From qubits and actions to the Pauli-Schrödinger equation

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Here I show that a classical or quantum bit state plus one simple operation, an *action*, are sufficient ingredients to derive a quantum dynamical equation that rules the sequential changes of the state. Then, by assuming that a freely moving massive particle is the qubit carrier, it is found that both, the particle position in physical space and the qubit state, change in time according to the Pauli-Schrödinger equation. So, this approach suggests the following conjecture: because it carries one qubit of information the particle motion has its description enslaved by the very existence of the internal degree of freedom. It is compelled to be no more described classically but by a wavefunction. I also briefly discuss the Dirac equation in terms of qubits.

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INTRODUCTION

Quantum mechanics (QM) can be instructed either by adopting the schemes proposed by its inventors (Born, Heisenberg, Schrödinger, Jordan) or, more rigorously, following Dirac, within the Hilbert space framework, or even using Feynman's path integral approach. As it is believed to be pedagogically more appealing, almost all textbooks prefer to begin with the nonrelativistic approach, discussing the wave-particle dualism, wavefunctions, Schrödinger equation, Hilbert space, noncommutative operators, etc. However, looking at the emblematic dynamical equations of Schrödinger and Dirac, one notes that the Schrödinger equation (SE) is less fundamental than Dirac's relativistic equation, because this one contains inherently the internal degree of freedom spin, while it is absent in the former. In between there is the Pauli-Schrödinger equation (PSE), which was derived by Pauli when he applied the low energy approximation in Dirac equation. Although being nonrelativistic yet the PSE is more complete than SE because the spin is inherently present, while in the SE the spin must be added as an extra degree of freedom. See references in [1] for a detailed discussion.

In the beginning of the 1980's the possibility of quantum computation was foreseen by people like Benioff, Feynman and Deutsch [2–4], and their work influenced a mini-revolution that began in the 1990ś, which threw a new look in QM, mainly in its interpretation and potentialities to explain new phenomena; in the last 15 years we have witnessed huge theoretical developments along with ingenious experiments involving single or few atoms or molecules, electrons and photons. So, the understanding of quantum physics has widen, shaping the new arena called *quantum information theory (QIT)* that borrowed many concepts of classical information theory. In this context it seems to exist a recognition [5] that QM is a special kind of information theory immersed in Hilbert space, and characterized by a reversible logic [6–8].

In that connection, by using elementary concepts of communication theory, as bits and gates, however represented in the framework of Hilbert space, I will show here that a quantum evolution equation for one classical bit (*Cbit*, as defined in [9]) or quantum bit (*qubit*) of information can be derived. Then, by asking what could be the *carrier* of one qubit (or spin 1/2), the natural choice is a particle of mass m characterized by its kinetic energy, and this information is introduced in the qubit dynamical equation. This procedure is sufficient to derive the PSE that rules the time evolution of both, the qubit and the particle, its carrier. So, it ceases to be a particle in the Newtonian sense to become a hybrid compelled to display wave properties and described by a wavefunction. The qubit/spin evolution acquires an ascendancy over the particle motion, being at the root of the observed quantum properties of matter. Last but not least, the Dirac equation and its solution are briefly discussed in terms of qubits.

CBITS AND ACTIONS

In classical information theory the numbers in $\mathbb{Z}_2 = \{0, 1\}$ are associated to bits, as in a relay or in a memory storage device. One can go one step further and associate a particular representation to the numbers 0 and 1: a column matrix for one classical bit of information, the Cbit state, $1 \longrightarrow \binom{1}{0}$ and $0 \longrightarrow \binom{0}{1}$, as like the states "up" and "down" for the spin 1/2. These states can be written in the more familiar form of Dirac's kets $|x\rangle$, $|\bar{x}\rangle$ ($\bar{x} = 1 - x$) for $\{x, \bar{x}\} \in \mathbb{Z}_2$, and the bras are the transposed, so $\mathcal{H}_2^{\times} \equiv (\langle 1|, \langle 0|)$ is the dual space of $\mathcal{H}_2 \equiv (|0\rangle, |1\rangle)$. The qubit $\binom{a}{b}$ is a generalization of the Cbit, with a and b being complex numbers. The simplest operators to be used are the identity $\boldsymbol{I}$, $\boldsymbol{I}|x\rangle = |x\rangle$, and the NOT $\boldsymbol{X}$

that inverts the Cbit state, $\boldsymbol{X}\left|x\right\rangle = \left|\bar{x}\right\rangle$. So the 4-uple $\mathcal{Q} = \left\{\mathbb{Z}_2, \mathcal{H}_2, \mathcal{H}_2^{\times}, \mathcal{L}_2\right\}$, $\mathcal{L}_2 = \{\boldsymbol{I}, \boldsymbol{X}\}$ plus the field of complex numbers $\mathbb{C}$ are sufficient tools for my purposes.

The *action* $\boldsymbol{U}(\alpha,\beta) \equiv \alpha\boldsymbol{I} + \beta\boldsymbol{X}$ is a linear map of a Cbit or qubit into a qubit, $\boldsymbol{U}(\alpha,\beta)\left|x_0\right\rangle \rightarrow \left|x_1\right\rangle = \alpha\left|x_0\right\rangle + \beta\left|\bar{x}_0\right\rangle$, for arbitrary parameter α, β in $\mathbb{C}$. Let's first restrict the parameters values to two numbers: $\beta = \bar{\alpha} = 1-\alpha$ and $\{\alpha,\bar{\alpha}\} \in \mathbb{Z}_2$, such that $\boldsymbol{U}_\alpha\left|x_0\right\rangle \equiv \alpha\left|x_0\right\rangle + \bar{\alpha}\left|\bar{x}_0\right\rangle$ is still a Cbit because $\alpha\bar{\alpha} = 0$. The actions $\{\boldsymbol{U}_0, \boldsymbol{U}_1\}$ form a group: (a) the unit element is $\boldsymbol{I} = \boldsymbol{U}_0$, while $\boldsymbol{X} = \boldsymbol{U}_1$; (b) the inverse is $\boldsymbol{U}_\alpha^{-1} = \boldsymbol{U}_\alpha$; (c) the product of two elements is an element in the group $\boldsymbol{U}_{\alpha_2}\boldsymbol{U}_{\alpha_1} = \boldsymbol{U}_\beta = \beta\boldsymbol{I} + \bar{\beta}\boldsymbol{X}$ with $\beta = \alpha_2\alpha_1 + \bar{\alpha}_2\bar{\alpha}_1$, and $\bar{\beta} = \overline{\alpha_2\alpha_1 + \bar{\alpha}_2\bar{\alpha}_1} = \alpha_2\bar{\alpha}_1 + \bar{\alpha}_2\alpha_1$; (d) the associative property $\left(\boldsymbol{U}_{\alpha_3}\boldsymbol{U}_{\alpha_2}\right)\boldsymbol{U}_{\alpha_1} = \boldsymbol{U}_{\alpha_3}\left(\boldsymbol{U}_{\alpha_2}\boldsymbol{U}_{\alpha_1}\right)$ holds, and the elements are unitary $\boldsymbol{U}_\alpha^{\dagger} = \boldsymbol{U}_\alpha^{-1}$.

Sequential n actions

$$\boldsymbol{U}_n(\vec{\alpha}) \equiv \prod_{j=1}^{n}\left(\alpha_j\boldsymbol{I} + \bar{\alpha}_j\boldsymbol{X}\right), \tag{1}$$

applied on a Cbit maps it into another Cbit, $\boldsymbol{U}_n(\vec{\alpha})\left|x_0\right\rangle = \left|x_n\right\rangle$, going through the intermediary states $\{\left|x_1\right\rangle, \left|x_2\right\rangle, \left|x_3\right\rangle, ..., \left|x_{n-1}\right\rangle\}$. Each set of numbers $h_n = \{\alpha_n, ..., \alpha_3, \alpha_2, \alpha_1\}$ defines one *history*, or *trajectory*. One can also write $\boldsymbol{U}(\alpha_1)\left|x_0\right\rangle$ by changing the label of the Cbit state, and by a trivial formal manipulation it is simple to show that $\boldsymbol{U}_{\alpha_1}\left|x_0\right\rangle = \left|x_1\right\rangle = \left|\alpha_1 x_0 + \bar{\alpha}_1\bar{x}_0\right\rangle$, identifying the label as the mapped bit $x_1 \equiv \alpha_1 x_0 + \bar{\alpha}_1\bar{x}_0$. By induction $x_n = \alpha_n x_{n-1} + \bar{\alpha}_n\bar{x}_{n-1}$ for $n = 1, 2, 3, ...$ Also holds the transitivity property expressed by the composition law

$$\boldsymbol{U}_n\left(\vec{\alpha}^{(2)}\right)\boldsymbol{U}_m\left(\vec{\alpha}^{(1)}\right) = \boldsymbol{U}_{n+m}(\vec{\alpha}). \tag{2}$$

The sequence of actions (1) is reversible since each one is unitary, then $\left|x_0\right\rangle = \boldsymbol{U}_n^{-1}(\vec{\alpha})\left|x_n\right\rangle$. The reverse history is given by the sequence $h_n^{-1} = \{\alpha_1, ..., \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$. In summary, (1) carries the evolution $\left|x_0\right\rangle \longrightarrow \left|x_n\right\rangle$, and $\boldsymbol{U}_n^{-1}(\vec{\alpha}) = \prod_{j=n}^{1}\left(\alpha_j\boldsymbol{I} + \bar{\alpha}_j\boldsymbol{X}\right)$ does the inverse path, $\left|x_n\right\rangle \longrightarrow \left|x_0\right\rangle$. Formally, $\boldsymbol{U}_n(\vec{\alpha})$ and $\boldsymbol{U}_n^{-1}(\vec{\alpha})$ are the same since each factor in Eq. (1) commutes with all others.

Coefficients on a circle of unit radius

I now assume the parameters α and β in $\boldsymbol{U}(\alpha,\beta)$ to be real, with $\alpha\beta \neq 0$ and ask $\alpha^2 + \beta^2 = 1$, so $(\alpha,\beta) \in \tilde{\mathbb{R}}_2$ is the set of all real numbers on a circle of radius 1. As so, acting on a Cbit one gets a qubit, $\boldsymbol{U}(\alpha,\beta)\left|x_0\right\rangle = \alpha\left|x_0\right\rangle + \beta\left|\bar{x}_0\right\rangle$. Two consecutive operations give $\boldsymbol{U}(\alpha_2,\beta_2)\boldsymbol{U}(\alpha_1,\beta_1) = \boldsymbol{U}(\alpha_3,\beta_3)$, and as $\alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2 = 1$, it follows that $\alpha_3^2 + \beta_3^2 = 1 + 4\alpha_2\alpha_1\beta_2\beta_1 \neq 1$, so $(\alpha_3,\beta_3) \notin \tilde{\mathbb{R}}_2$ and $\boldsymbol{U}(\alpha_3,\beta_3)$ is *not* an element of the group, unless one of the four coefficients is zero, therefore any probabilistic interpretation for α^2 and β^2 fails. Moreover, the inverse action is $\boldsymbol{U}^{-1}(\alpha,\beta) = \tilde{\alpha}\boldsymbol{I} + \tilde{\beta}\boldsymbol{X}$, where the new parameters $\tilde{\alpha} = \alpha/\left(\alpha^2 - \beta^2\right)$ and $\tilde{\beta} = -\beta\left(\alpha^2 - \beta^2\right)$ are *not* in $\tilde{\mathbb{R}}_2$. Due to the reality of α and β, $\boldsymbol{U}(\alpha,\beta)$ is a self-adjoint operator $\boldsymbol{U}^{\dagger}(\alpha,\beta) = \boldsymbol{U}(\alpha,\beta)$ however it is not unitary since $\boldsymbol{U}^{\dagger}(\alpha,\beta) \neq \boldsymbol{U}^{-1}(\alpha,\beta)$. Although the norm $\left\|\boldsymbol{U}(\alpha,\beta)\left|x_0\right\rangle\right\| = 1$ is parameter independent, this is not true for the inverse $\left\|\boldsymbol{U}^{-1}(\alpha,\beta)\left|x_0\right\rangle\right\| = \left|\alpha^2 - \beta^2\right|^{-1}$. Thus, if we want to construct an evolution operator $\boldsymbol{U}_n\left(\vec{\alpha},\vec{\beta}\right) = \prod_{j=1}^{n}\left(\alpha_j\boldsymbol{I} + \beta_j\boldsymbol{X}\right)$, with $\alpha_j^2 + \beta_j^2 = 1$, that is also reversible, we are in trouble. Since the inverse of $\boldsymbol{U}\left(\alpha_j,\beta_j\right)$ is $\boldsymbol{U}\left(\tilde{\alpha}_j,\tilde{\beta}_j\right)$, for a sequence of n inverse actions we have $\boldsymbol{U}_n^{-1}\left(\vec{\alpha},\vec{\beta}\right) = \prod_{j=n}^{1}\boldsymbol{U}\left(\tilde{\alpha}_j,\tilde{\beta}_j\right)$, however as $\tilde{\alpha}_j^2 + \tilde{\beta}_j^2 = \left(\alpha_j^2 - \beta_j^2\right)^{-2} \neq 1$, therefore normalization is not possible.

Invertibility and complex coefficients

In order to establish the invertibility of $\boldsymbol{U}(\alpha,\beta)$, the domain of α and β must be be extended to the field of complex numbers because the conditions $\left|\alpha\right|^2 + \left|\beta\right|^2 = 1$ and $\alpha^2 - \beta^2 = 1 \Longrightarrow \left|\beta\right|^2 + \beta^2 = 0$ must be satisfied. This happens for α real and $\beta = -i\left|\beta\right|$, a pure imaginary. Since one is left with one free parameter, a natural parametrization is $\alpha = \cos\xi$ and $\beta = -i\sin\xi$ (ξ real), thus $\boldsymbol{U}(\alpha,\beta) \equiv \boldsymbol{U}(\xi) = \cos\xi\ \boldsymbol{I} - i\sin\xi\ \boldsymbol{X}$ is a unitary operator mapping a Cbit or a qubit into a qubit, $\boldsymbol{U}(\xi)\left|x_0\right\rangle = \cos\xi\ \left|x_0\right\rangle - i\sin\xi\ \left|\bar{x}_0\right\rangle$. So, the complex nature of $\boldsymbol{U}(\xi)$ is due to its invertibility property. A sequence of actions

$$\boldsymbol{U}_n\left(\vec{\xi}\right) = \prod_{j=1}^{n}\left(\cos\xi_j\ \boldsymbol{I} - i\sin\xi_j\ \boldsymbol{X}\right), \tag{3}$$

on a Cbit $\left|x_0\right\rangle$ takes it to the qubit $\boldsymbol{U}_n\left(\vec{\xi}\right)\left|x_0\right\rangle = A_n\left(\vec{\xi}\right)\left|x_0\right\rangle + B_n\left(\vec{\xi}\right)\left|\bar{x}_0\right\rangle = \left|\psi_n\right\rangle$, with coefficients $A_n\left(\vec{\xi}\right) = \cos\left(\sum_{j=1}^{n}\xi_j\right)$ and $B_n\left(\vec{\xi}\right) = -i\sin\left(\sum_{j=1}^{n}\xi_j\right)$. The parameters ξ_j are undetermined and their sum is $\phi_n = \sum_{j=1}^{n}\xi_j$, so one can write (3) in the compact form

$$\boldsymbol{U}_n\left(\vec{\xi}\right) \Longrightarrow \boldsymbol{U}(\phi_n) = \exp\left[-i\phi_n\ \boldsymbol{X}\right], \tag{4}$$

where ϕ_n is interpreted as a register parameter, it sets the ordering of the actions. Due to the indetermination of the parameters ξ_j nothing can be said about the intervals between consecutive actions, see Figure (1-a), the vertical bars stand for each action, they can be distributed at will, although obeying an ordered sequence. Imposing the composition law (2) one has $\boldsymbol{U}(\phi_n)\boldsymbol{U}(\phi_m) =$

FIG. 1: (a) Undetermined intervals between sequences of actions. (b) Uniformization of the intervals.

$\boldsymbol{U}\left(\phi_{n+m}\right)$, and the form (4) implies $\phi_n + \phi_m = \phi_{n+m}$; as so, necessarily and uniquely ϕ_n must be linear in n, $\phi_n = n\bar{\xi}$ with $\bar{\xi}$ some parameter. Thus $\boldsymbol{U}(\phi_n)$ becomes $\boldsymbol{U}\left(n\bar{\xi}\right) = \exp\left[-in\bar{\xi}\ \boldsymbol{X}\right]$, which stands for a sequence of actions, or an evolution. The previously undetermined intervals between actions become equally spaced, see Figure (1-b), characterizing the *uniformization* of their distribution. In order to turn the distribution dense I shall look for a differential equation for $\boldsymbol{U}\left(n\bar{\xi}\right)$ by taking first the difference between two consecutive values of n and then dividing by $\bar{\xi}$,

$$\frac{\boldsymbol{U}\left(\left(n+1\right)\bar{\xi}\right) - \boldsymbol{U}\left(n\bar{\xi}\right)}{\bar{\xi}} = \left(\frac{e^{-i\bar{\xi}\boldsymbol{X}} - 1}{\bar{\xi}}\right)\exp\left[-in\bar{\xi}\ \boldsymbol{X}\right].$$

The limit to a continuous parameter is obtained for $n \gg 1$ and $\bar{\xi} \ll 1$, keeping however the product $n\bar{\xi} = \tau$ finite. A linear differential equation results, $id\boldsymbol{U}(\tau)/d\tau = \boldsymbol{X}\boldsymbol{U}(\tau)$, and $\boldsymbol{U}(\tau) = e^{-i\tau\boldsymbol{X}}$, where τ is the continuous ordering parameter of the actions, or a local *time* in arbitrary units, that should be set according to the clock to be used. Writing $|x_\tau\rangle = \boldsymbol{U}(\tau)|x_0\rangle$ the evolution equation $id|x_\tau\rangle/d\tau = \boldsymbol{X}|x_\tau\rangle$ says how a qubit evolves due to the action of $\boldsymbol{X}$, which is the generator of the changes.

Defining a more general generator, $\boldsymbol{G} = \mu\boldsymbol{I} + \nu\boldsymbol{X}$, μ and ν being two real parameters, the evolution equation writes

$$i\frac{d|\psi_\tau\rangle}{d\tau} = \boldsymbol{G}|\psi_\tau\rangle, \tag{5}$$

with $\boldsymbol{U}(\tau) = e^{-i\tau\mu\boldsymbol{I}}e^{-i\tau\nu\boldsymbol{X}}$ for the evolution operator. Differently from the factor $e^{-i\tau\nu\boldsymbol{X}}$ that do really affect the evolution of a qubit, the phase factor $e^{-i\tau\mu\boldsymbol{I}}$ is apparently no significant because, besides a global phase factor, it does not entail any change when acting on Cbit or qubit. The eigenvalues and eigenstates of $\boldsymbol{G}$ are respectively $G_{\pm1} = \mu \pm \nu$, $|x_{\pm1}\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. A general solution to Eq. (5) is $|\psi_\tau\rangle = \sum_{\sigma=\pm1} e^{-iG_\sigma\tau}c_\sigma|x_\sigma\rangle$ where $G_\sigma = \mu + \sigma\nu$. Now conjecturing about the qubit carrier, I assume it a massive particle [10] and the parameter μ is chosen to represent its energy; thus the change $\boldsymbol{X} \rightarrow \boldsymbol{G}$ is important because it allows the introduction of that particle property. $\boldsymbol{G}$ can be identified as a hamiltonian, and for an arbitrary initial condition the mean value is $\langle\psi_\tau|\boldsymbol{G}|\psi_\tau\rangle = \mu + \nu\left(|c_{+1}|^2 - |c_{-1}|^2\right)$; while μ is the particle kinetic energy, the second term is the qubit energy that exists only when it is coupled to some field ($\nu \neq 0$).

THE QUBIT CARRIER AND THE PAULI-SCHRÖDINGER EQUATION

The spatial localization of the carrier must be introduced into Eq. (5), thus for a qubit state $|\psi_0\rangle = a_0|x_0\rangle + b_0|\bar{x}_0\rangle$ the parameters a_0, b_0 should depend on the position q, namely, $|\psi_0(q)\rangle = a_0(q)|x_0\rangle + b_0(q)|\bar{x}_0\rangle$ becomes the state of the whole system, with normalization $\int dq\ \ |a_0(q)|^2 + \int dq\ \ |b_0(q)|^2 = 1$. The qubit state is correlated to the particle position that influences its probability outcomes $|a_0(q)|^2$ and $|b_0(q)|^2$. Coordinate dependence should also be present in the generator, so $\boldsymbol{G}(q) = \mu(q)\boldsymbol{I} + \nu\boldsymbol{X}$ and the parameter ν is assumed q-independent because interaction between both degrees of freedom is not considered. The evolved state is $\boldsymbol{U}(\tau)|\psi_0(q)\rangle = |\psi(q,\tau)\rangle = a_\tau(q)|x_0\rangle + b_\tau(q)|\bar{x}_0\rangle$, with amplitudes

$$a_\tau(q) = e^{-i\tau\mu(q)}\left(a_0(q)\cos\nu\tau - ib_0(q)\sin\nu\tau\right) \tag{6}$$
$$b_\tau(q) = e^{-i\tau\mu(q)}\left(-ia_0(q)\sin\nu\tau + b_0(q)\cos\nu\tau\right). \tag{7}$$

with $a_0(q)$ and $b_0(q)$ as initial values. So, the qubit was merged with the spatial motion of its carrier within a single equation, meaning that the joint evolution – the qubit sequence of actions as well as the change in the spatial configuration of the carrier – is measured by a single clock. To determine the parameters $a_0(q)$ and $b_0(q)$ they should obey some differential equation for the variable q, then $\mu(q)$ must depend also on $\partial/\partial q$ and/or its powers. However, instead of trying to guess the functional form, it is better to take advantage of the available information from hamiltonian mechanics, so I define μ as the kinetic energy of a non-relativistic particle, $\mu \implies T(p) = p^2/2m$, where p is the linear momentum in some reference frame. Eq. (5) becomes $i\kappa_0 d\left|\tilde{\psi}(p,\tau)\right\rangle/d\tau = \left[T(p)\boldsymbol{I} + \varepsilon_0'\nu\boldsymbol{X}\right]\left|\tilde{\psi}(p,\tau)\right\rangle$. Since $T(p)$ has units of energy, the second term in brackets should also have the same units. So the constants κ_0 and ε_0', have both units of energy. One can also choose some unit to measure the dimensionless time τ, $\tau = t/t_0$, so the dynamical equation becomes

$$ih_0\frac{d\left|\tilde{\psi}(p,t)\right\rangle}{dt} = \left[T(p)\boldsymbol{I} + \varepsilon_0\boldsymbol{X}\right]\left|\tilde{\psi}(p,t)\right\rangle, \tag{8}$$

where $h_0 = \kappa_0 t_0$, $\varepsilon_0 = \varepsilon_0' t_0 \nu$. Note that the constant h_0 has units of energy $\times$ time and ε_0 has units of energy. An arbitrary initial condition assumes that the particle

momentum and the qubit state are correlated and the probability amplitude

$$\left|\tilde{\psi}_0\left(p,0\right)\right\rangle = \left|\tilde{\psi}_0\left(p\right)\right\rangle = \tilde{a}_0\left(p\right)\left|x_0\right\rangle + \tilde{b}_0\left(p\right)\left|\bar{x}_0\right\rangle, \quad (9)$$

depends on the particle momentum and it contains all the available information. In momentum space the evolution operator is $\boldsymbol{U}\left(t\right) = \exp\left[-it\left(T\left(p\right)\boldsymbol{I} + \varepsilon_0\nu\boldsymbol{X}\right)/h_0\right]$ and the solution to Eq. (8) is

$$\left|\tilde{\psi}\left(p,t\right)\right\rangle = e^{-itT(p)/h_0}\left[\tilde{a}_t\left(p\right)\left|x_0\right\rangle + \tilde{b}_t\left(p\right)\left|\bar{x}_0\right\rangle\right]$$

with $\tilde{a}_t\left(p\right) = \cos\left(\varepsilon_0\nu t/h_0\right)\tilde{a}_0\left(p\right) - i\sin\left(\varepsilon_0\nu t/h_0\right)\tilde{b}_0\left(p\right)$ and $\tilde{b}_t\left(p\right) = \cos\left(\varepsilon_0\nu t/h_0\right)\tilde{b}_0\left(p\right) - i\sin\left(\varepsilon_0\nu t/h_0\right)\tilde{a}_0\left(p\right)$, and the particle mean energy is $\left\langle\tilde{\psi}\left(p,t\right)\right|\boldsymbol{H}\left(p\right)\left|\tilde{\psi}\left(p,t\right)\right\rangle = T\left(p\right) + 2\varepsilon_0 Re\left(\tilde{a}_0^*\left(p\right)\tilde{b}_0\left(p\right)\right)$. Since coordinate and momentum are conjugated variables the statevector in coordinate representation is

$$\left|\psi\left(q,t\right)\right\rangle = \psi_{x_0}\left(q,t\right)\left|x_0\right\rangle + \psi_{\bar{x}_0}\left(q,t\right)\left|\bar{x}_0\right\rangle,$$

and $\psi_{x_0}\left(q,t\right)$, $\psi_{\bar{x}_0}\left(q,t\right)$ are the amplitudes associated to the Cbits $\left|x_0\right\rangle$, $\left|\bar{x}_0\right\rangle$; they can be written as Fourier transforms

$$\psi_{\binom{x_0}{\bar{x}_0}}\left(q,t\right) = \int\frac{dp}{2\pi}e^{ipq/h_1}e^{-itT(p)/h_0}\binom{\tilde{a}_t\left(p\right)}{\tilde{b}_t\left(p\right)}. \quad (10)$$

The constant h_1 is introduced to set the correct dimensionality, it has the same units as h_0, nonetheless nothing can be said about being the same constant, unless confirmed by experiment. In Eq. (10)

$$\binom{\tilde{a}_t\left(p\right)}{\tilde{b}_t\left(p\right)} = \int dq' e^{-ipq'/h_1}\binom{a_t\left(q'\right)}{b_t\left(q'\right)}, \quad (11)$$

with $a_t\left(q'\right) = \cos\left(\varepsilon_0\nu t/h_0\right)a_0\left(q'\right) - i\sin\left(\varepsilon_0\nu t/h_0\right)b_0\left(q'\right)$ and $b_t\left(q'\right) = \cos\left(\varepsilon_0\nu t/h_0\right)b_0\left(q'\right) - i\sin\left(\varepsilon_0\nu t/h_0\right)a_0\left(q'\right)$. So even not existing a direct interaction between the qubit and its carrier, the probability for measuring the qubit in Cbit $\left|x_0\right\rangle$, or $\left|\bar{x}_0\right\rangle$, becomes affected by its position.

Using Eqs. (10) and (11) and manipulating Eq. (8) it is not hard to verify that one can substitute the c-number p by the derivative $-ih_1\partial/\partial q$, and we can rewrite that equation as

$$ih_0\frac{\partial\left|\psi\left(q,t\right)\right\rangle}{\partial t} = \left[\frac{1}{2m}\left(-ih_1\frac{\partial}{\partial q}\right)^2\boldsymbol{I} + \varepsilon_0\nu\boldsymbol{X}\right]\left|\psi\left(q,t\right)\right\rangle. \quad (12)$$

The terms in brackets stand for the particle and qubit hamiltonian in coordinate and matrix representation, so the parameter μ becomes determined. In the presence of an energy conserving potential $V\left(q\right)$ the PSE takes its familiar form, with hamiltonian $\boldsymbol{H} = H_0\boldsymbol{I} + \varepsilon_0\nu\boldsymbol{X}$ and $H_0 = \left[\left(-ih_1/(2m)\right)\left(\partial^2/\partial^2 q\right) + V\left(q\right)\right]$. The particle described by Eq. (12) has now blurred classical properties (it looses the sharp trajectory it has in phase space), its best representation is a wavefunction and the appearance of quantum properties are due to the qubit it is carrying. Any further generalization is trivial and immediate: (1) from 1-D to 3-D in spacial coordinates, $\partial/\partial q \to \nabla$ and (2) since any 2×2 matrix can be expanded in the basis formed by the unit matrix $\boldsymbol{I}$ and Pauli matrices (σ_x, σ_y, and σ_z), then $\nu\boldsymbol{X} \longrightarrow \vec{\nu}\cdot\vec{\boldsymbol{\sigma}}$.

DIRAC EQUATION: TWO QUBITS OF INFORMATION

Few words about Dirac equation $i\hbar\partial\left|\Psi_D\left(t\right)\right\rangle/\partial t = \boldsymbol{H}_D\left|\Psi_D\left(t\right)\right\rangle$, its hamiltonian is $\boldsymbol{H}_D = c\vec{\boldsymbol{\alpha}}\cdot\vec{p} + mc^2\boldsymbol{\beta}$ and the four dimension-4 matrices $\vec{\boldsymbol{\alpha}}$, $\boldsymbol{\beta}$ satisfy the relations $\boldsymbol{\alpha}_k\boldsymbol{\alpha}_l + \boldsymbol{\alpha}_l\boldsymbol{\alpha}_k = 2\boldsymbol{I}\delta_{kl}$, $\vec{\boldsymbol{\alpha}}\boldsymbol{\beta} + \boldsymbol{\beta}\vec{\boldsymbol{\alpha}} = \boldsymbol{0}$ and $\boldsymbol{\beta}^2 = 0$. These matrices can be expressed as tensor products of dimension-2 matrices, each one acting on its own qubit, $\vec{\boldsymbol{\alpha}} = \boldsymbol{X_1}\otimes\left(\boldsymbol{X_2},i\boldsymbol{Y_2},\boldsymbol{Z_2}\right) = \boldsymbol{X}_1\otimes\vec{\boldsymbol{\sigma}}_2$, so $c\vec{\boldsymbol{\alpha}}\cdot\vec{p} = \boldsymbol{X}_1\otimes\left(c\vec{p}\cdot\vec{\boldsymbol{\sigma}}_2\right)$ and $\boldsymbol{\beta} = \boldsymbol{Z}_1\otimes\boldsymbol{I}_2$. Thus, Dirac's hamiltonian can be written as tensor products acting on independent D-2 Hilbert subspaces $\boldsymbol{H}_D = \boldsymbol{Z}_1\otimes\left(mc^2\boldsymbol{I}_2\right) + \boldsymbol{X}_1\otimes\left(c\vec{p}\cdot\vec{\boldsymbol{\sigma}}_2\right)$. Squaring $\boldsymbol{H}_D$ one gets the relativistic energy $\left(\boldsymbol{H}_D\right)^2 = E_p^2\ \boldsymbol{I}_1\otimes\boldsymbol{I}_2$, where $E_p^2 = m^2c^4 + c^2\vec{p}^{\,2}$. The time-dependent equation reduces into direct products of 2×2 matrices

$$\left[\boldsymbol{I}_1\otimes\left(i\hbar\frac{\partial}{\partial t}\boldsymbol{I}_2\right) - \boldsymbol{Z}_1\otimes\left(mc^2\boldsymbol{I}_2\right) - \boldsymbol{X}_1\otimes\left(c\vec{p}\cdot\vec{\boldsymbol{\sigma}}_2\right)\right] \times\ \left|\Psi_D\left(t\right)\right\rangle = 0,$$

which is invariant under Lorentz transformation. The solutions are

$$\begin{aligned}\left|\Psi^\lambda\left(\vec{p},t\right)\right\rangle &= N_\lambda e^{-i\lambda tE_p}\left[\left|1\right\rangle_1\left|\varphi\left(\vec{p}\right)\right\rangle_2\right. \\ &+\ \left.\left|0\right\rangle_1\frac{c\vec{p}\cdot\vec{\boldsymbol{\sigma}}_2}{mc^2+\lambda E_p}\left|\varphi\left(\vec{p}\right)\right\rangle_2\right], \quad (13)\end{aligned}$$

where

$$\left|\varphi\left(\vec{p}\right)\right\rangle_2 = \begin{pmatrix}\varphi_+\left(\vec{p}\right)\\ \varphi_-\left(\vec{p}\right)\end{pmatrix}_2$$

with $\lambda = \pm 1$ standing for positive and negative energy solutions and N_λ is a normalization constant. The qubit 2 in Eq. (13) represents the particle state whereas the Cbit 1 is apparently ancillary, it works as a selector: the projector $\left(\left|1\right\rangle\left\langle 1\right|\right)_1$ selects the nonrelativistic component $\left|\varphi\left(\vec{p}\right)\right\rangle_2$ while $\left(\left|0\right\rangle\left\langle 0\right|\right)_1$ projects the relativistic complement. Also interesting is that all the γ_μ matrices have the structure of the direct product of two-qubit operators $\gamma^0 = \boldsymbol{Z}_1\otimes\boldsymbol{I}_2$, $\gamma^1 = i\boldsymbol{Y}_1\otimes\boldsymbol{X}_2$, $\gamma^2 = -\boldsymbol{Y}_1\otimes\boldsymbol{Y}_2$, $\gamma^3 = i\boldsymbol{Y}_1\otimes\boldsymbol{Z}_2$.

CONCLUDING REMARKS

As long as the qubit is not probed (for a spin, there is no external magnetic field), $\nu = 0$, Eq. (12) reduces to two uncoupled Schrödinger equations

$$ih_0 \frac{\partial}{\partial t} \begin{pmatrix} \psi_{x_0}(q,t) \\ \psi_{\bar{x}_0}(q,t) \end{pmatrix} = H_0 \begin{pmatrix} \psi_{x_0}(q,t) \\ \psi_{\bar{x}_0}(q,t) \end{pmatrix}, \qquad (14)$$

for two wavefunctions $\psi_{x_0}(q,t)$, $\psi_{\bar{x}_0}(q,t)$ in one qubit/spin eigenstate. The two equations are redundant thus the relevant information resides in one of them only, going to the usual spinless SE. Although classical physics was crucial to arrive at Eq. (14), by setting $\nu = 0$ the particle motion is not ruled (back) by classical physics (Hamilton equations) but by the usual SE. So why do quantum properties of the particle still persist even when the correlation between a qubit/spin and its carrier is broken? The answer is that even not being activated the qubit/spin is still carried by the particle, the internal degree of freedom and the particle make one single object, although not entangled. One is left with an equation (SE) that does not keep any clue about the presence of a qubit/spin, nonetheless it is still there although not manifestly evident. That´s why the SE can be used without any mention to spin if not needed; otherwise, this internal degree of freedom must be appended in order to explain observed phenomena. In conclusion, because it is carrying one qubit of information, to the observer the particle shifts its behavior from the classical picture, it acquires wave properties with a probabilistic character where the uncertainty relations represent one facet.

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Figure captions

Fig. 1. (a) Undetermined intervals between sequences of actions. (b) Uniformization of the intervals.

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