Nonlinear friction in quantum mechanics

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The effect of nonlinear friction forces in quantum mechanics is studied via dissipative Madelung hydrodynamics. A new thermo-quantum diffusion equation is derived, which is solved for the particular case of quantum Brownian motion with a cubic friction.

Nonlinear friction forces are a problem in classical Brownian motion for a long time [1]. They are described either by Langevin or by Fokker-Planck equations [2-4]. However, the rigorous generalized Langevin equation is linear, which points out that nonlinear friction forces possess a macroscopic hydrodynamic origin [1]. For this reason, they are not present in the modern quantum theory of open systems [5]. The scope of the present paper is to investigate the nonlinear friction effect on quantum mechanics. The analysis is based on a dissipative Madelung quantum hydrodynamics.

A quantum particle in vacuum is described by the Schrödinger equation

$$i\hbar\partial_t\psi = -\hbar^2\partial_x^2\psi / 2m + U\psi \qquad (1)$$

where m is the particle mass and U is an external potential. The complex wave function can be generally presented in its polar form $\psi = \sqrt{\rho}\exp(iS/\hbar)$, where ρ is the probability density and $S/\hbar$ is the wave function phase. Introducing this Madelung presentation in Eq. (1) results in the following two equations [6]

$$\partial_t\rho = -\partial_x(\rho V) \qquad m\partial_t V + mV\partial_x V = -\partial_x p_Q / \rho - \partial_x U \qquad (2)$$

corresponding to the imaginary and real parts, respectively. The first equation is a continuity one with $V \equiv \partial_x S / m$ being the hydrodynamic-like velocity in the probability space. The second equation is a macroscopic force balance, where the quantum effects are completely included in the quantum pressure $p_Q \equiv -(\hbar^2 / 4m)\rho\partial_x^2 \ln\rho$. Note that the latter depends both on the local density ρ and its spatial derivatives and, hence, the Madelung hydrodynamics is a non-local theory.

The Madelung presentation of the Schrödinger equation opens a door for introduction of dissipative forces in quantum mechanics. The friction force of a particle in a classical environment depends naturally on the particle velocity. Hence, one can add a macroscopic friction force $f(V)$ in the force balance (2) to obtain

$$m\partial_t V + mV\partial_x V = -\partial_x(p_Q + k_B T\rho)/\rho - \partial_x U + f(V) \tag{3}$$

Here the new pressure term accounts for the osmotic thermal pressure due to the environment temperature T. Thus one arrives to a dissipative Madelung hydrodynamics. At strong friction the inertial terms on the left-hand-site of Eq. (3) can be neglected as compared to the friction force and the hydrodynamic-like velocity can be expressed in the form $V = f^{-1}(\partial_x\mu)$, where f^{-1} is the inverse function of f and $\mu = Q + k_B T\ln\rho + U$ is the local chemical potential. The chemical potential term $Q \equiv -\hbar^2\partial_x^2\sqrt{\rho} / 2m\sqrt{\rho}$, corresponding to the quantum pressure via the Gibbs-Duhem relation $dp_Q = \rho dQ$, is in fact the Bohm quantum potential. While the latter is an icon in the de Broglie-Bohm theory, the symbol of the Madelung hydrodynamics is p_Q. Introducing now this expression for V into the continuity equation (2) results in a generalized nonlinear diffusion equation

$$\partial_t\rho = -\partial_x[\rho f^{-1}(\partial_x\mu)] \tag{4}$$

The equilibrium solution of Eq. (4) corresponds to $V = 0$ or a constant chemical potential, which is in accordance to the rules of thermodynamics.

Equation (4) is valid for arbitrary friction forces. Usually the friction force is well approximated by the expression $f(V) = -b_1 V - b_3 V^3$ with two friction coefficients, a linear one b_1 and a cubic one b_3 [7]. At low velocity the cubic term becomes negligible and, hence, $f^{-1}(\partial_x \mu) = -\partial_x \mu / b_1$. Thus Eq. (4) reduces to a quantum Smoluchowski equation [8]

$$\partial_t \rho = \partial_x [\rho \partial_x (Q + U) / b_1 + D \partial_x \rho] \tag{5}$$

where $D = k_B T / b_1$ is the classical Einstein diffusion constant. The solution of Eq. (5) for a free particle at zero temperature is a Gaussian distribution density with dispersion obeying the sub-diffusive quantum law $\sigma^2 = \hbar \sqrt{t / m b_1}$ [8]. In the opposite case of a fast particle the cubic term dominates the friction force and $f^{-1}(\partial_x \mu) = -\sqrt[3]{\partial_x \mu / b_3}$. Thus Eq. (4) acquires the following strongly nonlinear form

$$\partial_t \rho = \partial_x [\rho \sqrt[3]{\partial_x (Q + k_B T \ln \rho + U) / b_3}] \tag{6}$$

For a classical particle moving in a biquadratic external potential $U = Kx^4 / 4$ the solution of Eq. (6) reads $\rho = \Gamma(3/4) \exp(-x^4 / 4\sigma^4) / \pi\sigma$, where the average displacement evolves in time according to the equation

$$F(1/3, 1/3; 4/3; K\sigma^4 / k_B T) \sqrt[3]{K\sigma^4 / k_B T} = (4/3) \sqrt[3]{K / b_3}\, t \tag{7}$$

Here Γ and F are the gamma and hypergeometric functions, respectively. The plot of Eq. (7) is shown in Fig. 1. As is seen, initially the evolution is super-diffusive, than passes through a normal diffusive regime and ends with a sub-diffusive part. At infinite time $\sigma_\infty^4 = k_B T / K$ and the probability density reduces to the equilibrium Boltzmann distribution. In the case of a free classical particle with a cubic friction force Eq. (7) provides a super-diffusive classical law $\sigma^2 = \sqrt{64 k_B T t^3 / 27 b_3}$. Hence, the nonlinear friction accelerates the particle diffusion, which is, however, non-Gaussian.

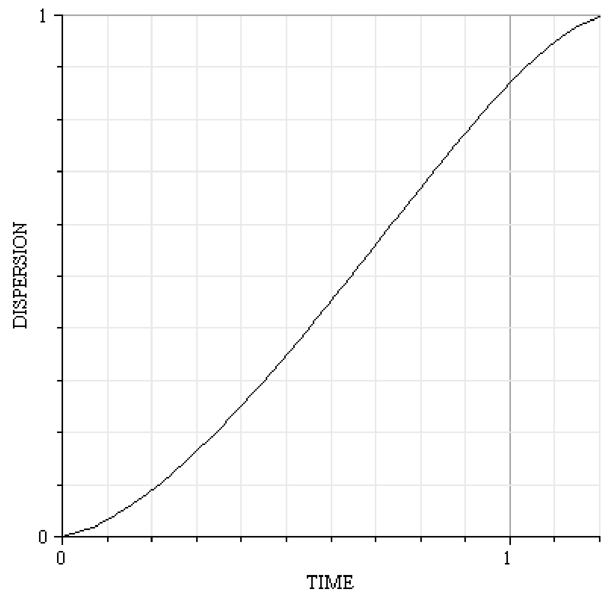

Fig. 1 Dimensionless dispersion $\sqrt{K / k_B T}\sigma^2$ vs. dimensionless time $(4/3)\sqrt[3]{K / b_3}t$.

In the case of free quantum diffusion at zero temperature Eq. (6) reduces to

$$\partial_t \rho = \partial_x (\rho \sqrt[3]{\partial_x Q / b_3}) = -\partial_x [\rho \sqrt[3]{\hbar^2 (\partial_x \ln \rho \partial_x^2 \ln \rho + \partial_x^3 \ln \rho) / 4mb_3}] \qquad (8)$$

At large x one can neglect the third-derivative term in the brackets of Eq. (8) and the solution of the remaining equation is $\rho = 3\sqrt[6]{3}\Gamma(2/3)\exp(-|x|^3 / 3\sigma^3) / 4\pi\sigma$. Surprisingly, the corresponding displacement obeys a normal diffusive law $\sigma^2 = \sqrt[3]{4\hbar^2 / mb_3}t$. This unexpected result shows that the quantum sub-diffusivity compensate the super-diffusivity originating from the cubic friction in such a way that the final result corresponds formally to the classical Einstein law with a novel quantum diffusion constant $\sqrt[3]{\hbar^2 / 2mb_3}$. The distribution density above is, however, non-Gaussian again.

Generally, it is possible to find the inverse function of the complete nonlinear friction force $f(V) = -b_1 V - b_3 V^3$ and to perform the corresponding analysis of Eq. (4). The physical transparency will suffer, however, due to mathematical complications. We expect the appearance of many sub- and super-diffusive regimes, which alternatively can be formally described via fractal diffusion equations [9]. In the case of diffusion is structured environment it is expected that the friction coefficients b_1 and b_3 will depend on the local particle position x [10]. This will not change, however, the validity of the general diffusion equation (4). Moreover, any more advanced model for the local chemical potential μ could be directly employed in Eq. (4).

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