Entanglement evolution of continuous variable quantum states

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We show when one of the modes of a initially bipartite Gaussian pure state interacts with a Gaussian noisy channel $I \otimes \$$, the evolution of entanglement can be simply factorized by the product of two factors that depend on the environment and the initial state, respectively. These two factors are the entanglement quantity of the initial pure state and the entanglement quantity of the mixed state generated by performing the map $I \otimes \$$ on the maximal two-mode squeezed state.

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Introduction. Quantum entanglement places the central important in quantum information processing(QIP)[1]. Therefore, the study of properties about quantum entanglement has drawn much interest for a long time[2–4]. Although initially QIP was studied with discrete quantum states, it was then extended to the continuous variable (CV) quantum states[5, 6]. A well known example is the continuous variable quantum teleportation (CVQT)[7–10]. Among the CV states, Gaussian states are a type of most often used states in practice. For example, a two-mode squeezed vacuum state is used as the entanglement source in CVQT. Therefore, it is an important topic to study the entanglement property with Gaussian states and properties of Gaussian operations. Some example results under this topics are whether a Gaussian state is entangled[11–14], how to quantify the entanglement of a Gaussian state[15, 16], Gaussian operation and entanglement purification[17, 18], the entanglement sudden death[19, 20], the characterization of Gaussian maps[21], and so on.

Recently, Konrad et al[22] finds a simple and general factorization law for the entanglement evolution of a 2×2 pure state $|\chi\rangle = \sqrt{\omega}|00\rangle + \sqrt{1-\omega}|11\rangle$ on passage a noisy channel on one mode, say $I \otimes \$$. In particular, they give an important formula for the entanglement quantity with entanglement concurrence:

$$\begin{aligned} & C[I \otimes \$(|\chi\rangle\langle\chi|)] \\ = \; & C[(|\chi\rangle\langle\chi|)] \cdot C[I \otimes \$(|\phi^+\rangle\langle\phi^+|)] \end{aligned} \quad (1)$$

and $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Their results have been experimentally tested very recently[23].

Gaussian states seem to be the most often used entanglement resource in CV QIP. A Gaussian channel often appears in CV QIP, e.g., we often take Gaussian operations to the quantum states, there can be Gaussian noise in CV QIP, and so on. Naturally, an interesting question arises is how the entanglement of a bipartite Gaussian pure state changes under Gaussian operations. In this Letter, we present the CV state version of the above factorization law in Eq.(1). We show that if we use the entanglement formation with the shortest distance measure for pure state, there is a similar entanglement factorization law as Eq.(1) for a bipartite Gaussian pure state, with one mode being taken Gaussian operation or Gaussian noisy channel.

Our goal and some definitions Using the entanglement measure E defined in this paper, we shall present a formula which relates entanglement quantity of state $I \otimes \$(|g(U,V,q)\rangle\langle g(U,V,q)|)$ and $I \otimes \$(|\phi^+\rangle\langle\phi^+|)$ in the form

$$\begin{aligned} & E[I \otimes \$(|g(U,V,q)\rangle\langle g(U,V,q)|)] \\ = \; & E[|g(U,V,q)\rangle\langle g(U,V,q)|] \cdot E[I \otimes \$(|\phi^+\rangle\langle\phi^+|)]. \end{aligned} \quad (2)$$

Here $|g(U,V,q)\rangle = U \otimes V|\chi(q)\rangle$ is any bipartite Gaussian pure state, state $|\chi(q)\rangle_{12} = \sqrt{1-q^2}\exp(q a_1^\dagger a_2^\dagger)|00\rangle$ $(-1 \leq q \leq 1)$ is a two-mode squeezed state, map $\$$ is a Gaussian channel which acts on one mode of the state. A Gaussian operation changes a Gaussian state to another Gaussian state only. We define the maximally entangled state $|\phi^+\rangle_{12}$ as the simultaneous eigenstate of position difference $\hat{x}_1 - \hat{x}_2$ and momentum sum $\hat{p}_1 + \hat{p}_2$, with both eigenvalues being 0. Also, when $q = 1$, the state $|\chi(q)\rangle = |\phi^+\rangle$. For simplicity, we first consider $|g(U,V,q)\rangle = |\chi(q)\rangle$, a two-mode squeezed state. We also define $\rho^G(q_\alpha) = (I \otimes \$)(|\chi(q_\alpha)\rangle\langle\chi(q_\alpha)|)$.

Main theme. We define $\hat{T}_i(q_\alpha)$ operator[9]:

$$\hat{T}_i(q_\alpha) = \sum_{n=0}^{\infty} q_\alpha^n |n\rangle\langle n| = q_\alpha^{a_i^\dagger a_i} \quad (3)$$

This operator has an important mathematical property

$$\hat{T}_i(q_\alpha)(a_i^\dagger, a_i)\hat{T}_i^{-1}(q_\alpha) = (q_\alpha a_i^\dagger, a_i/q_\alpha) \quad (4)$$

which shall be used latter in this paper. For simplicity, we sometimes omit the subscripts of states and operators provided that the omission does not affect the clarity.

Mathematically, without considering normalization, we find $|\chi(q = q_a q_b)\rangle = \hat{T}(q_a) \otimes I|\chi(q_b)\rangle$. Since, the operator $\hat{T}(q_a)\otimes I$ and the map $I\otimes\$$ commute, there is:

$$\begin{aligned}\rho^{out} =& I\otimes\$(|\chi(q)\rangle\langle\chi(q)|)\\ =& I\otimes\$(\hat{T}(q_a)\otimes I|\chi(q_b)\rangle\langle\chi(q_b)|\hat{T}^\dagger(q_a)\otimes I)\\ =& \hat{T}(q_a)\otimes I(I\otimes\$(|\chi(q_b)\rangle\langle\chi(q_b)|))\hat{T}^\dagger(q_a)\otimes I\\ =& \hat{T}(q_a)\otimes I\rho^G(q_b)\hat{T}^\dagger(q_a)\otimes I\end{aligned} \tag{5}$$

Therefore, we only need to calculate the entanglement quantity of the state as given by Eq.(5). In what follows we do this calculation.

Our entanglement measure. As shown below, using the entanglement measure defined in this work, for any two-mode squeezed vacuum state $|\chi(q)\rangle$, the amount of entanglement is $E(|\chi(q)\rangle)=q^2$.

For a pure state $|\chi(q)\rangle$, we use the shortest distance measure[2], i.e. we calculate $d=1-|\langle\psi\psi'|\chi(q)\rangle|^2$ for all possible $|\psi\rangle\otimes|\psi'\rangle$, the smallest value is defined as the entanglement measure $E(|\chi(q)\rangle)$. Using such a measure, one immediately finds

$$E(|\chi(q)\rangle)=|q|^2. \tag{6}$$

We use this formula for entanglement measure of pure states, for any mixed state, we use the entanglement formation[15, 24]. In particular, for the Gaussian state ρ^s with a covariance matrix in the standard form already, it can be written in the following convex form[15]

$$\begin{aligned}\rho^s \;=\; & \int d^2\beta_1 d^2\beta_2 P(\beta_1,\beta_2)\\ & \hat{D}(\beta_1,\beta_2)|\chi(q_0)\rangle\langle\chi(q_0)|\hat{D}^\dagger(\beta_1,\beta_2)\end{aligned} \tag{7}$$

where $P(\beta_1,\beta_2)$ is positive definite. Also, the entanglement quantity of ρ^s is $E(\rho^s)=E(|\chi(q_0)\rangle)=|q_0|^2$ if we cannot find another convex formula for ρ^s in the above form with a smaller q_0.

Standard form of covariance matrix. To use the decomposition result of Ref.[15], we need first transform ρ^{out} by local unitaries so that its characteristic function (or covariance matrix) is changed into the standard form[15]. We consider the covariance matrix M_G for the density matrix ρ^G first. Since only one mode of $|\phi^+\rangle$ is performed by \$ and \$ is a Gaussian map, the real symmetric matrix M_G must have the following form:

$$M_G=\begin{pmatrix} aI_2 & S^t\\ S & Q\end{pmatrix}. \tag{8}$$

Where I_2 is 2×2 unity matrix, S and Q are 2×2 real matrices, Q is symmetric, S^t is the transpose of S. As has been shown in Ref.[11], such type of matrix can be diagonalized in

$$\begin{pmatrix} M_1 & 0\\ 0 & M_2\end{pmatrix} M_G \begin{pmatrix} M_1^t & 0\\ 0 & M_2^t\end{pmatrix}=\begin{pmatrix} a & 0 & c & 0\\ 0 & a & 0 & c'\\ c & 0 & b & 0\\ 0 & c' & 0 & b\end{pmatrix} \tag{9}$$

Here X^t is the transpose matrix of X, $\det M_1=\det M_2=1$, and M_1 is *orthogonal*. These mean, we can realize M_1, M_2 by local unitaries $\hat{R}_1$, $\hat{U}_2$. Therefore, the covariance matrix of

$$\rho^s=\hat{R}_1\otimes\hat{U}_2\rho^G(q_b)\hat{R}_1^\dagger\otimes\hat{U}_2^\dagger \tag{10}$$

must be in the standard form as in Eq.(9). Note that $\hat{R}_1$ must be in the form of $e^{i\theta a_1^\dagger a_1}$, because M_1 is orthogonal. Hence we find

$$\begin{aligned}\tilde{\rho}^{out} \;=\;& \hat{R}_1\otimes\hat{U}_2\rho^{out}\hat{R}_1^\dagger\otimes\hat{U}_2^\dagger\\ =\;& T(q_a)\otimes I\rho^s T^\dagger(q_a)\otimes I.\end{aligned} \tag{11}$$

Here we have used the fact that $\hat{T}(q_a)$ and $\hat{R}_1$ *commute*. Since $\tilde{\rho}^{out}$ is obtained by taking local unitaries to ρ^{out}, its entanglement quantity must be same with that of ρ^{out}. We now only need to study the amount of entanglement of state $T(q_a)\otimes I\rho^s T^\dagger(q_a)\otimes I$. Since the covariance matrix of ρ^s is in the standard form of Eq.(9), we can use the decomposition form given by Ref.[15] for this operator.

Proof of Eq.(2). Using the entanglement measure as defined earlier in this paper, we now show if the entanglement quantity for ρ^G (or ρ^s) is $|q_0|^2$, the entanglement quantity of $\tilde{\rho}^{out}$ must be $|q_a|^2|q_0|^2$, for, otherwise the entanglement quantity of ρ^s is *not* $|q_0|^2$.

If the amount of entanglement of ρ^s is $|q_0|^2$, we cannot find another convex with $|\chi(q_0')\rangle$ and $|q_0'|<|q_0|$ in the format of Eq.(7). That is to say, if $|q_0'|<|q_0|$, with whatever positive definite functional $P_1(\beta_1,\beta_2)$,

$$\begin{aligned}\rho^s \;\neq\; & \int d^2\beta_1 d^2\beta_2 P_1(\beta_1,\beta_2)\hat{D}(\beta_1,\beta_2)\\ & |\chi(q_0')\rangle\langle\chi(q_0')|\hat{D}^\dagger(\beta_1,\beta_2).\end{aligned} \tag{12}$$

Since ρ^s has the convex form of Eq.(7), we have

$$\begin{aligned}\tilde{\rho}^{out} \;=\;& \hat{T}(q_a)\otimes I\left[\int d^2\beta_1 d^2\beta_2 P(\beta_1,\beta_2)\hat{D}(\beta_1,\beta_2)\right.\\ & \left.|\chi(q_0)\rangle\langle\chi(q_0)|\hat{D}^\dagger(\beta_1,\beta_2)\right]\hat{T}^\dagger(q_a)\otimes I.\end{aligned} \tag{13}$$

We shall use the following fact

Fact 1: In general, any following (un-normalized) state

$$e^{\alpha_1 a_1^\dagger+\alpha_2 a_2^\dagger}e^{q_0 a_1^\dagger a_2^\dagger}|00\rangle \tag{14}$$

can be written into the form of

$$\hat{D}(\beta_1,\beta_2)|\chi(q_0)\rangle \tag{15}$$

and vice versa, where α_i,α_i^* and β_i,β_i^* are related by a certain linear transformation. Therefore Eq.(13) can be written in

$$\begin{aligned}\tilde{\rho}^{out} \;=\;& \hat{T}(q_a)\otimes I\left[\int d^2\beta_1 d^2\beta_2 P(\beta_1,\beta_2)e^{\alpha_1 a_1^\dagger+\alpha_2 a_2^\dagger}\right.\\ & \left.|\chi(q_0)\rangle\langle\chi(q_0)|e^{\alpha_1^* a_1+\alpha_2^* a_2}\right]\hat{T}^\dagger(q_a)\otimes I.\end{aligned} \tag{16}$$

Using the mathematical property given in Eq.(4), we have

$$\tilde{\rho}^{out} = \int d^2\beta_1 d^2\beta_2 P(\beta_1,\beta_2) e^{q_a\alpha_1 a_1^\dagger+\alpha_2 a_2^\dagger} |\chi(q_a q_0)\rangle\langle\chi(q_a q_0)| e^{q_a\alpha_1^* a_1+\alpha_2^* a_2}. \quad (17)$$

According to *Fact 1*, we have

$$\tilde{\rho}^{out} = \int d^2\beta_1 d^2\beta_2 P(\beta_1,\beta_2)\hat{D}(\beta_1',\beta_2') |\chi(q_a q_0)\rangle\langle\chi(q_a q_0)|\hat{D}^\dagger(\beta_1',\beta_2'). \quad (18)$$

Since $P(\beta_1,\beta_2)$ are positive definite, this formula is a convex form for $\tilde{\rho}^{out}$. In order to show that $E(\tilde{\rho}^{out}) = |q_a q_0|^2$, we need show that $\tilde{\rho}^{out}$ cannot have another convex formula in the above format with $|\chi(\eta q_a q_0)\rangle\langle\chi(\eta q_a q_0)|$ and $|\eta|<1$. To show this we only need the mathematical identity $\hat{T}^{-1}(q_a)\hat{T}(q_a)=I$. This means $\rho^s = (\hat{T}^{-1}(q_a)\hat{T}(q_a)\otimes I)\rho^s(\hat{T}^{-1}(q_a)\hat{T}(q_a)\otimes I)^\dagger$. Suppose we have another convex form for Eq.(18) with a certain positive definitive functional $P_1(\beta_1,\beta_2)$:

$$\tilde{\rho}^{out} = \int d^2\beta_1 d^2\beta_2 P_1(\beta_1,\beta_2)\hat{D}(\beta_1,\beta_2) |\chi(\eta q_a q_0)\rangle\langle\chi(\eta q_a q_0)|\hat{D}^\dagger(\beta_1',\beta_2') \quad (19)$$

and

$$|\eta| < 1 \quad (20)$$

Recall Eq.(11) we have

$$\rho^s = (\hat{T}^{-1}(q_a)\otimes I)\tilde{\rho}^{out}(\hat{T}^{-1}(q_a)\otimes I)^\dagger. \quad (21)$$

Using Eq.(19) and *Fact 1*, we further obtain

$$\rho^s = \int d^2\beta_1 d^2\beta_2 P_1(\beta_1,\beta_2)\hat{D}(\beta_1'',\beta_2'') |\chi(\eta q_0)\rangle\langle\chi(\eta q_0)|\hat{D}^\dagger(\beta_1'',\beta_2''). \quad (22)$$

Here we have used $\hat{T}^{-1}(q_a)\otimes I|\chi(\eta q_a q_0)\rangle = |\chi(\eta q_0)\rangle$. Eq.(22) means that

$$E(\rho^s) \le |\eta q_0|^2 < |q_0|^2 \quad (23)$$

But we have already assumed $E(\rho^s)=|q_0|^2$ in the beginning of this subsection. Therefore, Eq.(19) can never hold for any $|\eta|<1$. With this proof, based on Eq.(18), we conclude

$$E[I\otimes\$(|\chi(q=q_a q_b)\rangle\langle\chi(q=q_a q_b)|)] = |q_a q_0|^2 = E[|\chi(q_a)\rangle\langle\chi(q_a)|]\cdot E[I\otimes\$(|\chi(q_b)\rangle\langle\chi(q_b)|)]. \quad (24)$$

Obviously, if $q_b=1$ we have $|\chi(q_b)\rangle = |\phi^+\rangle$ and $q=q_a$.

Without loss of generality, any bipartite Gaussian pure state has the following form:

$$|g(U,V,q)\rangle = U\otimes V|\chi(q)\rangle \quad (25)$$

where U and V are local Bogoliubov transformation operators. We shall use the following fact.

Fact 2: For any local unitary operators U,V, we can always find another unitary operator $\mathcal{U}^C$ and $\mathcal{U}^D$so that

$$U\otimes V|\phi^+\rangle = I\otimes\mathcal{U}^C|\phi^+\rangle = \mathcal{U}^D\otimes I|\phi^+\rangle. \quad (26)$$

Proof: Any local Gaussian unitary operator can be decomposed into the product form of $\mathcal{R}(\theta')\mathcal{S}(r)\mathcal{R}(\theta)$ where $\mathcal{S}(r)$ is a squeezing operator as defined by $\mathcal{S}(r)(\hat{x},\hat{p})\mathcal{S}^\dagger(r) = (r\hat{x},\hat{p}/r)$, $\mathcal{R}(\theta)$ is a rotation operator defined by $\mathcal{R}(\theta)(a^\dagger,a)\mathcal{R}^\dagger(\theta) = (e^{-i\theta}a^\dagger, e^{i\theta}a)$. For any two-mode squeezed state $|\chi(q)\rangle$ we have $\mathcal{R}(\theta_1)\otimes\mathcal{R}(\theta_2)|\chi(q)\rangle = I\otimes\mathcal{R}(\theta_1+\theta_2)|\chi(q)\rangle$. For the maximally two-mode squeezed state $|\phi^+\rangle$ we have $\mathcal{S}(r)\otimes\mathcal{S}(r)|\phi^+\rangle = |\phi^+\rangle$, for, the both sides are the simultaneous eigenstates of position difference and momentum sum, with both eigenvalues being 0. This also means $\mathcal{S}(r)\otimes I|\phi^+\rangle = I\otimes\mathcal{S}^\dagger(r)|\phi^+\rangle$. Suppose $U=\mathcal{R}(\theta_A')\mathcal{S}(r_A)\mathcal{R}(\theta_A)$ and $V=\mathcal{R}(\theta_B')\mathcal{S}(r_B)\mathcal{R}(\theta_B)$, then

$$\mathcal{R}(\theta_A')\mathcal{S}(r_A)\mathcal{R}(\theta_A)\otimes\mathcal{R}(\theta_B')\mathcal{S}(r_B)\mathcal{R}(\theta_B)|\phi^+\rangle = I\otimes R(\theta_B')\mathcal{S}(r_B)\mathcal{R}(\theta_A+\theta_B)\mathcal{S}^\dagger(r_A)R(\theta_A')|\phi^+\rangle \quad (27)$$

This completes the proof of the first part of Eq.(26) and the second part is obvious.

By using Eq.(26), we have

$$\begin{aligned} & E[I\otimes\$(I\otimes V|\phi^+\rangle\langle\phi^+|I\otimes V^\dagger)] \\ =\; & E[I\otimes\$(\tilde{\mathcal{U}}^D\otimes I|\phi^+\rangle\langle\phi^+|\tilde{\mathcal{U}}^{D\dagger}\otimes I)] \\ =\; & E[I\otimes\$(|\phi^+\rangle\langle\phi^+|)] \end{aligned} \quad (28)$$

Then we get

$$\begin{aligned} & |g(U,V,q)\rangle\langle g(U,V,q)| \\ =\; & U\otimes V|\chi(q)\rangle\langle\chi(q)|U^\dagger\otimes V^\dagger \\ =\; & (U\otimes V)(\hat{T}(q)\otimes I) \\ & |\phi^+\rangle\langle\phi^+|(\hat{T}^\dagger(q)\otimes I)(U^\dagger\otimes V^\dagger) \end{aligned} \quad (29)$$

and

$$\begin{aligned} & E[I\otimes\$(|g(U,V,q)\rangle\langle g(U,V,q)|)] \\ =\; & E[(\hat{T}(q)\otimes I)I\otimes\$(I\otimes V \\ & |\phi^+\rangle\langle\phi^+|I\otimes V^\dagger)(\hat{T}^\dagger(q)\otimes I)] \\ =\; & E[(\hat{T}(q)\otimes I) \\ & I\otimes\$(|\phi^+\rangle\langle\phi^+|)(\hat{T}^\dagger(q)\otimes I)]. \end{aligned} \quad (30)$$

We have omitted U since entanglement quantity of any state does not change under any local unitary operation. In the second equality we have used Eq.(28). This gives rise to

$$\begin{aligned} & E[I\otimes\$(|g(U,V,q)\rangle\langle g(U,V,q)|)] \\ =\; & E[|g(U,V,q)\rangle]\cdot E[I\otimes\$(|\phi^+\rangle\langle\phi^+|)]. \end{aligned} \quad (31)$$

This proves our major conclusion.

It is easy to show that the above formula also leads to the following equation

$$\frac{E[I\otimes \$(|g(U,V,q)\rangle\langle g(U,V,q)|)]}{E[I\otimes \$(|g(U',V',q')\rangle\langle g(U',V',q')|)]} = \frac{E[|g(U,V,q)\rangle\langle g(U,V,q)|]}{E[|g(U',V',q')\rangle\langle g(U',V',q')|]}. \quad (32)$$

For any Gaussian map \$. This formula seems to be more useful in a real experimental verification where maximally entangled state is not available.

In summary, using the decomposition formula by Marians[15] and the property of $\hat{T}$[9], we present a formula for entanglement evolution for bipartite Gaussian pure state. Any Gaussian pure state $|g\rangle$, after acted by a Gaussian map $I\otimes \$$, its entanglement quantity is equivalent to the product of the entanglement of the initial state $|g\rangle$ and the entanglement quantity of state $I\otimes \$(|\phi^+\rangle\langle\phi^+|)$.

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