_X K(x,y)w(y)dy))dx$$
$$\geq \int_A ((\int_A K_2(x,y)w(y)dy) \wedge (\int_A K(x,y)w(y)dy))dx$$
$$\geq \int_A (\rho \wedge \eta)dx = (\rho \wedge \eta)\mu(A) > 0$$

for every $w \in L^1(X, \Sigma, \mu), w \geq 0, \| w \| = 1, w1_A = w$.

By Corollary 2.5, T is asymptotically stable. □

Examples *(1) Assume that $X = (0,\infty)$, let $\mathcal{B}$ be the σ-algebra of all the Lebesgue measurable subsets of $(0, +\infty)$, and let λ be the Lebesgue measure on $(0, +\infty)$. Consider the function $K : (0, +\infty) \times (0, +\infty) \to \mathbb{R}$,*

$$K(x,y) = \begin{cases} e^y \int_y^\infty \frac{e^{-z}}{z}dz \text{ whenever } 0 < x \leq y \\ e^y \int_x^\infty \frac{e^{-z}}{z}dz \text{ whenever } 0 < y < x. \end{cases}$$

It is well-known and easy to check that K is a stochastic kernel (see, for example, Lasota and Mackey [6]). Let $T : L^1((0,\infty), \beta, \lambda) \to L^1((0,\infty), \beta, \lambda)$ be the stochastic operator defined by K. The kernel K and the operator T have appeared in the study of a special case of the linear Boltzman equation known as the Tjon-Wu equation (see Lasota and Mackey [6] or Malczak [8]).

If we set $A = (0, 1]$, then $inf_{x,y\in A}K(x,y) \geq min\{inf_{y\in A}e^y \int_y^\infty \frac{e^{-z}}{z}dz, \inf_{x,y\in A} e^y \int_x^\infty \frac{e^{-z}}{z}dz\} \geq \int_1^\infty \frac{e^{-z}}{z}dz > 0$. Thus, T has property I of Theorem 4.1.

An easy computation shows (see [6]) that $u(x) = e^{-x}$ is an invariant weak order unit for T, and since $K(x,y) > 0$ for every $(x,y) \in (0,+\infty) \times (0,+\infty)$ (therefore, $\{Tf > 0\} = X$ for every $f \in L^1((0,+\infty),\mathcal{B},\lambda), f \geq 0, f \neq 0$) it follows that we can use Theorem 4.1 in order to conclude that T is strongly asymptotically stable.

(2) We will now consider a stochastic operator which was introduced in biology, in the study of the cell cycle by Hannsgen and Tyson [3] (see also Komorowski and Tyrcha [4], Lasota and Mackey [6], and Malczak [8]; we will be using here the approach of [4]).

Let $\sigma \in \mathbb{R}, 0 < \sigma < 1$, let $\alpha \in \mathbb{R}, \alpha > 0$, and let $K : [\sigma,\infty) \times [\sigma,\infty) \to \mathbb{R}$ be defined by

$$K(x,y) = \begin{cases} \frac{\alpha}{\sigma}(\frac{x}{\sigma})^{-1-\alpha} & \text{if } \sigma \leq y \leq 1 \\ \frac{\alpha}{\sigma}(\frac{x}{\sigma})^{-1-\alpha} y^{\alpha} & \text{if } 1 < y \leq \frac{x}{6} \\ 0 & \text{if } y > \frac{x}{\sigma}. \end{cases}$$

Let $\mathcal{B}$ be the σ-algebra of all Lebesgue measurable subsets of $[\sigma,\infty)$, and let λ be the Lebesgue measure on $[\sigma,\infty)$. It is easy to see that K is a stochastic kernel. Thus, it makes sense to consider the stochastic operator $T : L^1([\sigma,\infty),\mathcal{B},\lambda) \to L^1([\sigma,\infty),\mathcal{B},\lambda)$ defined by K.

If we set $A = [\sigma,1]$, then $inf_{(x,y)\in A\times A} K(x,y) = inf_{\sigma\leq x\leq 1} \frac{\alpha}{\sigma}.(\frac{x}{\sigma})^{-1-\alpha} = \alpha\sigma^{\alpha} > 0$; therefore, T satisfies I of Theorem 4.1.

We now prove that T satisfies III of Theorem 4.1. To this end, set $\omega(g) = inf\{\beta \in \mathbb{R} \mid \beta > \sigma$ and $\lambda((g^{-1}((0,\infty))) \cap [\sigma,\beta]) > 0\}$ for every $g \in L^1([\sigma,\infty),\mathcal{B},\lambda), g \geq 0, g \neq 0$, and let $f \in L^1([\sigma,\infty),\mathcal{B},\lambda), f \geq 0, f \neq 0,$.

If $\omega(f) = \sigma$, then $Tf(x) = \int_{\sigma}^{\frac{x}{\sigma}} K(x,y)f(y)dy > 0$ for every $x \geq \sigma$ since $\lambda(\{y \in [\sigma,\frac{x}{\sigma}] \mid f(y) > 0\}) > 0$ and $K(x,y) > 0$ for all $y \in [\sigma,\frac{x}{\sigma}]$. Thus, $\{Tf > 0\} = [\sigma,\infty)$ in this case.

If $\omega(f) > \sigma$, then $Tf(x) = \int_{\sigma}^{\frac{x}{\sigma}} K(x,y)f(y)dy > 0$ for every $x \in R, x \geq \sigma, \frac{x}{\sigma} > \omega(f)$; that is, $Tf(x) > 0$ whenever $x \in \mathbb{R}, x > max\{\sigma\omega(f),\sigma\}$. Thus, $\omega(Tf) \leq max\{\sigma\omega(f),\sigma\}$. In general, taking into consideration that $T^n f = T(T^{n-1}f)$, we obtain that $T^n f(x) > 0$ whenever $x > max\{\sigma\omega(T^{n-1}f),\sigma\}$; hence, $\omega(T^n f) \leq max\{\sigma\omega(T^{n-1}f),\sigma\}$

$$\begin{aligned} &\leq max\{\sigma max\{\sigma\omega(T^{n-2}f),\sigma\},\sigma\} \\ &= max\{\sigma^2\omega(T^{n-2}f),\sigma^2,\sigma\} = max\{\sigma^2\omega(T^{n-2}f),\sigma\} \end{aligned}$$

$$= \ldots = max\{\sigma^n \omega(f), \sigma\} \text{ for every } \quad n \in \mathbb{N}.$$

It follows that $\{Tf > 0\} \subseteq (max\{\sigma\omega(f), \sigma\}, \infty)$, and, in general $\{T^n f > 0\} \supseteq (max\{\sigma^n\omega(f), \sigma\}, \infty)$ for every $n \in \mathbb{N}$.

Thus,

$$\bigcup_{n=1}^{\infty} \{T^n f > 0\} \supseteq \bigcup_{n=1}^{\infty} (max\{\sigma^n \omega(f), \sigma\}, \infty) = (\sigma, \infty).$$

It follows that T satisfies III of Theorem 4.1.

A straightforward computation (see [3]; also see [4], [6], and [8]) shows that if $\alpha ln\sigma < -1$, then T has an invariant weak order unit; it follows that we can use Theorem 4.1 in order to obtain a new proof of the fact that T is strongly asymptotically stable whenever $\alpha ln\sigma < -1$.

Acknowledgement

The author would like to express his appreciation to the referee for several suggestions that have improved the exposition significantly.

References

[1] S. R. Foguel, "The Ergodic Theory of Markov Processes", Van Nostrand-Reinhold, New York, 1969.

[2] H. Gacki and A. Lasota, Markov operators defined by Volterra type integrals with advanced argument, Ann. Polon. Math. 51 (1990), 155-166.

[3] K. B. Hannsgen and J. J. Tyson, Cell growth and division: a deterministic/probabilistic model of the cell cycle, J. Math. Biology 23 (1986), 231-246.

[4] T. Komorowski and J. Tyrcha, Asymptotic properties of some Markov operators, Bull. Polish Acad. Sci., Math. 37 (1989), 221-228.

[5] A. Lasota, T. Y. Li, and J. A. Yorke, Asymptotic periodicity of the iterates of Markov operators, Trans. Amer. Math. Soc. 286 (1984), 751-764.

[6] A. Lasota and M. C. Mackey, "Chaos, Fractals and Noise," Second Edition, Springer, New York, 1994.

[7] M. Lin, Convergence of the iterates of a Markov operator, Z. Wahrscheinlichkeitstheorie verw. Geb. 29 (1974), 153-163.

[8] J. Malczak, An application of Markov operators in differential and integral equations, Rend. Sem. Mat. Univ. Padova 87 (1992), 281-297.

[9] S. Orey, An ergodic theorem for Markov chains, Z. Wahrscheinlichkeitstheorie verw. Geb. 1 (1962), 174-176.

[10] D. Ornstein and L. Sucheston, An operator theorem on L_1 convergence to zero with applications to Markov kernels, Ann. Math. Statist. 41 (1970), 1631-1639.

[11] J. Tyrcha, Asymptotic stability in a generalized probabilistic/deterministic model of the cell cycle, J. Math. Biol. 26 (1988), 465-475.

Radu ZAHAROPOL

Received 23.02.1999

Department of Mathematical Sciences,

Binghamton University (SUNY at Binghamton)

Binghamton, New York 13902-6000-USA

e-mail: radu@math.binghamton.edu