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Fuzzy Ideals in Gamma-Rings

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Abstract

The converse of [7, Theorem 3.3] is provided. For an Artinian Γ-ring, a few results are investigated.

Key words and phrases: (Artinian, Noetherian) Γ-ring, fuzzy left (right) ideal, level left (right) ideal.

1. Introduction

The notion of a fuzzy set in a set was introduced by L. A. Zadeh [8], and since then this concept have been applied to various algebraic structures. N. Nobusawa [6] introduced the notion of a Γ-ring, as more general than a ring. W. E. Barnes [1] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. W. E. Barnes [1], S. Kyuno [3] and J. Luh [5] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. Y. B. Jun and C. Y. Lee [2] applied the concept of fuzzy sets to the theory of Γ-rings. In [7], the present authors discussed characterizations of Noetherian Γ-rings by using fuzzy ideals, and gave a condition for a Γ-ring to be Artinian. As a continuation of the paper [7], in this paper, we investigate further results. In particular, we state the converse of Theorem 3.3 in [7].

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2. Preliminaries

Let M and Γ be two abelian groups. If for all $x, y, z \in M$ and all α, $\beta \in \Gamma$ the conditions

- $x\alpha y \in M$,
- $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z$,
- $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied, then we call M a *Γ-ring.* By a *right* (resp. *left*) *ideal* of a Γ-ring M we mean an additive subgroup U of M such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). For any subsets A and B of a Γ-ring M, by $A \subset B$ we exclude the possibility that $A = B$. A Γ-ring M is said to satisfy the *left* (*right*) *ascending chain condition* of left (right) ideals (or to be *left* (*right*) *Noetherian*) if every strictly increasing sequence $U_1 \subset U_2 \subset U_3 \subset \cdots$ of left (right) ideals of M is of finite length. A Γ-ring M is said to satisfy the *left* (*right*) *descending chain condition* of left (right) ideals (or to be *left* (*right*) *Artinian*) if every strictly decreasing sequence $V_1 \supset V_2 \supset V_3 \supset \cdots$ of left (right) ideals of M is of finite length. A Γ-ring M is *left* (resp. *right*) *Noetherian* if M satisfies the left (right) ascending chain condition on left (resp. right) ideals. A Γ-ring M is *left* (resp. *right*) *Artinian* if M satisfies the left (right) descending chain condition on left (resp. right) ideals.

We now review some fuzzy logic concepts. A fuzzy set μ in a Γ-ring M is called a *fuzzy left* (resp. *right*) *ideal* of M ([2]) if it satisfies

- $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$
- $\mu(x\gamma y) \geq \mu(y)$ (resp. $\mu(x\gamma y) \geq \mu(x)$)

for all $x, y \in M$ and $\gamma \in \Gamma$. We note from [2] that if μ is a fuzzy left (right) ideal of a Γ-ring M then $\mu(0) \geq \mu(x)$ for all $x \in M$.

Note from Jun and Lee [2, Theorem 3] that *a fuzzy set μ in a Γ-ring M is a fuzzy left (right) ideal of M if and only if for every $t \in [0,1]$, the set*

$$U(\mu; t) := \{x \in M \mid \mu(x) \geq t\}$$

is a left (right) ideal of M when it is nonempty. We call $U(\mu; t)$ the *level left (right) ideal* of M with respect to μ.

3. Main Results

In what follows, the terms "(fuzzy, level) ideal" and "Artinian (Noetherian) Γ-ring" mean "(fuzzy, level) left ideal" and "left Artinian (Noetherian) Γ-ring", respectively.

For a fuzzy set μ in M and $t \in [0,1]$, we define

$$U_\mu^t := \mu^{-1}\Big((t,1]\Big) \text{ and } V_\mu^t := \mu^{-1}\Big([t,1]\Big).$$

Theorem 3.1. *If μ is a fuzzy ideal of a Γ-ring M, then for each $t \in [0, \mu(0)]$, U_μ^t and V_μ^t are ideals of M.*

Proof. Let $x, y \in U_\mu^t$. Then $\mu(x-y) \geq \min\{\mu(x), \mu(y)\} > t$ and so $x - y \in U_\mu^t$. Let $x \in M$, $y \in U_\mu^t$ and $\gamma \in \Gamma$. Then $\mu(x\gamma y) \geq \mu(y) > t$, and thus $x\gamma y \in U_\mu^t$. Hence U_μ^t is an ideal of M. Note that $V_\mu^t = U(\mu; t)$ which is an ideal of M.

Theorem 3.2. *Let w be a fixed element of a Γ-ring M. If μ is a fuzzy ideal of M, then the set*

$$\mu^w := \{x \in M \mid \mu(x) \geq \mu(w)\}$$

is an ideal of M.

Proof. Let $x, y \in \mu^w$. Then $\mu(x-y) \geq \min\{\mu(x), \mu(y)\} \geq \mu(w)$, which implies that $x - y \in \mu^w$. Now let $x \in M$, $y \in \mu^w$ and $\gamma \in \Gamma$. Then $\mu(x\gamma y) \geq \mu(y) \geq \mu(w)$, and so $x\gamma y \in \mu^w$. Therefore μ^w is an ideal of M.

Corollary 3.3. ([2, Theorem 1]) *If μ is a fuzzy ideal of a Γ-ring M, then the set*

$$U := \{x \in M \mid \mu(x) = \mu(0)\}$$

is an ideal of M.

Lemma 3.4. ([4, Corollary 2]) *If a Γ-ring M is Artinian, then M is Noetherian.*

Lemma 3.5. *Let μ be a fuzzy ideal of a Γ-ring M and let $s, t \in \operatorname{Im}(\mu)$. Then $U(\mu; s) = U(\mu; t)$ if and only if $s = t$.*

Proof. Straightforward.

Lemma 3.6. ([7, Theorem 3.3]) *If every fuzzy ideal of a* Γ*-ring* M *has finite number of values, then* M *is Artinian.*

Combining Lemmas 3.4 and 3.6, we have the following corollary.

Corollary 3.7. *If for any fuzzy ideal* μ *of a* Γ*-ring* M*,* μ *is finite valued, then* M *is Noetherian.*

We discuss the converse of Lemma 3.6.

Theorem 3.8. *If a* Γ*-ring* M *is Artinian, then every fuzzy ideal of* M *is finite valued.*

Proof. Let a Γ-ring M be Artinian and let μ be a fuzzy ideal of M. Suppose that $\mathrm{Im}(\mu)$ is infinite. Note that every subset of $[0,1]$ contains either a strictly increasing or strictly decreasing infinite sequence. Hence $\mathrm{Im}(\mu)$ has a strictly increasing or strictly decreasing sequence. Let $t_1 < t_2 < t_3 < \cdots$ be a strictly increasing sequence in $\mathrm{Im}(\mu)$. Then $U(\mu; t_1) \supset U(\mu; t_2) \supset U(\mu; t_3) \supset \cdots$ is a strictly descending chain of ideals of M. Since M is Artinian, there exists a natural number i such that $U(\mu; t_i) = U(\mu; t_{i+n})$ for all $n \geq 1$. Since $t_i \in \mathrm{Im}(\mu)$ for all i, it follows from Lemma 3.5 that $t_i = t_{i+n}$ for all $n \geq 1$. This is a contradiction. If $t_1 > t_2 > t_3 > \cdots$ is a strictly decreasing sequence in $\mathrm{Im}(\mu)$, then $U(\mu; t_1) \subset U(\mu; t_2) \subset U(\mu; t_3) \subset \cdots$ is an ascending chain of ideals of M. Since M is Noetherian by Lemma 3.4, there exists a natural number j such that $U(\mu; t_j) = U(\mu; t_{j+n})$ for all $n \geq 1$. Since $t_j \in \mathrm{Im}(\mu)$ for all j, by Lemma 3.5 we have $t_j = t_{j+n}$ for all $n \geq 1$, which is also a contradiction. Hence $\mathrm{Im}(\mu)$ is finite.

Let $\mathcal{U}_\mu$ denote the family of all level ideals of M with respect to μ.

Theorem 3.9. *Let a* Γ*-ring* M *be Artinian and let* μ *be a fuzzy ideal of* M*. Then* $|\mathcal{U}_\mu| = |\mathrm{Im}(\mu)|$*.*

Proof. Since M is Artinian, it follows from Theorem 3.8 that $\mathrm{Im}(\mu)$ is finite. Let $\mathrm{Im}(\mu) = \{t_1, t_2, \cdots, t_n\}$, where $t_1 < t_2 < \cdots < t_n$. It is sufficient to show that $\mathcal{U}_\mu$ consists of level ideals of M with respect to μ for all $t_i \in \mathrm{Im}(\mu)$, that is, $\mathcal{U}_\mu = \{U(\mu; t_i) \mid 1 \leq i \leq n\}$. Obviously, $U(\mu; t_i) \in \mathcal{U}_\mu$ for all $t_i \in \mathrm{Im}(\mu)$. Let $0 \leq t \leq \mu(0)$ and let $U(\mu; t)$ be a level ideal of M with respect to μ. Assume that $t \notin \mathrm{Im}(\mu)$. If $t < t_1$, then clearly $U(\mu; t) = U(\mu; t_1)$, and so let $t_i < t < t_{i+1}$ for some i. Then $U(\mu; t_{i+1}) \subseteq U(\mu; t)$. Let $x \in U(\mu; t)$. Then $\mu(x) > t$ because $t \notin \mathrm{Im}(\mu)$, and so $\mu(x) \geq t_{i+1}$, that is, $x \in U(\mu; t_{i+1})$.

Hence $U(\mu;t) = U(\mu;t_{i+1})$, which shows that $\mathcal{U}_\mu$ consists of level ideals of M with respect to μ for all $t_i \in \mathrm{Im}(\mu)$. Therefore $|\mathcal{U}_\mu| = |\mathrm{Im}(\mu)|$.

If μ is a fuzzy ideal of a Γ-ring M and $\mathrm{Im}(\mu)$ is finite, then $|\mathcal{U}_\mu| = |\mathrm{Im}(\mu)|$ by Lemma 3.6 and Theorem 3.9. Let $\mathrm{Im}(\mu) = \{t_1, t_2, \cdots, t_n\}$, where $t_1 < t_2 < \cdots < t_n$. Then $\mathcal{U}_\mu = \{U(\mu;t_i) \mid 1 \leq i \leq n\}$. Now $t_i < t_j$ if and only if $U(\mu;t_i) \supset U(\mu;t_j)$. Thus we have the following chain of ideals:

$$M = U(\mu;t_1) \supset U(\mu;t_2) \supset \cdots \supset U(\mu;t_n).$$

Theorem 3.10. *Let a Γ-ring M be Artinian and let μ and ν be fuzzy ideals of M. Then $\mathcal{U}_\mu = \mathcal{U}_\nu$ and $\mathrm{Im}(\mu) = \mathrm{Im}(\nu)$ if and only if $\mu = \nu$.*

Proof. If $\mu = \nu$, then clearly $\mathcal{U}_\mu = \mathcal{U}_\nu$ and $\mathrm{Im}(\mu) = \mathrm{Im}(\nu)$. Suppose that $\mathcal{U}_\mu = \mathcal{U}_\nu$ and $\mathrm{Im}(\mu) = \mathrm{Im}(\nu)$. By Theorems 3.8 and 3.9, $\mathrm{Im}(\mu)$ and $\mathrm{Im}(\nu)$ are finite and $|\mathcal{U}_\mu| = |\mathrm{Im}(\mu)|$ and $|\mathcal{U}_\nu| = |\mathrm{Im}(\nu)|$. Let $\mathrm{Im}(\mu) = \{t_1, t_2, \cdots, t_n\}$ and $\mathrm{Im}(\nu) = \{s_1, s_2, \cdots, s_n\}$, where $t_1 < t_2 < \cdots < t_n$ and $s_1 < s_2 < \cdots < s_n$. Then $t_i = s_i$ for all i. We now prove that $U(\mu;t_i) = U(\nu;t_i)$ for all i. Note that $U(\mu;t_1) = M = U(\nu;t_1)$. Consider $U(\mu;t_2)$ and $U(\nu;t_2)$, and suppose $U(\mu;t_2) \neq U(\nu;t_2)$. Then $U(\mu;t_2) = U(\nu;t_k)$ for some $k > 2$ and $U(\mu;t_j) = U(\nu;t_2)$ for some $j > 2$. Now let $x \in M$ be such that $\mu(x) = t_2$. Then $\mu(x) < t_j$ for all $j > 2$. Since $U(\mu;t_2) = U(\nu;t_k)$, it follows that $x \in U(\nu;t_k)$ so that $\nu(x) \geq t_k > t_2$ for $k > 2$. Thus $x \in U(\nu;t_2) = U(\mu;t_j)$ and so $\mu(x) \geq t_j$ for some $j > 2$. This is a contradiction. Hence $U(\mu;t_2) = U(\nu;t_2)$. Continuing in this way, we get $U(\mu;t_i) = U(\nu;t_i)$ for all i. Now let $x \in M$ be such that $\mu(x) = t_i$ for some i. Then $x \notin U(\mu;t_j)$ for all $i+1 \leq j \leq n$, which implies that $x \notin U(\nu;t_j)$ for all $i+1 \leq j \leq n$. Hence $\nu(x) < t_j$ for all $i+1 \leq j \leq n$. Suppose that $\nu(x) = t_m$ for some $1 \leq m \leq i$. If $i \neq m$, then $x \notin U(\nu;t_i)$. On the other hand, $x \in U(\mu;t_i) = U(\nu;t_i)$ because $\mu(x) = t_i$. This is a contradiction, and thus $i = m$ and $\mu(x) = t_i = t_m = \nu(x)$. Consequently, $\mu = \nu$.

References

[1] W. E. Barnes, *On the Γ-rings of Nobusawa,* Pacific J. Math. **18** (1966), 411-422.

[2] Y. B. Jun and C. Y. Lee, *Fuzzy Γ-rings,* Pusan Kyongnam Math. J. (presently, East Asian Math. J.) **8(2)** (1992), 163-170.

[3] S. Kyuno, *On prime gamma rings,* Pacific J. Math. **75(1)** (1978), 185-190.

[4] S. Kyuno, *A gamma ring with the right and left unities,* Math. Japonica **24(2)** (1979), 191-193.

[5] J. Luh, *On the theory of simple* Γ*-rings,* Michigan Math. J. **16** (1969), 65-75.

[6] N. Nobusawa, *On a generalization of the ring theory,* Osaka J. Math. **1** (1964), 81-89.

[7] M. A. Öztürk, M. Uckun and Y. B. Jun, *Characterizations of Artinian and Noetherian gamma-rings in terms of fuzzy ideals,* Tr. J. of Mathematics (submitted).

[8] L. A. Zadeh, *Fuzzy sets,* Inform. and Control **8** (1965), 338-353.

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