Classification of the Entangled States $2 \times M \times N$

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Abstract

We extend the matrix decomposition method(MDM) in classifying the $2\times N\times N$ truly entangled states to $2 \times M \times N$ system under the condition of stochastic local operations and classical communication(SLOCC). It is found that the MDM is quite practical and convenient in operation for the asymmetrical tripartite states, and an explicit example of the classification of $2 \times 6 \times 7$ quantum system is presented.

1 Introduction

Entanglement is an essential feature of quantum theory, describing a quantum correlation that exhibits nonlocal properties. In the seminal work [1], Einstein, Podolsky, and Rosen (EPR) demonstrated through a Gedanken experiment that the quantum mechanics (QM) can not provide a complete description of the "physical reality" for two spatially separated but quantum mechanically correlated particles state which is now known as entangled state. The subsequent Bell theorem manifest the nonlocal character of the quantum correlation in the violation of Bell's inequalities [2]. As the quantum information science develops, the impact of entanglement goes far beyond the testing of the conceptual foundations of QM. Entanglement is now of central importance in the

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quantum information theory (QIT) and is thought as the key physical resource to realize quantum information tasks, such as quantum cryptography [3, 4], superdense coding [5, 6], and quantum computation [7], etc. This necessitates the qualitative and quantitative description of the entanglement [8]. However due to the lack of suitable tools for characterizing the entanglement, very limited quantum state space was explored in the quantum information theory.

In quantum information processing (QIP), two states are suited to implement the same task if they can be mutually converted by Stochastic Local Operations and Classical Communication (SLOCC) [9], and therefore they are said to be in the same equivalent class. For three qubits, known result is that there are two kinds of true tripartite entanglement classes for pure state, namely, GHZ and W states [9]. As the dimensions of each party increases nontrivial aspect shows up, i.e., non-local parameters may resides in the entangled states of $2 \times N \times N$ system when $N \geq 4$ [10, 11]. Many investigations concerned the classifications of $2 \times M \times N$ states has been done in [10, 12, 13]. In the Refs.[10, 12], an iterated method was introduced to determine all the inequivalent classes of the entangled states of $2 \times M \times N$ system based on the "range criterion", where the entanglement classification of the low dimension system is a prerequisite for the high dimensions ones. Practical classifications of dimensions up to $2 \times 4 \times 4$ and the related systems of $2 \times (M+4) \times (2M+4)$ were given in [10]. With the increasing of dimensions, the complexity of the method grows dramatically because of the iterated nature of their inequivalent proof of the entanglement classes. In a recent work [11] a novel method of classifying the pure state of $2 \times N \times N$ systems was introduced in which all the inequivalent true tripartite entanglement classes can be determined directly by using merely the elementary operations on the cubic grid form of the state.

The present work deals with the more general case: quantum state of $2 \times M \times N$ systems (pure state if not specified). We show that the method we introduced in [11] can be generalized to the classification of true entangled states of $2 \times M \times N$ systems. And therefore all the inequivalent classes can be generated directly and no followed-up inequivalence proof of these classes is needed. The content goes as follows, in section 2, by representing the $2 \times M \times N$ state in the form of matrix pairs, the $2 \times M \times N$ states are

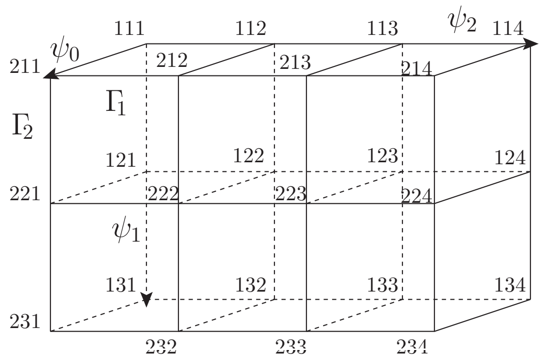

Figure 1: The cubic form for $2 \times 3 \times 4$ state.

divide into inequivalent sets under SLOCC. The detailed classification procedures with these inequivalent sets are presented in section 3 and a concrete example of classification of $2 \times 6 \times 7$ system is given. Finally, in section 4 we give some concluding remarks.

2 Matrix pair representation of $2 \times M \times N$ state

Adopt the conventions of [11], an arbitrary state of $2 \times M \times N$ can be written as

$$|\Psi_{2\times M\times N}\rangle = \mathrm{Tr}[\Gamma_{\{i,j,k\}}\,\psi_2 \otimes \psi_1^T \otimes \psi_0^T]\,, \tag{1}$$

where, ψ_0 represents the first qubit, ψ_1 and ψ_2 has the dimension of M and N separately; $\Gamma_{\{1,j,k\}}$ and $\Gamma_{\{2,j,k\}}$ are $M \times N$ complex matrices (we assume $M \leq N$ without loss of generalities). Then the state can be written in the following compact form

$$|\Psi_{2\times M\times N}\rangle = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} . \tag{2}$$

Clearly, to every state of $2 \times M \times N$, there is a form of Eq.(2) that corresponds to it, and a pictorial description of the state is straightforward, see Fig.(1).

The reduced density matrix of state $\Psi_{2\times M\times N}$ is defined as $\rho_\alpha = \mathrm{Tr}_{\beta\gamma}|\Psi\rangle\langle\Psi|$, where α, β, γ can be ψ_0, ψ_1, ψ_2. For three-partite systems, ture (genuine [9]) entanglement means that the determinant of the reduced density matrix of each partite is nonzero. For the $2 \times M \times N$ systems, this is equivalent to $r(\rho_{\psi_0}) = 2, r(\rho_{\psi_1}) = M, r(\rho_{\psi_2}) = N$. The density matrix in the form of the matrix pairs can be expressed as

$$\rho_{\psi_0,\psi_1,\psi_2} = (\Gamma_i)_{jk}(\Gamma_{i'})^*_{j'k'} \ , \tag{3}$$

where $i, i' = 1, 2; j, j' = 1, 2, \cdots, M; k, k' = 1, 2, \cdots, N$. Then the reduced density matrix (take ψ_2 as an example) is

$$\begin{aligned}\rho_{\psi_2} &= \mathrm{Tr}_{\psi_0,\psi_1}(\rho_{\psi_0,\psi_1,\psi_2}) \\ &= \sum_{ij} (\Gamma_i)^*_{jk'}(\Gamma_i)_{jk} \\ &= \sum_i \Gamma_i^\dagger \Gamma_i \ .\end{aligned} \tag{4}$$

We can infer that if $r(\rho_{\psi_2}) < N$ ($\det(\rho_{\psi_2}) = 0$), there will be ILOs that transform ρ_{ψ_2} to ρ'_{ψ_2} who has at least one column or one row of zeros. Without loss of generalities suppose the kth column of ρ'_{ψ_2} are zeros, for the element (k, k) of ρ'_{ψ_2} we would get

$$\begin{aligned}(\rho'_{\psi_2})_{kk} &= \sum_{ij} |(\Gamma'_i)_{jk}|^2 \\ &= 0 \ ,\end{aligned} \tag{5}$$

which indicates that $(\Gamma'_i)_{jk} = 0$ for all i and j. In the pictorial description of the $2 \times M \times N$ state, this corresponds to the case that the cubic grid has a whole plane of zero coefficients. Clearly in this condition the entanglement of $2 \times M \times N$ system reduces to the case of $2 \times M' \times N'$ with $M' < M$ or/and $N' < N$ which should in principle be considered as an entanglement system of $2 \times M' \times N'$.

3 Classification of $2 \times M \times N$ State

Two $2 \times M \times N$ states $\widetilde{\Psi}$ and Ψ are said to be SLOCC equivalent if they are connected via invertible local operator (ILO). That is $\widetilde{\Psi}$ is SLOCC equivalent to Ψ if

$$|\widetilde{\Psi}_{2\times M\times N}\rangle = T \otimes P \otimes Q \, |\Psi_{2\times M\times N}\rangle \ , \tag{6}$$

where T, P, Q are invertible complex matrices of dimension 2×2, $M \times M$, and $N \times N$ which act on ψ_0, ψ_1, ψ_2, respectively. Neglecting the extra factor of the determinant of matrices,

T, P, and Q correspond to the special linear groups of $SL(2,\mathbb{C}), SL(M,\mathbb{C}), SL(N,\mathbb{C})$ [9]. Take $|\Psi_{2\times M\times N}\rangle$ in the form of Eq.(2), the ILO operators T, P, Q in Eq.(6) can be written as

$$|\widetilde{\Psi}_{2\times M\times N}\rangle = \left(\begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array}\right)\left(\begin{array}{c} P\Gamma_1 Q \\ P\Gamma_2 Q \end{array}\right) . \tag{7}$$

From Eq.(2) and Eq.(7) we can see that the SLOCC equivalence of the quantum state turns to the connectivity of the matrix pairs (Γ_1, Γ_2) under the special linear transformations T, P, Q. Define the set that contains all the matrices pair (Γ_1, Γ_2) as C. The whole space of C can be partitioned into numbers of inequivalent sets with different n, l

$$C_{n,\,l} = \{(\Gamma_1,\Gamma_2)|\, r_{\max}(\alpha_1\Gamma_1+\beta_1\Gamma_2) = n, r_{\min}(\alpha_2\Gamma_1+\beta_2\Gamma_2) = l\} \;, \tag{8}$$

where $r_{\max}$ and $r_{\min}$ represent the the maximum and minimum rank of the matrices respectively, and we let r denote the rank of matrix hereafter; $\alpha_i, \beta_i \in \mathbb{C}$ and $|\alpha_i|+|\beta_i| \neq 0$; $l \in [0, n]$, $n \in [0, M]$.

3.1 Classification on sets $C_{n,l}$ with $n = M$

We start our classification of $C_{n,\,l}$ in $2\times M\times N$ system from the case $n = M$. Our aim is to construct the subsets $c_{M,l}$ which: (i), it includes representative elements of all the inequivalent entanglement classes; (ii), each inequivalent class has only one representative element in $c_{M,l}$.

$\forall$ $(\Gamma_1, \Gamma_2) \in C_{M,\,l}$ there always exists an ILO operator T that

$$T\left(\begin{array}{c} \Gamma_1 \\ \Gamma_2 \end{array}\right) = \left(\begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array}\right)\left(\begin{array}{c} \Gamma_1 \\ \Gamma_2 \end{array}\right) , \tag{9}$$

where $r(t_{11}\Gamma_1 + t_{12}\Gamma_2) = M$, $r(t_{21}\Gamma_1 + t_{22}\Gamma_2) = l$. So we just assume $r(\Gamma_1) = M$ and $r(\Gamma_2) = l$ without loss of generality. Two specific ILOs P and Q can transform (Γ_1, Γ_2) into the following form

$$\left(\begin{array}{c} \Gamma_1 \\ \Gamma_2 \end{array}\right) \xrightarrow{P,Q} \left(\begin{array}{c} \left(\begin{array}{cc} E_{M\times M} & \mathbb{0}_{M\times(N-M)} \end{array}\right) \\ \left(\begin{array}{cc} A_{M\times M} & B_{M\times(N-M)} \end{array}\right) \end{array}\right) . \tag{10}$$

where E is an unitary submatrix, $\mathbb{0}$ hereafter represents zero submatrix, A and B have the same dimensions as E, $\mathbb{0}$ separately, and all of them have the subscripts as their dimensions. If $(N-M)>M$, even if its submatrix B has the maximum rank of M, the right hand of Eq.(10) can still be further transformed by ILOs into

$$\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \xrightarrow{P,Q} \begin{pmatrix} \begin{pmatrix} E_{M\times M} & \mathbb{0}_{M\times(N-2M)} & \mathbb{0}_{M\times M} \end{pmatrix} \\ \begin{pmatrix} \mathbb{0}_{M\times M} & \mathbb{0}_{M\times(N-2M)} & E_{M\times M} \end{pmatrix} \end{pmatrix}, \tag{11}$$

In the form of the cubic grid (Fig.(1)), this corresponds to that $(N-2M)$ vertical planes in the middle of the cube are zero planes, which is actually an entangled states of $2\times M\times 2M$ according to the statement below Eq.(5). Thus here we only consider the case $M\geq N/2$.

For arbitrary matrix pair with the form of the right hand of Eq.(10), we can implement the following transformation by ILOs

$$\begin{pmatrix} \begin{pmatrix} E_{m\times m} & \mathbb{0}_{m\times(n-m)} \end{pmatrix} \\ \begin{pmatrix} A_{m\times m} & B_{m\times(n-m)} \end{pmatrix} \end{pmatrix} \xrightarrow{\text{step i}} \begin{pmatrix} \begin{pmatrix} E_{1A'} & \mathbb{0}_{1B'} & \mathbb{0}_{1a} \\ \mathbb{0}_{1b} & E_{1'} & \mathbb{0}_{1E'} \end{pmatrix} \\ \begin{pmatrix} A' & B' & \mathbb{0}_{2a} \\ \mathbb{0}_{2b} & \mathbb{0}_{2c} & E' \end{pmatrix} \end{pmatrix}, \tag{12}$$

where $r(E')=r(B)$ and E' is submatrix with typical form of $(\mathbb{0}_{r(E')\times((n-m)-r(E'))}, E_{r(E')\times r(E')})$, A', $E_{1A'}$ are also square submatrices with the dimensions $(m-r(E'))\times(m-r(E'))$; the rest of the matrices are partitioned accordingly, i.e., $\mathbb{0}_{1B'}$, B' have the dimension $(m-r(E'))\times r(E')$, $\mathbb{0}_{1a}$, $\mathbb{0}_{2a}$ have the dimension of $(m-r(E'))\times(n-m)$, $\mathbb{0}_{1b}$, $\mathbb{0}_{2b}$ have the dimension $r(E')\times(m-r(E'))$, $E_{1'}$, $\mathbb{0}_{2c}$ have the dimension $r(E')\times r(E')$. After this transformation $\Gamma_1=(E_{M\times N},\mathbb{0}_{M\times(N-M)})$ is unchanged, Γ_2 becomes a quasidiagonal matrix and we named this procedure step i.

Next we repartitioned the matrices on the left hand side of Eq.(12) as follows

$$\begin{pmatrix} \begin{pmatrix} E_{1A'} & \mathbb{0}_{1B'} & \mathbb{0}_{1a} \\ \mathbb{0}_{1b} & E_{1'} & \mathbb{0}_{1E'} \end{pmatrix} \\ \begin{pmatrix} A' & B' & \mathbb{0}_{2a} \\ \mathbb{0}_{2b} & \mathbb{0}_{2c} & E' \end{pmatrix} \end{pmatrix} \xrightarrow{\text{step ii}} \begin{pmatrix} \left(\begin{array}{cc|c} E_{1A'} & \mathbb{0}_{1a} & \mathbb{0}_{1b} \\ \hline \mathbb{0}_{1c} & E_{1'} & \mathbb{0}_{1d} \end{array}\right) \\ \left(\begin{array}{cc|c} A' & B' & \mathbb{0}_{2a} \\ \hline \mathbb{0}_{2b} & \mathbb{0}_{2c} & E' \end{array}\right) \end{pmatrix}. \tag{13}$$

This is named as step ii. Consider the submatrix B', if it is not identically zero we can perform the transformation of step i on the left top submatrices of Eq.(13)

$$\begin{pmatrix} \begin{pmatrix} E_{1A'} & \mathbb{0}_{1a} \end{pmatrix} \\ \begin{pmatrix} A' & B' \end{pmatrix} \end{pmatrix} \xrightarrow{\text{step i}} \begin{pmatrix} \begin{pmatrix} E_{1A''} & \mathbb{0}_{1B''} & \mathbb{0}_{1a'} \\ \mathbb{0}_{1b'} & E_{1''} & \mathbb{0}_{1E''} \end{pmatrix} \\ \begin{pmatrix} A'' & B'' & \mathbb{0}_{2a} \\ \mathbb{0}_{2b'} & \mathbb{0}_{2c'} & E'' \end{pmatrix} \end{pmatrix}. \tag{14}$$

This procedure can be done repeatedly (suppose repeat n times), until the $B^{(n)}$ is identically zero or has zero dimension. We can get that the matrix pair (Γ_1, Γ_2) can be transformed into the following form

$$\Gamma_1 \to \left(\begin{array}{c|ccccc} E_{1A^{(n)}} & \mathbb{0} & \mathbb{0} & \cdots & \mathbb{0} & \mathbb{0} \\ \hline \mathbb{0} & E_{1(n-1)} & \mathbb{0} & \cdots & \mathbb{0} & \mathbb{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \cdots & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \cdots & E_{1'} & \mathbb{0} \end{array}\right) \equiv \left(\begin{array}{cc} E & \mathbb{0} \\ \mathbb{0} & E_1 \end{array}\right) , \tag{15}$$

$$\Gamma_2 \to \left(\begin{array}{c|ccccc} A^{(n)} & B^{(n)} = \mathbb{0} & \mathbb{0} & \cdots & \mathbb{0} & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & E^{(n-1)} & \cdots & \mathbb{0} & \mathbb{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \cdots & E'' & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \cdots & \mathbb{0} & E' \end{array}\right) \equiv \left(\begin{array}{cc} SJS^{-1} & \mathbb{0} \\ \mathbb{0} & E_2 \end{array}\right) , \tag{16}$$

where the transformed Γ_1 is just $(E_{M\times M}, \mathbb{0}_{M\times(N-M)})$, and $E_{1,2}$ are defined according to the partition lines.

As a concrete example here we show how this whole procedure is proceeded on the sets of $C_{4,l}$ of $2\times 4\times 6$ state. The transformation of Eq.(12) is start with

$$\left(\begin{array}{c} \left(\begin{array}{cc} E_{4\times 4} & \mathbb{0}_{4\times 2} \end{array}\right) \\ \left(\begin{array}{cc} A_{4\times 4} & B_{4\times 2} \end{array}\right) \end{array} \right) \xrightarrow{\text{step i}} \left(\begin{array}{c} \left(\begin{array}{ccc} E_{1A'} & \mathbb{0}_{1B'} & \mathbb{0}_{1a} \\ \mathbb{0}_{1b} & E_{1'} & \mathbb{0}_{1c} \end{array}\right) \\ \left(\begin{array}{ccc} A' & B' & \mathbb{0}_{2a} \\ \mathbb{0}_{2b} & \mathbb{0}_{2c} & E' \end{array}\right) \end{array} \right) , \tag{17}$$

where

$$\left(\begin{array}{c} \left(\begin{array}{ccc} E_{1A'} & \mathbb{0}_{1B'} & \mathbb{0}_{1a} \\ \mathbb{0}_{1b} & E_{1'} & \mathbb{0}_{1c} \end{array}\right) \\ \left(\begin{array}{ccc} A' & B' & \mathbb{0}_{2a} \\ \mathbb{0}_{2b} & \mathbb{0}_{2c} & E' \end{array}\right) \end{array} \right) = \left(\begin{array}{c} \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \\ \left(\begin{array}{cc|cc|cc} \times & \times & + & + & 0 & 0 \\ \times & \times & + & + & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} \right) , \tag{18}$$

Here the rank of $B_{4\times 2}$ must be 2 due to the argument below Eq.(11). The step ii goes as follows

$$\left(\begin{array}{c} \left(\begin{array}{ccc} E_{1A'} & \mathbb{0}_{1B'} & \mathbb{0}_{1a} \\ \mathbb{0}_{1b} & E_{1'} & \mathbb{0}_{1c} \end{array}\right) \\ \left(\begin{array}{ccc} A' & B' & \mathbb{0}_{2a} \\ \mathbb{0}_{2b} & \mathbb{0}_{2c} & E' \end{array}\right) \end{array} \right) \xrightarrow{\text{step ii}} \left(\begin{array}{c} \left(\begin{array}{cc|c} E_{1A'} & \mathbb{0}_{1B'} & \mathbb{0}_{1a} \\ \hline \mathbb{0}_{1b} & E_{1'} & \mathbb{0}_{1c} \end{array}\right) \\ \left(\begin{array}{cc|c} A' & B' & \mathbb{0}_{2a} \\ \hline \mathbb{0}_{2b} & \mathbb{0}_{2c} & E' \end{array}\right) \end{array} \right) . \tag{19}$$

Next we repeat the step i to the up-left submatrices of the right hand side of Eq.(19). This iteration of step i depends on the rank of B'.

(1), $r(B')=0$. In this case the matrix pair (Γ_1,Γ_2) become

$$\left(\begin{array}{c}\Gamma_1\\ \Gamma_2\end{array}\right) \xrightarrow{T,P,Q} \left(\begin{array}{c} \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \\ \left(\begin{array}{cc|cc|cc} \times & \times & 0 & 0 & 0 & 0\\ \times & \times & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array}\right). \tag{20}$$

And there are three different forms, i.e.,

$$(1.1)\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix},(1.2)\begin{bmatrix} \lambda&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix},(1.3)\begin{bmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix},\tag{21}$$

correspond to two Jordan canonical forms of A', $J=\begin{bmatrix}\lambda & 0\\ 0 & 0\end{bmatrix} J=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, and a zero matrix $A'=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

(2), $r(B')=1$. In this case

$$\left(\begin{array}{ccc} A' & B' & \mathbb{0}\\ \mathbb{0} & \mathbb{0} & E' \end{array}\right) \xrightarrow{\text{step i}} \left(\begin{array}{cc|cc|c} \times & \times & 0 & 0 & \mathbb{0}\\ 0 & 0 & 0 & 1 & \mathbb{0}\\ \hline \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} & E' \end{array}\right). \tag{22}$$

$$\left(\begin{array}{cc|cc|c} \times & \times & 0 & 0 & \mathbb{0}\\ 0 & 0 & 0 & 1 & \mathbb{0}\\ \hline \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} & E' \end{array}\right) \xrightarrow{\text{step ii}} \left(\begin{array}{cc|cc} A'' & B'' & \mathbb{0} & \mathbb{0}\\ \hline 0 & 0 & E'' & \mathbb{0}\\ \mathbb{0} & \mathbb{0} & \mathbb{0} & E' \end{array}\right), \tag{23}$$

where A'', B'' are matrices of 1×1 and $E''=(0,1)$. Again apply step i on $(A''\ B'')$ we have

(2.1), $r(B'')=0$

$$\left(\begin{array}{cc|cc} A'' & B'' & \mathbb{0} & \mathbb{0}\\ \hline 0 & 0 & E'' & \mathbb{0}\\ \mathbb{0} & \mathbb{0} & \mathbb{0} & E' \end{array}\right) \xrightarrow{\text{step i}} \left(\begin{array}{cc|cc} \times & 0 & \mathbb{0} & \mathbb{0}\\ \hline 0 & 0 & E'' & \mathbb{0}\\ \mathbb{0} & \mathbb{0} & \mathbb{0} & E' \end{array}\right). \tag{24}$$

(2.2) $r(B'') = 1$

$$\left(\begin{array}{cc|cc} A'' & B'' & \mathbb{0} & \mathbb{0} \\ \hline 0 & 0 & E'' & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & E' \end{array}\right) \xrightarrow{\text{step i}} \left(\begin{array}{cc|cc} 0 & 1 & \mathbb{0} & \mathbb{0} \\ \hline 0 & 0 & E'' & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & E' \end{array}\right) . \tag{25}$$

For Eq.(24), A'' is equivalent to the case of $A'' = 0$ according to *theorem 1* of [11]. For Eq.(25), in the next step of step ii, $B^{(3)}$ will be a matrix of dimension zero, and satisfies $r(B^{(3)}) = 0$, thus the procedure is stopped. We get two inequivalent forms of Γ_2

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{26}$$

(3). $r(B') = 2$. In this case

$$\left(\begin{array}{ccc} A' & B' & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & E' \end{array}\right) \xrightarrow{\text{step i}} \left(\begin{array}{cc|cc|c} 0 & 0 & 1 & 0 & \mathbb{0} \\ 0 & 0 & 0 & 1 & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} & E' \end{array}\right) . \tag{27}$$

Thus here is only one class, where Γ_2 has just the form of Eq.(27). In the following, we shall see that these six cases correspond to the six inequivalent entanglement classes in $2 \times 4 \times 6$ systems, which agrees with the result of Ref.[12].

In all, for every $(\Gamma_1, \Gamma_2) \in C_{M,\,l}$, there exists an ILO transformation that make

$$\left(\begin{array}{c} \Gamma_1' \\ \Gamma_2' \end{array}\right) = T \otimes P \otimes Q \left(\begin{array}{c} \Gamma_1 \\ \Gamma_2 \end{array}\right) . \tag{28}$$

Here Γ_1' has the form of Eq.(15), and $\Gamma_2' = \left(\begin{array}{cc} J & 0 \\ 0 & E_2 \end{array}\right)$ has the form of Eq.(16). Eq.(28) maps $C_{M,l}$ to $c_{M,l}$, where $c_{M,\,l} \subseteq C_{M,\,l}$ and

$$c_{M,l} = \{(\Gamma_1, \Gamma_2) | \Gamma_1 = \left(\begin{array}{cc} E & \mathbb{0} \\ \mathbb{0} & E_1 \end{array}\right), \Gamma_2 = \left(\begin{array}{cc} J & \mathbb{0} \\ \mathbb{0} & E_2 \end{array}\right); (\Gamma_1, \Gamma_2) \in C_{M,l}\} . \tag{29}$$

Thus we have separated the classification of $C_{M,l}$ into two procedures: 1, the construction of E_2 matrix; 2, classification of J. And for the second procedure, we have already complete the classification in [11]. We have the following theorem

Theorem 1 $\forall$ $(\Gamma_1, \Gamma_2) \in c_{M,\,l}$, *the set* $c_{M,\,l}$ *is of the classification of* $C_{M,\,l}$. (i) *if two states are SLOCC equivalent then they can be transformed into the same matrix vector*

(Γ_1, Γ_2)*;* (ii) *this matrix vector is unique in* $c_{M,l}$*, that is if* (Γ_1, Γ_2') *is SLOCC equivalent with* (Γ_1, Γ_2)*, then* $(\Gamma_1, \Gamma_2') = (\Gamma_1, \Gamma_2)$*,* ($\Gamma_2' = \Gamma_2$ *means that* $E_2 = E_2'$ *and their Jordan forms of* J *are equivalent under the condition of theorem 1 Ref.[11]*)

Proof:

(i) The proof of this statement is straightforward, since in every step of transformation only invertible operators take part in.

(ii) Suppose

$$\begin{pmatrix} \Gamma_1 \\ \Gamma_2' \end{pmatrix} = T' \otimes P' \otimes Q' \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} . \tag{30}$$

First we show that the T' transformations can always be replaced by ILO operators P_0^{-1}, Q_0^{-1}, i.e.,

$$\begin{bmatrix} t_{11}' & t_{12}' \\ t_{21}' & t_{22}' \end{bmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} P_0^{-1}\Gamma_1 Q_0^{-1} \\ P_0^{-1}\Gamma_2 Q_0^{-1} \end{pmatrix} . \tag{31}$$

Because Γ_2 in $(\Gamma_1, \Gamma_2) \in c_{M,l}$ has a form of direct sum of J and E_2 as shown in the definition (29). Thus when the dimension of J does not equal zero, there are no zeroes in pivot of T' and the left hand side of Eq.(31) can be separated into two parts

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \begin{pmatrix} E \\ J \end{pmatrix} , \tag{32}$$

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} , \tag{33}$$

where $\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}$ is the LU decomposition of T' [14]; E_1 has the same dimension as E_2.

For the J sub-matrix we have proved [11] there exists P_J, Q_J which make

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} P_J E Q_J \\ P_J J Q_J \end{pmatrix} = \begin{pmatrix} E \\ J \end{pmatrix} , \tag{34}$$

For the $E_{1,2}$ parts, there exist operators that

$$P_y \begin{pmatrix} E_1 + \lambda' E_2 \\ E_2 \end{pmatrix} Q_y = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} ,$$

$$P_x \begin{pmatrix} E_1 \\ E_2 + \lambda E_1 \end{pmatrix} Q_x = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} , \tag{35}$$

where $\lambda' = \frac{\beta}{\alpha}$. It is simple to verify that such kind of $P_{x,y}, Q_{x,y}$ satisfying the equations does exist (see Appendixes of [11] for detailed derivations). Thus $P_C = P_x P_y$ and $Q_C = Q_y Q_x$ will make

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} P_C E_1 Q_C \\ P_C E_2 Q_C \end{pmatrix} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} . \tag{36}$$

Combine Eq.(34) and Eq.(36) we can get such $P_0 = P_J \oplus P_C$, $Q_0 = Q_J \oplus Q_C$ that satisfy the following equation

$$\left[\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array} \right] \left[\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right] \begin{pmatrix} P_0 \Gamma_1 Q_0 \\ P_0 \Gamma_2 Q_0 \end{pmatrix} = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} , \tag{37}$$

which is just Eq.(31).

In one word, Eq.(30) can always be written as

$$\begin{aligned} \begin{pmatrix} \Gamma_1 \\ \Gamma_2' \end{pmatrix} &= P' P_0^{-1} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} Q_0^{-1} Q' \\ &= P'' \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} Q'' \end{aligned} \tag{38}$$

The effective transformation P'', Q'' that keep Γ_1 invariant must be of the form

$$P'' = P'' \, ; Q'' = \begin{pmatrix} P''^{-1} & \mathbb{0} \\ X & Y \end{pmatrix} , \tag{39}$$

where $\mathrm{Det}(Y) \neq 0$. The transformation of Γ_2 reads

$$P'' \Gamma_2 Q'' = P''(A, B) Q'' = (P'' A P''^{-1} + P'' B X, P'' B Y) , \tag{40}$$

where A is the $M \times M$ submatrix, and B is the $M \times (N - M)$ submatrix. Since P'' and Y both are ILO operators, the rank of submatrix B, is unchanged and it can be further transformed to form of Eq.(12)

$$\begin{pmatrix} A' & B' & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & E' \end{pmatrix} . \tag{41}$$

We get that if two states are SLOCC equivalent then E' block of Γ_2' and Γ_2 must be identical. In Eq.(41) we see that E' block is diagonalized in Γ_2 which guarantee that we can proceed our proof to E'' ($E^{(3)}$, $E^{(4)}$ and so on) by leaving the E' block unchanged. The rest steps of this proof are just like argument after Eq.(40), and finally we can get that if (Γ_1, Γ_2') is SLOCC equivalent with (Γ_1, Γ_2) then Γ_2' and Γ_2 have the same canonical form in the set of Eq.(29). (However there exists the special case that the dimension of J equals zero, in this case there can be zero elements in the pivot of the nonsingular square matrix T'. T' can then be decomposed as decomposed as [14]

$$\left(\begin{array}{cc} t'_{11} & t'_{12} \\ t'_{21} & t'_{22} \end{array} \right) = P_{T'} \cdot \left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array} \right) \cdot \left(\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) , \tag{42}$$

where $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$, $P_{T'} = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]$ and both matrices on the righthand side of above equation are nonsingular. It can be show that $P_{T'}$ can be compensated by some operators P_z, Q_z which act on Γ_1 and Γ_2, i.e.,

$$\left(\begin{array}{c} E_1 \\ E_2 \end{array} \right) = P_T \left(\begin{array}{c} P_z E_1 Q_z \\ P_z E_2 Q_z \end{array} \right) , \tag{43}$$

see Appendixes of [11]) Q.E.D.

3.2 Classification on sets $C_{n,l}$ with $n = M - 1$

For the set of $c_{M-1,l} \in C_{M-1,l}$, the construction of $c_{M-1,l}$ is essentially the same as that $c_{N-1,l}$ of $2 \times N \times N$ systems in [11]. Here for the sake of convenience we only take a $2 \times 7 \times 8$ state as a demonstration, i.e., $C_{M-1,l} = C_{6,l}$. The matrix pair (Γ_1, Γ_2) can be transformed into the following form

$$\left(\begin{array}{c} \Gamma_1 \\ \Gamma_2 \end{array} \right) \xrightarrow{T,P,Q} \left(\begin{array}{c} \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\ \left(\begin{array}{ccc|ccccc} \times & \times & \times & 0 & c_{04} & c_{05} & 0 & 0 \\ \times & \times & \times & 0 & c_{14} & c_{15} & 0 & 0 \\ \times & \times & \times & 0 & c_{24} & c_{25} & 0 & 0 \\ \hline r_{30} & r_{31} & r_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right) \end{array} \right) , \tag{44}$$

where Γ_2 can then be expressed as

$$\Gamma_2' = \left(\begin{array}{c|ccc} A & \mathbb{0} & c & \mathbb{0} \\ \hline r & 0 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 0 & \mathbb{0} & E \\ \mathbb{0} & 1 & \mathbb{0} & \mathbb{0} \end{array}\right) \equiv \begin{pmatrix} A & c \\ r & B \end{pmatrix}. \tag{45}$$

Further simplification can be proceeded according to the vector(or submatrices) c, r. There are four cases in general, i.e., (1), $(c = 0, r = 0)$; (2), $(c \neq 0, r = 0)$; (3), $(c = 0, r \neq 0)$; (4), $(c \neq 0, r \neq 0)$. Here $c \neq 0$ means that $r(c) \geq 1$ and different ranks will result in different classes, i.e.,

$$\Gamma_2^{00} = \begin{pmatrix} \times & \times & \times & 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \ {}_1\Gamma_2^{10} = \begin{pmatrix} \times & \times & \times & 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{46}$$

$$\Gamma_2^{01} = \begin{pmatrix} \times & \times & 0 & 0 & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \ {}_1\Gamma_2^{11} = \begin{pmatrix} \times & 0 & \times & 0 & 0 & 0 & 0 & 0 \\ \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{47}$$

$${}_2\Gamma_2^{10} = \begin{pmatrix} \times & \times & \times & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \ {}_2\Gamma_2^{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{48}$$

Clearly, analogous with the set $C_{N-1,\,l}$ in Ref.[11], we can finally get the following set

$$c_{M-1,\,l} = \{(\Lambda, \Gamma)|\, r(\Gamma) = l; \Gamma = \begin{pmatrix} J & 0 \\ 0 & B \end{pmatrix};\ (\Lambda, \Gamma) \in C_{M-1,\,l}\}\ . \tag{49}$$

J represents the Jordan canonical form.

Theorem 2 $\forall\ (\Lambda,\Gamma) \in c_{M-1,\,l}$, *the set* $c_{M-1,\,l}$ *is of the classification of* $C_{M-1,\,l}$. (i) *suppose two states are SLOCC equivalent, they can be transformed into the same matrix vector* (Λ,Γ)*;* (ii) *this matrix vector is unique in* $c_{M-1,l}$*, that is suppose* (Λ,Γ') *is SLOCC equivalent with* (Λ,Γ)*,* $(\Lambda,\Gamma') = (\Lambda,\Gamma)$ ($\Gamma' = \Gamma$ *means* Js *are equivalent in the condition of theorem 1 and* $B' = B$*).*

The proof of this theorem is in the same manner as that of *theorem 2* in [11].

Following the procedure of $n = (M-1)$ and the method introduced in [11], we can further classify the more generally cases, i.e., $n = M - i$ with $i > 1$. Here we neglect the proof and only give a concrete example of the classification of this kind. We give a complete classification of $2 \times (M+5) \times (2M+5)$ for $M = 1$, i.e., $2 \times 6 \times 7$ state whose classification has not been presented in literature so far.

Classes of sets $c_{6,\,l}$: for all inequivalent classes in $c_{6,\,l}$, they have the same form of Γ_1 in the definition (29)

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{50}$$

So we only list the form of Γ_2s,

$$\begin{bmatrix} \times & \times & \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & \times & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \times & \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \times & \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{51}$$

$$\begin{bmatrix} \times & \times & 0 & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{52}$$

Here the square matrices of $\{\times\}_{n\times n}$ in Eq.(51, 52) consists of all the inequivalent classes of sets $c_{n,l}$ in $2\times n\times n$ states. For example the first matrix of Eq.(51) is made up by all the genuine entanglement classes of the sets $c_{5,l}$ in $2\times 5\times 5$ state and plus the one with $\{\times\}_{5\times 5}=0$, thus there are $(26+1)$ [15] inequivalent forms of this matrix.

Classes of set $c_{5,l}$: for all inequivalent classes in $c_{5,l}$, they has the same form of Λ

$$\Lambda=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\end{bmatrix}. \tag{53}$$

The different Γ_2s are

$$\begin{bmatrix}\times&\times&0&0&0&0&0\\\times&\times&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\\0&0&1&0&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\\0&0&1&0&0&0&0\end{bmatrix},\begin{bmatrix}0&1&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\\0&0&1&0&0&0&0\end{bmatrix}, \tag{54}$$

$$\begin{bmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\\0&0&1&0&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\1&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\\0&0&1&0&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\\0&0&1&0&0&0&0\end{bmatrix}. \tag{55}$$

$$\begin{bmatrix}0&0&0&0&0&0&0\\1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\\0&0&1&0&0&0&0\end{bmatrix}, \tag{56}$$

Same as that of $c_{6,l}$, $\{\times\}_{2\times 2}$ here has three different forms.

Thus we get $(26+1)+(13+1)+(5+1)+(2+1)+1+1+(2+1)+1+1+1+1+1+1=61$ inequivalent entanglement classes in $2\times 6\times 7$. It is clearly to see that this method is simple and effective, meanwhile each entangled state can be read out directly from the matrix pairs.

4 Conclusions

In summary, we have generalized our method of entanglement classification under stochastic local operations and classical communications to the more general case of $2 \times M \times N$ systems. Two examples of $2 \times 4 \times 6$ and $2 \times 6 \times 7$ are given where all their inequivalent entanglement classes are determined. Because the classification procedure is essentially a constructive algorithm, the method can serve as a powerful tool in practical entanglement classifications with the aid of computers. Most importantly a wide range of state space is explored which provide a rich resource for possible new applications in the quantum information theory.

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