Quantum noise and self-sustained radiation of $\mathcal{PT}$-symmetric systems

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(Dated: January 2010)

The observation that $\mathcal{PT}$-symmetric Hamiltonians can have real-valued energy levels even if they are non-hermitian has triggered intense activities, with experiments in particular focusing on optical systems, where hermiticity can be broken by absorption and amplification. For classical waves, absorption and amplification are related by time-reversal symmetry. This work shows that microreversibility-breaking quantum noise turns $\mathcal{PT}$-symmetric systems into self-sustained sources of radiation, which distinguishes them from ordinary, hermitian quantum systems.

PACS numbers: 42.50.Lc, 03.65.-w, 03.65.Nk

arXiv:1001.0539v1 [quant-ph] 4 Jan 2010

A frequent common factor in quantum systems with a non-hermitian Hamiltonian is the non-conservation of particle number, either because the system is open, or because of loss or gain in an absorbing or amplifying medium. Ignoring non-linear effects such as the feedback in a laser, such systems ordinarily do not possess stationary states; instead, they only support decaying quasi-bound states with complex energy, where the imaginary part $\mathrm{Im}\,E = -1/2\tau$ (setting $\hbar \equiv 1$) accounts for particle loss with decay rate $1/\tau$. A notable exception are non-hermitian systems that are invariant under joint parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) symmetry [1]. These $\mathcal{PT}$-symmetric systems generically possess a set of real-valued energy levels, as well as complex energy levels that occur in complex-conjugate pairs. Systems with entirely real spectrum define a consistent unitary extension of quantum mechanics [2, 3]. This observation has led to intense research efforts delivering a new theoretical perspective on systems as varied as quantum field theories and complex crystals (reviewed in Ref. [4]), while experimental realizations in particular focus on optical systems where hermiticity can be violated by absorption and amplification [5].

For classical waves, amplification and absorption are strictly related by time reversal. Physically, the existence of stationary states with real energy can therefore be seen as a consequence of the balance of amplification and absorption in parity-related regions of $\mathcal{PT}$-symmetric system. At the heart of absorption and amplification, however, are noisy microscopic quantum processes (spontaneous and stimulated emission events, and stimulated absorption events) which effectively break time-reversal symmetry [6] (for the delicate reservoir engineering required to time-reverse spontaneous emission see [7]). In this work I show that the effects of quantum noise distinguish $\mathcal{PT}$-symmetric systems from hermitian quantum systems, and indeed suggest an alternative interpretation of the physics behind non-hermitian $\mathcal{PT}$-symmetry: (i) Accounting for quantum noise, $\mathcal{PT}$-symmetric systems with stationary states are self-sustained sources of radiation, fed by the pumping in the amplifying parts of the system. (ii) That the energy of these states is real means that the system is stabilized at the lasing threshold. (iii) When the system is sufficiently open, the emitted radiation breaks parity symmetry (i.e., the emission pattern is asymmetric). (iv) In the limit of a closed system the emitted radiation intensity approaches a constant value, and provides a direct measure of the non-hermiticity of the system. The internal energy density of radiation then diverges, which entails a practical limitation for the implementation of $\mathcal{PT}$ symmetry in closed systems.

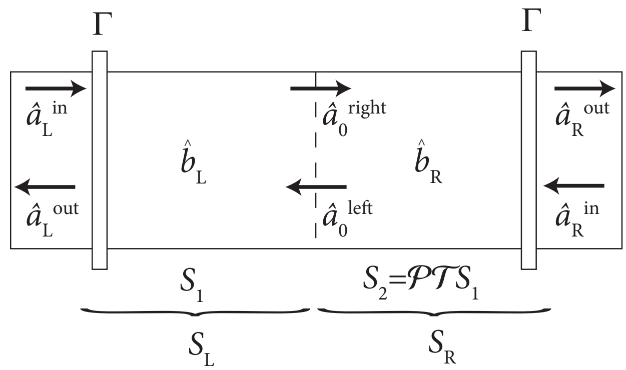

FIG. 1: Illustration of the scattering input-output approach to non-hermitian $\mathcal{PT}$-symmetric systems, defining the scattering input-output operators $\hat{a}$ and internal bosonic modes $\hat{b}$ in different parts of the system. Semi-transparent mirrors with transmission probability Γ are introduced to study the limit $\Gamma \to 0$ of a closed system.

These conclusions are obtained by employing the quantum-optical input-output formalism in its scattering formulation [8–10]. The scattering approach also provides insight into $\mathcal{PT}$ symmetry for classical waves [11], which defines the starting point of this paper.

Scattering approach to non-hermitian $\mathcal{PT}$-symmetric systems.—Probing the internal dynamics of an optical system by external radiation naturally leads to the scattering scenario depicted in Figure 1. The relation $a^{\mathrm{out}} = Sa^{\mathrm{in}}$ between incoming and outgoing wave amplitudes is provided by the scattering matrix $S(E) = \begin{pmatrix} r & t' \\ t & r \end{pmatrix}$, which contains blocks describing reflection (r, r') and transmission (t, t') when probed from the left or right, respectively. Each block consists of an $N \times N$-dimensional matrix, where N is the number of modes at each entrance. The poles of the scattering matrix determine the energies of quasibound states, which turn into bound states as the system is closed off.

In general, the scattering matrix fulfills the following two reciprocity relations: The Onsager relation $S(\gamma, -B, E) = S^T(\gamma, B, E)$, and the relation $S(-\gamma, B, E) = [S^\dagger(\gamma, B, E^*)]^{-1}$ of classical microreversibility. Here, γ and B characterize two possible sources of broken time-reversal symmetry: absorption/amplification ($\gamma > 0/\gamma < 0$), which contribute an imaginary symmetric (nonhermitian) term to the Hamiltonian H, and magneto-optical effects (B), which contribute an imaginary antisymmetric (but still hermitian) term.

Conventional time-reversal $\mathcal{T}H = H^*$ transforms solutions according to $\mathcal{T}\psi = \psi^*$, which interchanges incoming and outgoing states, and therefore transforms the scattering matrix according to

$$\mathcal{T}S(\gamma, B, E) = [S^*(\gamma, B, E)]^{-1} = S(-\gamma, -B, E^*). \quad (1)$$

Assuming that energy is real, a system has $\mathcal{T}$ symmetry, $\mathcal{T}S = S$ hence $S^* = S^{-1}$, if $S(\gamma, B) = S(-\gamma, -B)$, which requires $\gamma = B = 0$ [12]. Parity $\mathcal{P}H(x) = H(-x)$ transforms solutions according to $\mathcal{P}\psi(x) = \psi(-x)$, which exchanges the left and right leads and yields

$$\mathcal{P}S(\gamma, B, E) = \sigma_x S(\gamma, B, E)\sigma_x, \quad (2)$$

where σ_x is a Pauli matrix. The $\mathcal{PT}$ operation on the scattering matrix is therefore given by

$$\begin{aligned} \mathcal{PT}S(\gamma, B, E) &= \sigma_x[S^*(\gamma, B, E)]^{-1}\sigma_x \quad (3) \\ &= \sigma_x S(-\gamma, -B, E^*)\sigma_x. \quad (4) \end{aligned}$$

For hermitian systems, $\mathcal{PT}$-symmetry implies $S = \sigma_x S^T \sigma_x$ [13]. For non-hermitian systems, $\mathcal{PT}$ symmetry implies the additional condition $\mathcal{P}\gamma = -\gamma$ [e.g., $\gamma(x) = -\gamma(-x)$ if $\mathcal{P}$ is reflection about the xy plane], i.e., there is a balance of absorption and amplification in parity-related regions.

Let us now explore from the scattering perspective how real-energy bound states appear in $\mathcal{PT}$-symmetric systems. As shown in Fig. 1, such systems can be constructed by joining two regions, where the left region, with scattering matrix $S_1 = \begin{pmatrix} r_1 & t_1' \\ t_1 & r_1' \end{pmatrix}$, is $\mathcal{PT}$-symmetric to the right region, $S_2 = \mathcal{PT}S_1$, which using standard block-inversion formulas can be written as

$$S_2 = \begin{pmatrix} \frac{1}{(r_1' - t_1 r_1^{-1} t_1')^*} & (r_1'^{-1} t_1)^* \frac{1}{(t_1' r_1'^{-1} t_1 - r_1)^*} \\ (r_1^{-1} t_1')^* \frac{1}{(t_1 r_1^{-1} t_1' - r_1')^*} & \frac{1}{(r_1 - t_1' r_1'^{-1} t_1)^*} \end{pmatrix}. \quad (5)$$

Bound states can be studied by closing the system off by mirrors with small transmission probability $\Gamma \ll 1$, described by a scattering matrix

$$S_\Gamma = -\begin{pmatrix} \sqrt{1-\Gamma} & i\sqrt{\Gamma} \\ i\sqrt{\Gamma} & \sqrt{1-\Gamma} \end{pmatrix}. \quad (6)$$

Including the mirrors, the scattering matrix of the left half of the system can then be written as

$$S_L = -\begin{pmatrix} \frac{r_1 + \sqrt{1-\Gamma}}{1 + r_1\sqrt{1-\Gamma}} & \frac{i t_1' \sqrt{\Gamma}}{1 + r_1\sqrt{1-\Gamma}} \\ \frac{i t_1 \sqrt{\Gamma}}{1 + r_1\sqrt{1-\Gamma}} & \frac{t_1 t_1' \sqrt{1-\Gamma}}{1 + r_1\sqrt{1-\Gamma}} - r_1' \end{pmatrix}, \quad (7)$$

while the scattering matrix $S_R = \mathcal{PT}S_L$ of the right half again follows from symmetry. These scattering matrices relate amplitudes of in- and outgoing modes (defined in Fig. 1) according to

$$\begin{pmatrix} a_L^{\rm out} \\ a_0^{\rm right} \end{pmatrix} = S_L \begin{pmatrix} a_L^{\rm in} \\ a_0^{\rm left} \end{pmatrix}, \quad \begin{pmatrix} a_0^{\rm left} \\ a_R^{\rm out} \end{pmatrix} = S_R \begin{pmatrix} a_0^{\rm right} \\ a_R^{\rm in} \end{pmatrix}. \quad (8)$$

The scattering matrix of the composed system is obtained by algebraically eliminating the amplitudes $a_0^{\rm left}$ and $a_0^{\rm right}$ at the interface between both regions. For $\Gamma \to 0$, these amplitudes become singular when

$$\det {\rm Im}\,(r_L') = \det \left[{\rm Im}\left(r_1' - \frac{t_1 t_1'}{1 + r_1}\right)\right] = 0, \quad (9)$$

which is the quantization condition of the closed system.

The quantization condition (9) requires that the N real column vectors of ${\rm Im}\,(r_L')$ be linearly dependent, which generically can be achieved by varying a single real parameter (identifying this as a problem of co-dimension one). Therefore, the system typically possesses a number of bound states with real energy, even if the Hamiltonian is not hermitian.

Quantum noise.—The scattering approach can be extended to include quantum noise by passing from wave amplitudes $a^{\rm in}$, $a^{\rm out}$ to bosonic annihilation operators $\hat{a}^{\rm in}$, $\hat{a}^{\rm out}$, respectively. This defines the scattering variant of the input-output formalism [8–10], which has been used to describe systems that are exclusively absorbing or amplifying. To adapt the approach to $\mathcal{PT}$-symmetric systems, where both effects are combined, we formally separate the absorbing regions from the amplifying regions, and then join them together similar to the description of classical waves, given above.

For definiteness let us assume that the left half of the system is purely absorbing. For this part, the input-output scattering relations then take the form

$$\begin{pmatrix} \hat{a}_L^{\rm out} \\ \hat{a}_0^{\rm right} \end{pmatrix} = S_L \begin{pmatrix} \hat{a}_L^{\rm in} \\ \hat{a}_0^{\rm left} \end{pmatrix} + Q_L \hat{b}_L, \quad (10)$$

which connects the in- and outgoing modes to bosonic operators $\hat{b}$ representing the medium. These operators appear because both $\hat{a}^{\rm in}$ and $\hat{a}^{\rm out}$ have to satisfy standard canonical commutation relations, dictating that the coupling matrix Q_L satisfies the fluctuation-dissipation theorem $Q_L Q_L^\dagger = 1 - S_L S_L^\dagger$ [9]. In the right half of the system, where the medium is amplifying, we have

$$\begin{pmatrix} \hat{a}_0^{\rm left} \\ \hat{a}_R^{\rm out} \end{pmatrix} = S_R \begin{pmatrix} \hat{a}_0^{\rm right} \\ \hat{a}_R^{\rm in} \end{pmatrix} + Q_R \hat{b}_R^\dagger, \tag{11}$$

where the commutation relations now dictate coupling to creation operators, with $Q_R Q_R^\dagger = S_R S_R^\dagger - 1$. By assumption, the operators $\hat{b}_L^\dagger$ and $\hat{b}_R$ commute with $\hat{a}^{\rm in}$; however, according to Eqs. (10) and (11) they do not commute with $\hat{a}^{\rm out}$, which is a manifestation of broken micro-reversibility in quantum optics.

We can now describe the full $\mathcal{PT}$-symmetric system by algebraically eliminating the interface operators $\hat{a}_0^{\rm left}$ and $\hat{a}_0^{\rm right}$. In absence of any incoming radiation, the intensity emitted to the left and right is then given by $I_L(E) = \frac{1}{2\pi}\langle \hat{a}_L^{\rm out\dagger}\hat{a}_L^{\rm out}\rangle$, $I_R(E) = \frac{1}{2\pi}\langle \hat{a}_R^{\rm out\dagger}\hat{a}_R^{\rm out}\rangle$, respectively, which can be evaluated assuming $\langle \hat{b}_L^\dagger \hat{b}_L\rangle = 1$ (ground-state population in the absorbing regions), $\langle \hat{b}_R^\dagger \hat{b}_R\rangle = 0$ (total population inversion in the amplifying regions; these conditions minimize the quantum noise).

Let us first consider the case of a single-mode resonator ($N = 1$) with purely ballistic internal dynamics and absorption in the left region [14], described by scattering matrices

$$S_1 = \begin{pmatrix} 0 & t_1 \\ t_1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1/t_1^* \\ 1/t_1^* & 0 \end{pmatrix}, \tag{12}$$

where $|t_1| < 1$. Including the mirrors, the total scattering matrix is

$$S = \begin{pmatrix} \frac{\sqrt{1-\Gamma}\left(t_1^{*2} - t_1^2\right)}{t_1^2(1-\Gamma) - t_1^{*2}} & \frac{|t_1|^2\Gamma}{t_1^2(1-\Gamma) - t_1^{*2}} \\ \frac{|t_1|^2\Gamma}{t_1^2(1-\Gamma) - t_1^{*2}} & \frac{\sqrt{1-\Gamma}\left(t_1^{*2} - t_1^2\right)}{t_1^2(1-\Gamma) - t_1^{*2}} \end{pmatrix}, \tag{13}$$

and the quantization condition (9) for the closed resonator takes the form $\operatorname{Im} t_1^2 = 0$. Following the quantum-optical procedure described above we find that this resonator emits radiation of intensity

$$I_L(E) = \frac{\Gamma\left(|t_1|^{-2} - 1\right)\left(1 - \Gamma + |t_1|^2\right)}{\pi|(t_1/t_1^*)^2 - 1 + \Gamma|^2}, \tag{14}$$

$$I_R(E) = \frac{\Gamma\left(1 - |t_1|^2\right)\left(1 - \Gamma + |t_1|^{-2}\right)}{\pi|(t_1/t_1^*)^2 - 1 + \Gamma|^2}. \tag{15}$$

Since $|t_1| < 1$ this gives $I_R > I_L$, the difference being

$$\Delta I(E) = I_R(E) - I_L(E) = \frac{\Gamma^2\left(|t_1|^{-1} - |t_1|\right)^2}{\pi|(t_1/t_1^*)^2 - 1 + \Gamma|^2}. \tag{16}$$

Therefore, the emission from the right exit, close to the amplifying region of the medium, is larger than the emission from the left exit, close to the absorbing region of the medium (formally, and up to a sign, I_L and I_R are related by the transformation $t_1 \to 1/t_1^*$).

The overall output intensity to both sides can be written as

$$I(E) = I_L(E) + I_R(E) = \frac{\Gamma(2-\Gamma)\left(|t_1|^{-2} - |t|^2\right)}{\pi|(t_1/t_1^*)^2 - 1 + \Gamma|^2}. \tag{17}$$

Close to quantization in the closed system, [$\Gamma \ll 1$, $E \approx E_0$, where E_0 fulfills the quantization condition $\operatorname{Im} t_1^2(E_0) = 0$], the emission pattern becomes symmetric and approaches a Lorentzian of the form

$$I_L(E) = I_R(E) = \frac{\Gamma\left(|t_0|^{-2} - |t_0|^2\right)}{\pi|2i\tau(E - E_0) + \Gamma|^2}. \tag{18}$$

Here $t_0 = t_1(E_0)$, while $\tau = 2\operatorname{Im} t_1^{-1} dt_1/dE|_{E=E_0}$ is the transmission delay time of propagation between the two mirrors. The full width at half maximum is given by $\Delta E = \Gamma/\tau$. While this width shrinks to zero as the system is closed off, remarkably the total intensity

$$I_{\rm tot} = \int I(E)\, dE = \frac{|t_0|^{-2} - |t_0|^2}{\tau} \tag{19}$$

remains finite, and can be interpreted as a direct measure of the degree of non-hermiticity of the system (for ballistic transport, hermiticity implies $|t_0| = 1$, for which the intensity vanishes).

In the more general case of a single-mode resonator with backscattering (where r_1 and r_1' are finite), compact expressions can still be obtained as long as the leakage remains small ($\Gamma \ll 1$), implying according to Eq. (7) that $|r_L + 1|, |t_L|, |t_L'| \ll 1$. The emission pattern is then still symmetric, with intensity

$$I_L(E) = I_R(E) = \frac{1}{2\pi}\frac{(1 - |r_L'|^2)|t_L'|^2}{|2(\operatorname{Im} r_L') - i t_L t_L'|^2}. \tag{20}$$

Linearization around the quantization condition again reveals a Lorentzian line shape, with line width $\Delta E = \operatorname{Re}\{d[(\operatorname{Im} r_L')/t_L t_L']/dE\}^{-1}$. Accounting for the scaling (7) of scattering coefficients with Γ, the total intensity $I_{\rm tot} \propto (1 - |r_L'|^2)$ again remains finite as $\Gamma \to 0$. In the hermitian case, this limit would imply $|r_L'| = 1$, such that the intensity vanishes. Therefore, the emitted radiation is still a direct measure of the degree of non-hermiticity of the system.

Following the general formalism described above, the observations for one-dimensional scattering can be directly extended to the general case of $\mathcal{PT}$-symmetric systems with many modes, for which compact expressions are no longer available. In particular, the emitted intensity generally remains finite even in the limit of a closed system. Because the expectation values $\langle \hat{a}_0^{\rm left\dagger}\hat{a}_0^{\rm left}\rangle \propto \Gamma^{-1}$, $\langle \hat{a}_0^{\rm right\dagger}\hat{a}_0^{\rm right}\rangle \propto \Gamma^{-1}$ of the internal operators generally diverge in this limit, this is accompanied by a diverging internal energy density, which can be interpreted as the source of this radiation. In practice, this necessarily leads to saturation in the amplifying parts of the system and therefore identifies an obstacle for the implementation of strict $\mathcal{PT}$ symmetry in closed optical systems. For open systems, the internal energy density becomes finite, while the emitted intensity remains a direct measure of the non-hermiticity of the system.

Conclusions.—It has been observed that non-hermitian $\mathcal{PT}$-symmetric systems with an entirely real spectrum define a consistent unitary theory of quantum mechanics [2, 3]. This fascinating prospect can be formalized using the concept of quasi-hermiticity, which introduces a new scalar product based on a generalised conjugation operation $\mathcal{C}$, satisfying $\mathcal{C}^2 = 1$, $[\mathcal{C}, H] = [\mathcal{C}, \mathcal{PT}] = 0$. The present paper demonstrates that accounting for quantum noise, non-hermitian $\mathcal{PT}$-symmetric systems are physically distinct from ordinary hermitian quantum systems because they emit self-sustained radiation of an intensity which is a direct measure of non-hermiticity. To understand the relation to quantum noise on a fundamental level, it suffices to consider the canonical commutation relations for the input and output operators. These commutation relations are only invariant under unitary transformations, which constraints the possibility to introduce alternative scalar products. From a practical perspective, the self-sustained radiation can be used as an indicator of successfully implemented non-hermitian $\mathcal{PT}$-symmetry in open systems, while the diverging internal energy density identifies a practical obstacle for its implementation in closed systems.

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