Optimal reconstruction of states in qutrits system

Fei Yan *,[1] Ming Yang†,[1] and Zhuo-Liang Cao[2]

[1]*Key Laboratory of Opto-electronic Information Acquisition and Manipulation, Ministry of Education, School of Physics and Material Science, Anhui University, Hefei 230039, People's Republic of China*

[2]*Department of Physics & Electronic Engineering, Hefei Normal University, Hefei 230061, People's Republic of China*

Based on the mutually unbiased measurements, an optimal tomographic scheme for the single-qutrit states and two-qutrit states is presented explicitly. Because the mutually-unbiased-bases based state reconstruction process is free of information waste, we call our scheme the optimal one. That is to say, by optimal we mean the number of required conditional operations reaches the minimum in our tomographic scheme for the states of qutrits system. We will pay our special attention on how to realize those different mutually unbiased measurements, i.e. how to decompose each transformation that connects each mutually unbiased basis with the standard computational basis. We found that all those transformations can be decomposed into several basic implementable single- and two-qutrit unitary operations. This will help the experimental scientists to realize the most economical reconstruction of quantum states in qutrits system.

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I. INTRODUCTION

The quantum state of a system is a fundamental concept in quantum mechanics, because a quantum state can be described by a density matrix, which contains all the information we can obtain about that system. A main task for implementing quantum computation is to reconstruct the density matrix of an unknown state. Quantum state reconstruction [1, 2], or quantum state tomography, are a set of useful tools to reconstruct the density matrix of an unknown state. The technique was first developed by Stokes to determine the polarization state of a light beam [3]. Recently, Minimal qubit tomography process has been proposed by Řeháček *et al*, where only four measurement probabilities are needed for fully determining a single qubit state, rather than the six probabilities in the standard procedure [4]. But the implementation of this tomography process requires measurement of N-particle correlations[5].The statistical reconstruction of biphotons states based on mutually complementary measurements has been proposed by Bogdanov *et al*[6, 7]. Ivanov *et al* investigated a method to determine an unknown mixed qutrit state from nine independent fluorescence signals[8]. Moreva *et al* paid attention to experimental problem of realization of the optimal protocol for polarization ququarts state tomography [9]. In 2009, Taguchi *et al* developed the single scan tomography of spatial three-dimensional (qutrits) state based on the effect of realistic measurement operators from the experimental results[10]. Allevi *et al* studied the implementation of a reconstruction scheme for determining the Wigner function and the density matrix for coherent and thermal states by on/off single photon avalanche photodetectors[11].

In order to obtain the full information about the system we need to perform a series of measurements on a large number of identical prepared copies of the system. Because the measurement results will not be independent of each other, there may be redundancy in the these results in the previously used quantum tomography process [12]. This will cause a resources waste. If we can totally remove this redundancy, the reconstruction process will become an optimal one. So to design an optimal set of measurements for totally removing the redundancy is of fundamental significance.

Mutually unbiased bases (MUBs) have been used in a variety of topics in quantum mechanics [13–15, 17–36]. MUBs are defined by the property that the squared overlap between a vector in one basis and all basis vectors in the other bases are equal. That is to say the detection over a particular basis state does not give away any information about the state if it is measured in another basis. Ivanovic first introduced the concept of MUBs to the problem of quantum state determination [13], and proved the existence of such bases in the prime-dimension system by an explicit construction. Some times later it had been shown by Wootters and Fields that measurements in this special class of bases, i.e. mutually unbiased measurements (MUMs) provide a minimal as well as optimal way of complete specification of an unknown density matrix [14]. They proved that the maximal numbers of MUBs is $d + 1$ in prime-dimension system. This result also applies to the prime-power-dimension system.

MUBs have a special role in determining quantum states, not only forming a minimal set of measurement bases but also providing an optimal way for determining a quantum state [13–16]. Recently an optimal tomographic reconstruction scheme was proposed by Klimov *et al* for the case of determining a state of multiqubit quantum system based on MUMs via trapped ions [37]. However, the use of three-level systems instead of two-level systems has been proven to be more secure against a symmetric attack on a quantum key distribution protocol, where the measurement bases are MUBs [38, 39]. Quantum tomography in high dimensional (qudit) systems has been proposed and the number of required measurements is $d^{2n} - 1$ with d being the dimension of the qudit system and n being the number of the qudits [12]. As shown before, this tomography process is not an optimal one,

*Electronic address: yanfei1599@163.com
†Corresponding Author Electronic address: mingyang@ahu.edu.cn

and there is a big redundancy among the measurement results there. To remove this redundancy, we will propose an optimal tomography process for qutrits states. This optimal quantum tomography process is the MUBs-based qutrits states tomography, and the number of required measurements is greatly reduced. A d-dimensional quantum system is represented by a positive semidefinite Hermitian matrix ρ with unit trace in d-dimensional Hilbert space, which is specified by d^2-1 real parameters. A nondegenerate measurement performed on such a system provides $d-1$ independent probabilities. So it shows that in general one requires at least $d+1$ different orthogonal measurements for fully determining an unknown ρ. In qutrit system, for n-qutrit tomography it only needs 3^n+1 measurements. Through analysis, one can find that the tomography process for qubit system can not be generalized to qudit system in a trivial way. The MUBs-based tomography process for qubit system proposed by Klimov *et al* can not be directly applied to qutrit system [37]. This is because that the entanglement feature of the MUBs of qubit system is totally different from that of qutrit system. So, we will study the physical implementation of an optimal tomographic scheme for the case of determining the states of qutrit system, including single-qutrit states and two-qutrit states, based on the MUMs.

This paper is arranged as follows. In the next section we introduce the MUBs in a d-dimensional system ($d=p^n$ $(p\neq 2)$) and show how to reconstruct an unknown state by MUMs. Here p is a prime number. In section 3 we briefly review the general method to reconstruct a qutrit state, where the number of measurements is 8. However we will show that the MUBs-based qutrit tomography only needs 4 measurements, which means the number of measurements is reduced. Reduction for single qutrit case is not obvious, so in section 4 we will extend one-qutrit case to two-qutrit case. We find that for the two-qutirt system the number of measurements is only 10 for determining all the elements of the density operator rather than $3^4-1=80$ measurements of the scheme proposed in Ref. [12]. It means a great reduction of the experimental complexity. So we conclude that the optimal measurements on the unknown qutrit states are the MUMs. The last section is the conclusion.

II. MUTUALLY UNBIASED BASES AND MUTUALLY UNBIASED MEASUREMENTS

As shown by Wootters, Fields [14] and Klappenecker, Rotteler [19], in finite field language, the first MUB in a $d=p^n$ $(p\neq 2)$-dimensional quantum system is the standard basis B^0 given by the vector $(a_k^{(0)})_l=\delta_{kl}, k,l\in \mathcal{F}_{p^n}$, where the superscript denotes the basis, k the vector in the basis, l the component and $\mathcal{F}_{p^n}$ is the field with p^n elements. The other d MUBs are denoted B^r which consists of vectors $(a_k^{(r)})_l$ defined by[14]:

$$(a_k^{(r)})_l=\frac{1}{\sqrt{d}}\omega^{\mathcal{T}r(r\cdot l^2+k\cdot l)}, r,k,l\in \mathcal{F}_{p^n}, r\neq 0, \quad (1)$$

where $\omega=exp(2\pi i/p)$ and the $\mathcal{T}r$ in Eq.(1) is defined as

$$\mathcal{T}r\theta=\theta+\theta^p+\theta^{p^2}+\cdots+\theta^{p^{n-1}}. \quad (2)$$

The set of mutually unbiased projectors can be given by

$$P_k^{(r)}=|a_k^{(r)}\rangle\langle a_k^{(r)}|. \quad (3)$$

It is worth noticing that $|a_k^{(r)}\rangle$ contains the computational basis B^0. Here

$$Tr(P_j^{(s)}P_k^{(r)})=\frac{1}{d}(1-\delta_{sr}+d\delta_{sr}\delta_{jk}). \quad (4)$$

Then the measurement probabilities given by

$$\omega_k^{(r)}=Tr(P_k^{(r)}\rho) \quad (5)$$

completely determine the density operator of an unknown d-dimensional system [13]:

$$\rho=\Sigma_{r=0}^{d}\Sigma_{k=0}^{d-1}\omega_k^{(r)}P_k^{(r)}-I. \quad (6)$$

For instance, in a qutrit system, there are three MUBs beside the computational basis $B^0=\{|0\rangle,|1\rangle,|2\rangle\}$, in the following form:

$$\begin{aligned} B^1 \ : \ \{|a_0^{(1)}\rangle &= \frac{1}{\sqrt{3}}(|0\rangle+|1\rangle+|2\rangle), \\ |a_1^{(1)}\rangle &= \frac{1}{\sqrt{3}}(|0\rangle+\omega|1\rangle+\omega^*|2\rangle), \\ |a_2^{(1)}\rangle &= \frac{1}{\sqrt{3}}(|0\rangle+\omega^*|1\rangle+\omega|2\rangle)\}; \end{aligned} \quad (7)$$

$$\begin{aligned} B^2 \ : \ \{|a_0^{(2)}\rangle &= \frac{1}{\sqrt{3}}(\omega|0\rangle+|1\rangle+|2\rangle), \\ |a_1^{(2)}\rangle &= \frac{1}{\sqrt{3}}(|0\rangle+\omega|1\rangle+|2\rangle), \\ |a_2^{(2)}\rangle &= \frac{1}{\sqrt{3}}(|0\rangle+|1\rangle+\omega|2\rangle)\}; \end{aligned} \quad (8)$$

$$\begin{aligned} B^3 \ : \ \{|a_0^{(3)}\rangle &= \frac{1}{\sqrt{3}}(\omega^*|0\rangle+|1\rangle+|2\rangle), \\ |a_1^{(3)}\rangle &= \frac{1}{\sqrt{3}}(|0\rangle+\omega^*|1\rangle+|2\rangle), \\ |a_2^{(3)}\rangle &= \frac{1}{\sqrt{3}}(|0\rangle+|1\rangle+\omega^*|2\rangle)\}, \end{aligned} \quad (9)$$

where $\omega=exp(2\pi i/3)$.

III. RECONSTRUCTION PROCESS FOR AN ARBITRARY SINGLE QUTRIT STATE

An unknown single qutrit state can be expressed as [12, 40]:

$$\rho=\frac{1}{3}\Sigma_{j=0}^{8}r_j\lambda_j, \quad (10)$$

TABLE I: The transformations connecting the MUBs with the standard computational basis for a qutrit system based on Fourier transforms and the phase operations.

Basis	Transformation
2	F^{-1}
3	$F^{-1}R$
4	$F^{-1}R^{-1}$

where the λ_j are the SU(3) generators and λ_0 is an identity operator [41]. The general method to reconstruct the qutrit state is to measure the expectation values of the λ operators [12], where $r_j = \langle\lambda_j\rangle = Tr[\rho\lambda_j]$. Thus one will find that the number of required measurements is 8. However, if we choose the MUMs to determine the qutrit state, the number of needed MUMs is only 4 rather than 8 of Ref. [12]. The four optimal set of MUBs have been presented by Eqs.(7,8,9) plus the standard computational basis in the preceding section. Each of the three MUBs in Eqs.(7,8,9) is related with the standard computational basis by a unitary transformation. These transformations have been listed in Table.I. Here, F denotes the Fourier transformation:

$$F|j\rangle = \frac{1}{\sqrt{3}}\sum_{l=0}^{2} exp(2\pi ilj/3)|l\rangle, j = 0, 1, 2, \quad (11)$$

and R denotes a phase operation:

$$R = |0\rangle\langle 0| + \omega|1\rangle\langle 1| + \omega|2\rangle\langle 2|. \quad (12)$$

If there are n qutrits, the number of MUM is $3^n + 1$, which is far less than $3^{2n} - 1$ in Ref. [12]. That is to say the use of MUMs can represent a considerable reduction in the operations and time required for performing the full state determination [37].

IV. RECONSTRUCTION PROCESS FOR AN ARBITRARY TWO-QUTRIT STATE

Now if we further extend one-qutrit case to two-qutrit case, the density matrix can be expressed as:

$$\rho_{12} = \frac{1}{9}\Sigma_{j,k=0} r_{jk}\lambda_j \otimes \lambda_k, \quad (13)$$

where $r_{jk} = \langle\lambda_j \otimes \lambda_k\rangle$. If we use the general method in Ref. [12] to fully determine the state, $d^{2n} - 1 = 3^4 - 1 = 80$ measurements will be needed. So much measurements will inevitably introduce redundant information of the state, which is obviously a resource waste. So here we will take advantage of the MUMs to reconstruct the two-qutrit state. It is easy to find that the nine elements of F_9 (finite field) are $\{0, \alpha, 2\alpha, 1, 1+\alpha, 1+2\alpha, 2, 2+\alpha, 2+2\alpha\}$ by using the irreducible polynomials $\theta^2 + \theta + 2 = 0$ [14]. Here we use the representation $\{|0\rangle, |\alpha\rangle, |2\alpha\rangle \cdots |2+2\alpha\rangle\}$ as the standard basis.

TABLE II: The decompositions of MUBs for two qutrits system based on Fourier transformations, two-level phase operations and controlled-NOT gates (X_{12}) [42] with the first particle as source and the second one as target. The subscript denotes the ith particles, $i = 1, 2$.

Basis	Decompositions
2	$F_1^{-1}R_1F_2^{-1}R_2$
3	$F_1^{-1}X_{12}F_2^{-1}R_2^{-1}$
4	$F_1^{-1}X_{12}R_1X_{12}F_2^{-1}R_2^{-1}$
5	$F_1^{-1}X_{12}X_{12}F_2^{-1}R_1^{-1}$
6	$F_1^{-1}R_1^{-1}F_2^{-1}R_2^{-1}$
7	$F_1^{-1}X_{12}X_{12}F_2^{-1}R_2$
8	$F_1^{-1}R_1^{-1}X_{12}F_2^{-1}R_2$
9	$F_1^{-1}R_1X_{12}F_2^{-1}$

One can find that there will be only $d^2 + 1 = 3^2 + 1 = 10$ MUMs to be done, which is much less than 80 of Ref. [12]. It means that the operations and time needed for the whole state determination is great reduced. The decompositions for all the MUBs of two-qutrit system have been listed in Table. II. In general, the fidelity of single logic gates can be greater than 99%, but nonlocal gates have a relatively lower fidelity. As shown in the Ref.[43], the fidelity of a practical CNOT gate can reach a value up to 0.926 for trapped ions system in Lab. Klimov *et al* have introduced the concept of the physical complexity of each set of MUBs as a function of the number of nonlocal gates needed for implementing the MUMs [37]. Here the fidelity value of the CNOT gates for qubits systems also can be used to evaluate the physical complexity of the MUBs of qutrits systems. Why we can say so is because of the following point. Although the systems involved here are three-state ones, all the operations used in our reconstruction process can be decomposed into effective two-state operations. So the complexity of the current tomography scheme is proportional to the number of the nonlocal gates used($C \propto 9$), which is a very important value in experimental realization.

V. CONCLUSION

We have explicitly presented an optimal tomographic scheme for the single-qutrit states and two-qutrit states based on the MUMs. Because the MUBs based state reconstruction process is free of information waste, the minimal number of required conditional operations are needed. So we call our qutrits tomographic scheme the optimal one. Here, we explicitly decompose each measurement into several basic single- and two-qutrit operations. Furthermore, all these basic operations have been proven implementable [42]. The physical complexity of a set of MUBs also has been calculated, which is an important threshold in experiment. We hope these decompositions can help the experimental scientists to realize the most economical reconstruction of quantum states in qutrits system in Lab.

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