Continuous-time quantum walks on the threshold network model

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Abstract It is well known that many real world networks have the power-law degree distribution (scale-free property). However there are no rigorous results for continuous-time quantum walks on such realistic graphs. In this paper, we analyze space-time behaviors of continuous-time quantum walks and random walks on the threshold network model which is a reasonable candidate model having scale-free property. We show that the quantum walker exhibits localization at the starting point, although the random walker tends to spread uniformly.

1 Introduction

Continuous-time quantum walks, which are the quantum counterparts of the classical random walks, have been widely studied on various deterministic graphs, such as the line [17], star graph [34,40], cycle graph [2,15,29], dendrimers [28], spidernet graphs [35], the Dual Sierpinski Gasket [1], direct product of Cayley graphs [36], quotient graphs [32], odd graphs [33], trees [16,18] and ultrametric spaces [19]. For further information, see reviews such as [20,39]. Also there are simulation based study of continuous-time quantum walks on probabilistic graphs, such as small-world networks [30] and Erdős-Rényi random graph [41]. However there are no rigorous results for continuous-time quantum walks on such probabilistic graphs. In this paper, we focus on the continuous-time quantum walk on a random graph called the threshold network model.

Many real world networks (graphs) are characterized by small diameters, high clustering, and power-law (scale-free) degree distributions [3, 4, 31]. The threshold network model belongs to the so-called hidden variable models [7, 38] and is known for being capable of generating scale-free networks. Their mean behavior [5, 7, 10, 11, 23, 37, 38] and limit theorems [9, 12, 13, 21] for the degree, the clustering coefficients, the number of subgraphs, and the average distance have been analyzed. The strong law of large numbers and central limit theorem for the rank of the adjacency matrix of the model with self-loops are given by [6]. Eigenvalues and eigenvectors of the adjacency matrix [14], the Laplacian matrix [26, 27] of the model have been studied. See also [8, 12, 13, 21, 22, 24, 25] for related works.

This paper is organized as follows. We define the threshold network model and give a brief review of the hierarchical structure of the graph in Section 2. In Section 3, we define the continuous-time quantum walk on the threshold network model and the special setting called the binary threshold model. The main results are presented in this section and the proofs are given in Section 4. Results on the continuous-time random walks on the models are obtained in Section 5. Summary is given in the last section.

2 Threshold network model

The *threshold network model* $\mathcal{G}_n(X, \theta)$ is a random graph on the vertex set $V = \{1, 2, \dots, n\}$. Let $\{X_1, X_2, \dots, X_n\}$ be independent copies of a random variable X with distribution $\mathbb{P}$. We draw an edge between two distinct vertices $i, j \in V$ if $X_i + X_j > \theta$ where $\theta \in \mathbb{R}$ is a constant called a threshold. Hereafter, we use $\mathbb{P}^\infty$ as the distribution of $\{X_i\}_{i=1}^\infty$.

Each sample graph $G \in \mathcal{G}_n(X, \theta)$ has a hierarchical structure described by the so-called creation sequence [8, 11]. Let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be a rearranged sequence of random variables $X_1, X_2, \dots, X_n$ in increasing order. If $X_{(1)} + X_{(n)} > \theta$, we have

$$\theta < X_{(1)} + X_{(n)} \le X_{(2)} + X_{(n)} \le \cdots \le X_{(n-1)} + X_{(n)},$$

which means that the vertex corresponding to $X_{(n)}$ is connected with the $n-1$ other vertices. Otherwise, we have

$$\theta \ge X_{(1)} + X_{(n)} \ge \cdots \ge X_{(1)} + X_{(3)} \ge X_{(1)} + X_{(2)},$$

which means that the vertex corresponding to $X_{(1)}$ is isolatedD We set $s_n = 1$ or $s_n = 0$ according as the former case or the latter occurs. Then, according to the case we remove the random variable $X_{(n)}$ or $X_{(1)}$, we continue similar procedure to define $s_{n-1}, \dots, s_2$. Finally, we set $s_1 = s_2$ and obtain a $\{0, 1\}$-sequence $\{s_1, s_2, \dots, s_n\}$, which is called the *creation sequence* of G and is denoted by S_G.

Given a creation sequence S_G let k_i and l_i denote the number of consecutive bits of 1's and 0's, respectively, as follows:

$$S_G = \{\overbrace{1, \dots, 1}^{k_1}, \overbrace{0, \dots, 0}^{l_1}, \overbrace{1, \dots, 1}^{k_2}, \overbrace{0, \dots, 0}^{l_2}, \dots, \overbrace{1, \dots, 1}^{k_m}, \overbrace{0, \dots, 0}^{l_m}\}. \tag{2.1}$$

It may happen that $k_1 = 0$ or $l_m = 0$, but we have $k_2, \dots, k_m, l_1, \dots, l_{m-1} \ge 1$ and $m \ge 1$. Moreover, by definition we have two cases: (a) $k_1 = 0$ (equivalently $s_1 = 0$) and $l_1 \ge 2$; (b) $k_1 \ge 2$ (equivalently, $s_1 = 1$).

For example, if $S_G = \{1, 1, 0, 0, 1, 0, 1, 0\}$ then $k_1 = 2,\ l_1 = 2,\ k_2 = 1,\ l_2 = 1,\ k_3 = 1,\ l_3 = 1$ and Fig. 1 shows the shape of G.

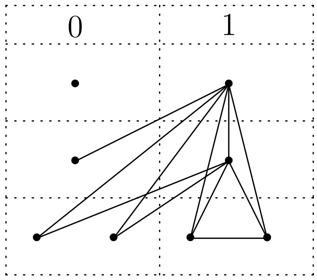

Figure 1: A threshold graph G corresponding to $S_G = \{1, 1, 0, 0, 1, 0, 1, 0\}$

The creation sequence S_G gives rise to a partition of the vertex set:

$$V = \bigcup_{i=1}^{m} V_i^{(1)} \cup \bigcup_{i=1}^{m} V_i^{(0)} \qquad |V_i^{(1)}| = k_i\,, \quad |V_i^{(0)}| = l_i\,.$$

The subgraph induced by $V_i^{(1)}$ is the complete graph of k_i vertices, and that induced by $V_i^{(0)}$ is the null graph of l_i vertices. Moreover, every vertex in $V_i^{(1)}$ (resp. $V_i^{(0)}$) is connected (resp. disconnected) with all vertices in

$$V_1^{(1)} \cup \cdots \cup V_i^{(1)} \cup V_1^{(0)} \cup \cdots \cup V_{i-1}^{(0)}\,.$$

In general, a graph possessing the above hierarchical structure is called a *threshold graph* [22].

3 Our model and results

Let A_G be the adjacency matrix and D_G be the diagonal matrix of degrees (the sum of the rows of A_G) of $G \in \mathcal{G}_n(X, \theta)$. Then the Laplacian matrix L_G of G is given by $L_G = D_G - A_G$. The time evolution operator $U_{n,t}^G$ of a continuous-time quantum walk on G is defined by

$$U_{n,t}^G = e^{itL_G} \equiv \sum_{k=0}^{\infty} \frac{(it)^k}{k!} L_G^k. \tag{3.2}$$

Let $\{\Psi_{n,t}^G\}_{t\geq 0}$ be the probability amplitude of the quantum walk, i.e., $\Psi_{n,t}^G = U_{n,t}^G \Psi_{n,0}^G$, and $X_{n,t}$ denotes the position of the quantum walker at time t. Then the probability that the quantum walker on G is in position $x \in V$ at time t with initial condition $\Psi_{n,0}^G$ is defined by

$$P_{n,t}^G(X_{n,t} = x) \equiv |\Psi_{n,t}^G(x)|^2,$$

where $\Psi_{n,t}^G = {}^T\big[\Psi_{n,t}^G(1) \quad \cdots \quad \Psi_{n,t}^G(n)\big]$. Here ${}^T A$ denotes the transpose of a matrix A.

The time evolution operator is obtained as follows:

THEOREM 3.1 *Suppose $G \in \mathcal{G}_n(X,\theta)$ is connected. The time evolution operator $U^G_{n,t}$ of the continuous-time quantum walk on G is given by*

$$\left(U^G_{n,t}\right)_{v,w} = \begin{cases} \left(U^G_{n,t}\right)^{(1,i)}_{v,w} & \text{if } v \in V^{(1)}_i,\ w \in V^{(1)}_{j+1} \cup V^{(0)}_j \ (j \leq i-1), \\ \left(U^G_{n,t}\right)^{(0,i)}_{v,w} & \text{if } v \in V^{(0)}_i, w \in V^{(1)}_j \cup V^{(0)}_j \ (j \leq i). \end{cases}$$

Here $A_{v,w}$ denotes the (v,w) element of a matrix A and

$$\begin{aligned}\left(U^G_{n,t}\right)^{(1,i)}_{v,w} &= \left(I_{V^{(1)}_i}(w) - \frac{1}{D_{k_i} - D_{l_i} + 1}\right) e^{it(D_{k_i}+1)} + \sum_{j=i+1}^{m} \frac{(D_{l_{j-1}} - D_{l_j}) e^{it(D_{k_j}+1)}}{(D_{k_j} - D_{l_{j-1}} + 1)(D_{k_j} - D_{l_j} + 1)} \\ &+ \sum_{j=i}^{m-1} \frac{(D_{k_{j+1}} - D_{k_j}) e^{itD_{l_j}}}{(D_{k_j} - D_{l_j} + 1)(D_{k_{j+1}} - D_{l_j} + 1)} + \frac{1}{n}, \\ \left(U^G_{n,t}\right)^{(0,i)}_{v,w} &= \left(I_{V^{(0)}_i}(w) - \frac{1}{D_{k_{i+1}} - D_{l_i} + 1}\right) e^{itD_{l_i}} + \sum_{j=i}^{m} \frac{(D_{l_{j-1}} - D_{l_j}) e^{it(D_{k_j}+1)}}{(D_{k_j} - D_{l_{j-1}} + 1)(D_{k_j} - D_{l_j} + 1)} \\ &+ \sum_{j=i+1}^{m-1} \frac{(D_{k_{j+1}} - D_{k_j}) e^{itD_{l_j}}}{(D_{k_j} - D_{l_j} + 1)(D_{k_{j+1}} - D_{l_j} + 1)} + \frac{1}{n}.\end{aligned}$$

Where D_{k_i} and D_{l_i} denote the degree of vertices in $V^{(1)}_i$ and $V^{(0)}_i$, respectively and $I_A(x)$ is the indicator function of a set A, i.e., $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ otherwise.

Theorem 3.1 shows that we can obtain the probability of the quantum walker in position $x \in V$ at time t for any initial conditions at least in principle. But in general cases, it is hard to obtain the probability. In this paper, we analyze behaviors of quantum walks starting from a vertex v, i.e., the case of

$$\Psi_{n,0}(s) = \begin{cases} 1, & \text{if } s = v, \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 3.2 *Suppose G is connected. The limit of the probability of the quantum walker starting from a vertex $v \in V^{(1)}_m$ is given by*

$$\lim_{n\to\infty} P^G_{n,t}(x) = \begin{cases} 1, & \text{if } x = v, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } \mathbb{P}^\infty\text{-almost every } G.$$

Because $P^G_{n,t}$ is not converge in $t \to \infty$, we study the time-averaged probability $\bar{P}^G_n(x)$ defined by

$$\bar{P}^G_n(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T P^G_{n,t}(X_{n,t} = x) dt.$$

THEOREM 3.3 *Suppose $G \in \mathcal{G}_n(X,\theta)$ is connected. The time-averaged probability $\bar{P}_n^G(x)$ of the quantum walker on G starting from a vertex $v \in V_m^{(1)}$ is*

$$\bar{P}_n^G(x) = \begin{cases} \left(1-\frac{1}{n}\right)^2 + \frac{1}{n^2}, & \text{if } x = v, \\ \frac{2}{n^2}, & \text{otherwise.} \end{cases}$$

In order to study more detailed properties of the quantum walk on the model, we focus on the threshold network model $\mathcal{G}_n(X,\theta)$ defined by Bernoulli trials with success probability $p \in (0,1)$, i.e., $\mathbb{P}(X=1) = 1 - \mathbb{P}(X=0) = p$, and a threshold $\theta \in [0,1)$. We call this model the *binary threshold model* $\mathcal{G}_n(p)$. For each $G \in \mathcal{G}_n(p)$, i.e., realization G of $\mathcal{G}_n(p)$, we consider a partition of the vertex set V:

$$V = V_G^{(1)} \cup V_G^{(0)}, \quad V_G^{(1)} = \{i : X_i = 1\}, \quad V_G^{(0)} = \{i : X_i = 0\}.$$

It is easy to see that the subgraph induced by $V_G^{(1)}$ is the complete graph on $k_G \equiv \sharp V_G^{(1)}$ vertices, and that induced by $V_G^{(0)}$ is the null graph on $l_G \equiv \sharp V_G^{(0)}$ vertices, where $\sharp A$ is the number of elements in a set A. Moreover, every vertex in $V_G^{(1)}$ is connected with all vertices in $V_G^{(0)}$. Note that this is the case of $m = 2$, $k_1 = l_2 = 0$, $k_2 = k_G$ and $l_1 = l_G$ in Eq. (2.1).

The time evolution operator of the continuous-time quantum walk on the binary threshold model is obtained as follows:

THEOREM 3.4 *The time evolution operator $U_{n,t}^G$ of the continuous-time quantum walk on $G \in \mathcal{G}_n(p)$ is given by*

$$U_{n,t}^G = \begin{bmatrix} U_{k_G,k_G} & U_{k_G,l_G} \\ U_{l_G,k_G} & U_{l_G,l_G} \end{bmatrix}.$$

The elements of $U_{n,t}^G$ are

$$\begin{aligned} U_{k_G,k_G} &= e^{int} I_{k_G} + \frac{1-e^{int}}{n} \mathbf{1}_{k_G,k_G}, \\ U_{k_G,l_G} &= \frac{1-e^{int}}{n} \mathbf{1}_{k_G,l_G}, \\ U_{l_G,k_G} &= \frac{1-e^{int}}{n} \mathbf{1}_{l_G,k_G}, \\ U_{l_G,l_G} &= e^{ik_G t} I_{l_G} + \left(\frac{1}{n} + \frac{k_G e^{int}}{n l_G} - \frac{e^{ik_G t}}{l_G} \right) \mathbf{1}_{l_G,l_G}, \end{aligned}$$

where $\mathbf{1}_{i,j}$ is the $i \times j$ matrix consisting of only 1 *and I_i is the $i \times i$ identity matrix.*

On the binary threshold model, a strong localization is observed at any starting point as follows:

PROPOSITION 3.5 *The limit of the probability of the quantum walker starting from a vertex $v \in V$ is given by*

$$\lim_{n\to\infty} P_{n,t}^G(x) = \begin{cases} 1, & \text{if } x = v, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } \mathbb{P}^\infty\text{-almost every } G.$$

Note that if the quantum walker starts from a vertex $v \in V_G^{(1)}$ then the statement of Proposition 3.5 is the same as Theorem 3.2.

The time averaged probability of the quantum walker on the binary threshold model is obtained as follows:

PROPOSITION 3.6 *The time-averaged probability $\bar{P}_n^G(x)$ of the quantum walker on $G \in \mathcal{G}_n(p)$ starting from a vertex $v \in V_G^{(0)}$ is given by*

$$\bar{P}_n^G(x) = \begin{cases} \left(1 - \frac{1}{l_G}\right)^2 + \left(\frac{k_G}{nl_G}\right)^2 + \frac{1}{n^2}, & \text{if } x = v, \\ \frac{1}{l_G^2} + \left(\frac{k_G}{nl_G}\right)^2 + \frac{1}{n^2}, & \text{if } x \in V_G^{(0)} \setminus \{v\}, \\ \frac{2}{n^2}, & \text{otherwise.} \end{cases}$$

Note that when the quantum walker starts from a vertex $v \in V_G^{(1)}$, we can use Theorem 3.3.

4 Proofs

The Laplacian matrix L_G of $G \in \mathcal{G}_n(p)$ is given by

$$L_G = \begin{bmatrix} nI_{k_G} - \mathbf{1}_{k_G,k_G} & -\mathbf{1}_{k_G,l_G} \\ -\mathbf{1}_{l_G,k_G} & kI_{l_G} \end{bmatrix}.$$

Eigenvalues and eigenvectors of L_G are known as follows [11, 26, 27]:

eigenvalue	eigenvectors
n	$\mathbf{v}_j \equiv \frac{1}{\sqrt{j(j+1)}} \begin{bmatrix} \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{n-j-1,1} \end{bmatrix} \ (1 \leq j \leq k_G - 1), \mathbf{v}_{k_G} \equiv \frac{1}{\sqrt{nk_G l_G}} \begin{bmatrix} l_G \mathbf{1}_{k_G,1} \\ -k_G \mathbf{1}_{l_G,1} \end{bmatrix},$
k_G	$\mathbf{w}_j \equiv \frac{1}{\sqrt{j(j+1)}} \begin{bmatrix} \mathbf{0}_{k_G,1} \\ \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{l_G-j-1,1} \end{bmatrix} \ (1 \leq j \leq l_G - 1),$
0	$\mathbf{w}_{l_G} \equiv \frac{1}{\sqrt{n}} \left[\mathbf{1}_{n,1}\right],$

where $\mathbf{0}_{i,j}$ is the $i \times j$ zero matrix. Note that the set of these eigenvectors forms an orthonormal basis of $\mathbb{R}^n$. Thus we can define a orthogonal matrix B_G corresponding to the eigenvectors as follows:

$$B_G = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_{k_G} & \mathbf{w}_1 & \dots & \mathbf{w}_{l_G} \end{bmatrix}. \tag{4.3}$$

Because $U_{n,t}^G$ is an $n \times n$ (finite) matrix, using Eqs. (3.2) and (4.3), it is represented by

$$U_{n,t}^G = B_G \begin{bmatrix} e^{int} I_{k_G} & \mathbf{0}_{k_G,l_G} & \mathbf{0}_{k_G,1} \\ \mathbf{0}_{l_G-1,k_G} & e^{ik_G t} I_{l_G-1} & \mathbf{0}_{l_G-1,1} \\ \mathbf{0}_{1,k_G} & \mathbf{0}_{1,l_G-1} & 1 \end{bmatrix} {}^T B_G.$$

It is easy to see that

$$U_{n,t}^G = \begin{bmatrix} e^{int} I_{k_G} + \left[\left(-\frac{1}{k_G} + \frac{l_G}{nk_G}\right) e^{int} + \frac{1}{n}\right] \mathbf{1}_{k_G,k_G} & \frac{1}{n}\left(1 - e^{int}\right) \mathbf{1}_{k_G,l_G} \\ \frac{1}{n}\left(1 - e^{int}\right) \mathbf{1}_{l_G,k_G} & e^{ik_G t} I_{k_G} + \left[-\frac{1}{l_G} e^{ik_G t} + \frac{k_G}{nl_G} e^{int} + \frac{1}{n}\right] \mathbf{1}_{l_G,l_G} \end{bmatrix}.$$

By using a relation $-1/k_G + l_G/nk_G = -1/n$, we obtain Theorem 3.4.

It is given by simple calculations that

$$\left|\left(1 - \frac{1}{n}\right) e^{int} + \frac{1}{n}\right|^2 = 1 - \frac{2}{n}\left(1 - \frac{1}{n}\right)\left(1 - \cos nt\right),$$

$$\left|\frac{1}{n}\left(1 - e^{int}\right)\right|^2 = \frac{1}{n^2}\left(2 - \cos nt\right),$$

$$\begin{aligned} \left|\left(1 - \frac{1}{l_G}\right) e^{ik_G t} + \frac{k_G e^{int}}{nl_G} + \frac{1}{n}\right|^2 &= \left(1 - \frac{1}{l_G}\right)^2 + \left(\frac{k_G}{nl_G}\right)^2 + \frac{1}{n^2} \\ &\quad + \frac{2k_G}{nl_G}\left(1 - \frac{1}{l_G}\right)\left(\cos l_G t + \frac{l_G}{k_G}\cos k_G t\right) + \frac{2k_G}{n^2 l_G}\cos nt, \end{aligned}$$

$$\begin{aligned} \left|-\frac{e^{ik_G t}}{l_G} + \frac{k_G e^{int}}{nl_G} + \frac{1}{n}\right|^2 &= \frac{1}{l_G^2} + \left(\frac{k_G}{nl_G}\right)^2 + \frac{1}{n^2} \\ &\quad - \frac{2k_G}{nl_G^2}\left(\cos l_G t + \frac{l_G}{k_G}\cos k_G t - \frac{l_G}{n}\cos nt\right). \end{aligned}$$

On the other hand, we see that $\lim_{n\to\infty} k_G/n = p$ and $\lim_{n\to\infty} l_G/n = 1 - p$, $\mathbb{P}^\infty$-almost surely by the strong law of large numbers for the i.i.d. Bernoulli sequence. Combining these facts and Theorem 3.4, we have Proposition 3.5. Also we obtain Proposition 3.6 immediately from

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \cos nt\, dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T \cos k_G t\, dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T \cos l_G t\, dt = 0.$$

In the case of $G \in \mathcal{G}_n(X, \theta)$ which is connected, eigenvalues and eigenvectors of L_G are also known as follows [11, 26, 27]:

eigenvalue	eigenvectors
$D_{k_i}+1$ $(2 \leq i \leq m)$	$\frac{1}{\sqrt{j(j+1)}}\begin{bmatrix} \mathbf{0}_{u_i+l_i} \\ \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{d_i+k_i-j-1,1} \end{bmatrix}$ $(1 \leq j \leq k_i-1)$, $\frac{1}{\sqrt{(k_i+d_i)k_i d_i}}\begin{bmatrix} \mathbf{0}_{u_i+l_i} \\ d_i\mathbf{1}_{k_i,1} \\ -k_i\mathbf{1}_{d_i,1} \end{bmatrix}$,
D_{l_i} $(2 \leq i \leq m-1)$	$\frac{1}{\sqrt{j(j+1)}}\begin{bmatrix} \mathbf{0}_{u_i,1} \\ \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{d_{i+1}-j-1,1} \end{bmatrix}$ $(1 \leq j \leq l_i-1)$, $\frac{1}{\sqrt{(k_i+d_i)l_i d_{i+1}}}\begin{bmatrix} \mathbf{0}_{u_i+l_i} \\ (k_i+d_i)\mathbf{1}_{l_i,1} \\ -l_i\mathbf{1}_{k_i+d_i,1} \end{bmatrix}$,
$D_{k_1}+1$	$\frac{1}{\sqrt{j(j+1)}}\begin{bmatrix} \mathbf{0}_{u_i+l_i} \\ \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{k_1-j-1,1} \end{bmatrix}$ $(j=1,\ldots,k_1-1)$, if $k_1 \neq 0$,
D_{l_1}	$\frac{1}{\sqrt{j(j+1)}}\begin{bmatrix} \mathbf{0}_{u_1,1} \\ \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{d_2-j-1,1} \end{bmatrix}$ $(j=1,\ldots,l_1-1)$,
D_{l_1}	$\frac{1}{\sqrt{(k_1+d_1)l_1 d_2}}\begin{bmatrix} \mathbf{0}_{u_1+l_1} \\ (k_1+d_1)\mathbf{1}_{l_1,1} \\ -l_1\mathbf{1}_{k_1+d_1,1} \end{bmatrix}$, if $k_1 \neq 0$,
0	$\frac{1}{\sqrt{n}}\left[\mathbf{1}_{n,1}\right]$,

where $u_i = \sum_{j>i}(k_j + l_j)$ and $d_i = \sum_{j<i}(k_j + l_j)$. By the same argument in the proof of Theorem 3.4 and the following relations:

$$\begin{aligned} k_1 &= D_{k_1} - D_{l_1} + 1 \text{ (if } k_1 \neq 0), \\ k_i &= D_{l_{i-1}} - D_{l_i} \ (2 \leq i \leq m), \\ l_i &= D_{k_{i+1}} - D_{k_i} \ (1 \leq i \leq m-1), \end{aligned}$$

we have Theorem 3.1.

Note that $D_{k_m} = n - 1$ and $D_{l_m} = 0$ by assumpution. Comparing Theorem 3.1 with Theorem 3.4, the transition probability of the walker on $G \in \mathcal{G}_n(p)$ starting from a vertex $v \in V_m^{(1)}$ is the same as that of on $G \in \mathcal{G}_n(p)$ starting from a vertex $v \in V_G^{(1)}$. Thus we have Theorems 3.2 and 3.3.

5 Classical Case

The time evolution operator $\mathcal{U}_{n,t}^G$ of a continuous-time random walk on $G \in \mathcal{G}_n(X, \theta)$ is defined by

$$\mathcal{U}_{n,t}^G = e^{-tL_G} \equiv \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} L_G^k.$$

Let $\{\mathcal{P}_{n,t}^G\}_{t \geq 0}$ be the probability distribution of the random walk, i.e., $\mathcal{P}_{n,t}^G = \mathcal{U}_{n,t}^G \mathcal{P}_{n,0}^G$, and $Y_{n,t}$ denotes the position of the random walker at time t. By the same observation in the previous section, we have the same results for $\mathcal{U}_{n,t}^G$ as Theorems 3.1 and 3.4 by exchanging it of $U_{n,t}^G$ for t. Using these results, we have the following:

Proposition 5.1 *The limit of the probability that the random walker starting from a vertex $v \in V_m^{(1)}$ is given by*

$$\lim_{n \to \infty} n \mathcal{P}_{n,t}^G(y) = 1, \quad \textit{for all } y \in V, \textit{ for } \mathbb{P}^{\infty}\textit{-almost every } G.$$

Proposition 5.2 *The long-time limit of the probability of the random walk on $G \in \mathcal{G}_n(p)$ starting from a vertex $v \in V$ is given by*

$$\lim_{t \to \infty} \mathcal{P}_{n,t}^G(y) = \frac{1}{n}, \quad \textit{for all } y \in V.$$

We can also estimate the time-averaged probability $\bar{\mathcal{P}}_n^G(y)$. By simple calculation, we have $\bar{\mathcal{P}}_n^G(y) = 1/n$ for a random walk on $G \in \mathcal{G}_n(p)$ starting from a vertex $v \in V$.

6 Summary

In this paper, we study the continuous-time quantum and random walks on the threshold network model. By comparing Theorem 3.2 with Proposition 5.1, we have quite different limit behaviors in $n \to \infty$ for the two types of walks starting from a vertex which degree equals $n-1$. Although quantum walkers exhibit strong localization at the starting point, random walkers tend to spread uniformly.

Theorem 3.3 and Proposition 3.6 show that the time-averaged probabilities of quantum walkers are not the uniform distribution (different from random walks). Furthermore, the time-averaged probability shows localization at starting point as $n \to \infty$. In the case of the binary threshold model, the rate of convergence are slightly different in the two starting points. Indeed, we obtain $\lim_{n\to\infty} n(1-\bar{P}_n^G(v_1)) = 2 < \lim_{n\to\infty} n(1-\bar{P}_n^G(v_0)) = 2/(1-p)$, $\mathbb{P}^\infty$-almost surely for $v_0 \in V_G^{(0)}$ and $v_1 \in V_G^{(1)}$. A study covering the more general setting is now in progress.

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