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# BALANCING WITH DISTRIBUTED REFLEX DELAY

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**Abstract.** From a dynamics view-point, human self-balancing is a very complex process. The vertical position of the body is unstable in the gravitational space, and the human control tries to stabilize it when we stand still. The dynamic study of the control of the inverted pendulum provides important conclusions for balancing. The role of the delay of human reflexes is emphasized in this report. Different kinds of time delays are investigated. The critical delays of reflexes are determined with discrete delays, and also with delays distributed over the past. The numerical values fit well the experimentally determined ones.

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*Keywords*: time delay, balancing, linear stability, reflex, labyrinth

## 1. Introduction

The dynamics of human balancing is still not understood perfectly. Recently, neurologists [1] have started using balancing experiments to test the behavior of the human brain, to identify and quantify certain disorders. Such an experiment could be, for example, how long a patient can balance a half a meter long stick on his/her fingertip. From the results of these experiments, one can draw conclusions on the ability to concentrate and on the speed of reflexes.

Clearly, there is a unique correlation between the speed of reflexes and the performance of balancing. Self-balancing of a human body is much more complex than balancing a stick on a fingertip, but the role of time delay in the control is the same, and the arising vibration frequencies are also in the same range. The number of fall-overs of elderly people in retirement homes is proven to be correlated with the increased reflex time and increased threshold of perception [2]. It is also well-known, that the consumption of alcohol increases the reflex delay and jeopardizes not only self-balancing, but also any other human control aiming at stabilization, like driving a car. The loss of stability leads to oscillations about the desired but unstable equilibrium. The vibration frequencies are also measured and known not only on patients but also in sportsmen when balancing is important like in target-shooting [3].

In this report, a mechanical model of the inverted pendulum is given and analyzed in the presence of delayed PD control that imitates a spring-damper support. Simple formulae are derived for the critical time delays where balancing is still possible. The results are compared for different time delay models: discrete delays and delays distributed over the past with certain realistic weight functions imitating probability distributions. In unstable cases, the range of vibration frequencies is also determined. The results are compared with the qualitative behavior of the human balancing organ, the labyrinth.

Finally, a qualitative explanation is given for the chaotic nature of successful balancing. Precise measurements show that there are small-amplitude irregular oscillations even in cases when balancing is considered to be successful. These small oscillations can be modeled as chaotic ones. The dynamic models presenting chaos may give a new insight into the classical problem of the energy consumption of balancing, and also into the stochastic nature of fall-overs.

## 2. **Delayed dynamic model of balancing**

Figure 1 shows the simplest possible mechanical model of balancing a stick. The model has 2 degrees of freedom (DoF), the corresponding general coordinates are the horizontal displacement $x$ of the lowest point of the stick, and the angle $\varphi$ between the stick and the vertical direction. The horizontal force $Q$ controls the beam of mass $m$ and length $l$.

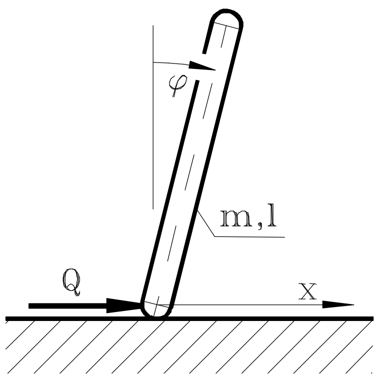


Figure 1. Mechanical model of balancing a beam

The derivation of the system of nonlinear differential equations is given in [4]. The variational system is obtained by linearisation at the upper vertical position $\varphi = 0$.

Its matrix differential equation assumes the form

$$\begin{pmatrix} m & \frac{1}{2}ml \\ \frac{1}{2}ml & \frac{1}{3}ml^2 \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}mgl \end{pmatrix} \begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix} \tag{2.1}$$

Dot stands for differentiation with respect to time. The control force is considered in the form of a simple PD controller

$$Q(t) = P \int_{-\infty}^{0} w_n(\vartheta)\varphi(t+\vartheta)\mathrm{d}\vartheta + D \int_{-\infty}^{0} w_n(\vartheta)\dot{\varphi}(t+\vartheta)\mathrm{d}\vartheta \tag{2.2}$$

where the weight function

$$w_n(\vartheta) = (-1)^n \frac{n^{n+1}}{\tau^{n+1} n!} \vartheta^n \mathrm{e}^{n\vartheta/\tau}, \quad \vartheta \in (-\infty, 0] \tag{2.3}$$

describes the weights of the delayed values of the angle and angular velocity of the beam in the control. Its integral between $-\infty$ and 0 is 1, and its maximum is at $-\tau$ for the parameters $n > 0$ (see Figure 2).

The delay of the operator's reflexes is characterized by the value $\tau$, while the deviation of the reflex delay is characterized by the integer $n$: the larger it is, the smaller the deviation is. In some sense, the weight function characterizes the probability distribution of the somewhat stochastically varying reflex delay around $\tau$. As $n$ tends to infinity, the weight function converges to the Dirac delta function (which tends to infinity at $-\tau$ and is zero elsewhere):

$$\lim_{n\to\infty} w_n(\vartheta) = \delta(\vartheta + \tau) \tag{2.4}$$

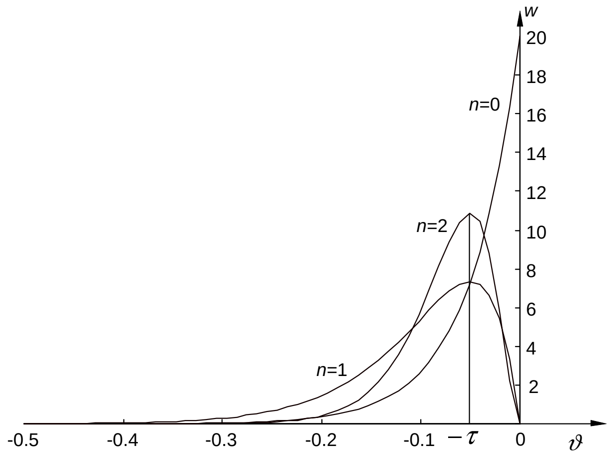


Figure 2. Weight function with respect to the past for $\tau = 0.05$ [s]

All this means that the horizontal control force $Q$ applied by the operator's hand at the lowest point of the stick must be proportional to the sum of the delayed values of the perceived angle and angular velocity of the beam. If this force is substituted into the linear system (2.1) of differential equations, the coordinate $x$ can be eliminated and a scalar delay differential equation (DDE) is obtained for the angle in the form:

$$\ddot{\varphi}(t)-\frac{6g}{l}\varphi(t)+\frac{6}{ml}D\int_{-\infty}^{0}w_n(\vartheta)\dot{\varphi}(t+\vartheta)\mathrm{d}\vartheta+\frac{6}{ml}P\int_{-\infty}^{0}w_n(\vartheta)\varphi(t+\vartheta)\mathrm{d}\vartheta=0\,. \quad (2.5)$$

When the weight function is the Dirac delta function ($n\to\infty$), the DDE (2.5) simplifies to a discrete delay equation:

$$\ddot{\varphi}(t)-\frac{6g}{l}\varphi(t)+\frac{6}{ml}D\dot{\varphi}(t-\tau)+\frac{6}{ml}P\varphi(t-\tau)=0. \quad (2.6)$$

The DDE (2.5) and consequently, its special case (2.6), are satisfied by the trivial solution, which corresponds to the upper equilibrium position of the beam. However, its stability analysis is usually not trivial since the DDE is not an ordinary differential equation, its state space and spectrum are usually infinite dimensional.

## 3. Stability analysis

Several special cases are studied to analyze the stability of the controlled inverted pendulum. First, check the case when there is no time delay in the system:

$$\ddot{\varphi}(t)+\frac{6}{ml}D\dot{\varphi}(t)+\frac{6}{ml}(P-mg)\varphi(t)=0. \quad (3.1)$$

The well-known Routh-Hurwitz criterion [5] proves that in this case, asymptotic stability is equivalent to the positiveness of all the coefficients, that is,

$$D>0,\quad P>mg \quad (3.2)$$

guarantees that both characteristic roots $\lambda_{1,2}$ of the characteristic equation

$$\lambda^2+\frac{6}{ml}D\lambda+\frac{6}{ml}(P-mg)=0 \quad (3.3)$$

have negative real parts. This means that balancing should always be successful for any large gains $P$ and $D$ without time delay in the system.

If the characteristic time delay $\tau$ is positive in (2.5) with weights (2.3), the DDE is infinite dimensional and it has infinitely many characteristic roots satisfying the transcendental characteristic equation

$$\lambda^2-\frac{6g}{l}+\frac{6}{ml}D\iiint_{-\infty}^{0}w_n(\vartheta)\lambda e^{\lambda\vartheta}\mathrm{d}\vartheta+\frac{6}{ml}P\iiint_{-\infty}^{0}w_n(\vartheta)e^{\lambda\vartheta}\mathrm{d}\vartheta=0\,, \quad (3.4)$$

which is obtained by substituting the exponential trial solution $x(t)=Ae^{\lambda t}$ into the DDE (2.5). When the reflex delay has no deviation, that is the weight function is the Dirac delta, the characteristic equation assumes the simple form

$$\lambda^2-\frac{6g}{l}+\frac{6}{ml}D\lambda\mathrm{e}^{-\lambda\tau}+\frac{6}{ml}P\mathrm{e}^{-\lambda\tau}=0\,. \quad (3.5)$$

In order to have asymptotic stability for a set of control parameters $P$, $D$ and $\tau$, all the infinitely many characteristic roots must be located in the left half of the complex plane. The lengthy but exact stability analysis can be found in [6]. In order to simplify the stability analysis to a finite dimensional one, suppose that the time delay is a small parameter, and use the power series expansion of the exponential expressions in the characteristic function (3.5) till their $2^{nd}$ degree terms with respect to $\tau$ :

$$e^{-\lambda\tau} \approx 1 - \lambda\tau + \frac{1}{2}\lambda^2\tau^2 \qquad \Rightarrow \tag{3.6}$$

$$\frac{1}{ml}\left[\underbrace{3D\tau^2}_{a_3}\lambda^3 + \underbrace{(ml + 3P\tau^2 - 6D\tau)}_{a_2}\lambda^2 + \underbrace{6\,(D - P\tau)}_{a_1}\lambda + \underbrace{6\,(P - mg)}_{a_0}\right] = 0\,.$$

Although this approximation is not convergent [6], it gives a quite reliable quantitative result at this $2^{nd}$ order approximation. The Routh-Hurwitz stability conditions imply that

$$a_k > 0, k = 0, ..., 3 \text{ and } H_2 = a_1a_2 - a_0a_3 > 0 \;\Rightarrow$$

$$P > mg,\; D > P\tau,\; D < \frac{ml}{6\tau} + \frac{\tau}{2}P \quad \text{and}$$

$$\begin{pmatrix} P & D \end{pmatrix}\begin{pmatrix} -\tau^2 & \tau \\ \tau & -2 \end{pmatrix}\begin{pmatrix} P \\ D \end{pmatrix} + ml\begin{pmatrix} \frac{1}{3} & \frac{1}{3\tau} + \frac{g}{l}\tau \end{pmatrix}\begin{pmatrix} P \\ D \end{pmatrix} > 0 \tag{3.7}$$

The corresponding stability domain in the plane of the gain parameters is presented in Figure 3 for a fixed delay parameter. This domain is essentially separated from the axes, it is bordered by the vertical line $a_0 = 0$ and by the ellipse $H_2 = 0$. This means that there is no way to stabilize the inverted pendulum without using both signals, namely the angle and the angular velocity of the beam.

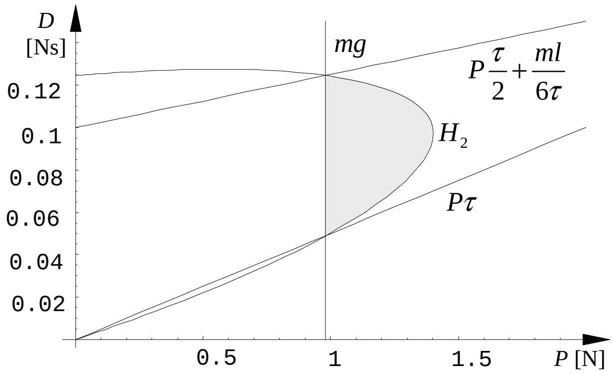


Figure 2. Stability domain (shaded) without reflex deviation for $m = 0.1$ [kg], $l = 0.3$ [m], $\tau = 0.05$ [s] with $n \to \infty$

If the deviation of the reflex delay is also modeled by the use of distributed time delays over the past, the weight function (2.3) can be applied in the equation of motion (2.5). The advantage of these polynomial-exponential weight functions is that the otherwise infinite dimensional system becomes finite dimensional. Consider the $n = 2$ case, that is,

$$w_2(\vartheta) = \frac{4}{\tau^3}\vartheta^2 e^{2\vartheta/\tau}, \quad \vartheta \in (-\infty, 0] \tag{3.8}$$

In this case, the integrals can be calculated in closed form in the characteristic function (3.4). The multiplication of the characteristic function by $\lambda^3$ results the polynomial

$$\lambda^5 + \frac{6}{\tau}\lambda^4 + \frac{6g}{l\tau^2}\left(2\frac{l}{g} - \tau^2\right)\lambda^3 + \frac{12g}{l\tau^3}\left(\frac{2l}{3g} - \tau^2\right)\lambda^2 +$$
$$\frac{48}{ml\tau^3}\left(D - \frac{3}{2}mg\tau\right)\lambda + \frac{48}{ml\tau^3}(P - mg) = \sum_{k=0}^{5} a_k\lambda^k = 0 \tag{3.9}$$

The Routh-Hurwitz criterion leads to the necessary and sufficient exponential stability conditions

$$a_k > 0, k = 0,1,2 \ \Rightarrow \ P > mg, \ D > \frac{3}{2}mg\tau, \ \tau < \sqrt{\frac{2}{3}\frac{l}{g}}$$
$$H_2 > 0 \ \Rightarrow \ D < \frac{7}{9}mg\tau + \frac{8ml}{27\tau} + \frac{mg^2\tau^3}{6l} + \frac{\tau}{6}P$$
$$H_3 > 0 \ \Rightarrow \begin{pmatrix} P & D \end{pmatrix}\begin{pmatrix} -3l\tau^3 & 18l\tau^2 \\ 18l\tau^2 & -108l\tau \end{pmatrix}\begin{pmatrix} P \\ D \end{pmatrix} +$$
$$m\begin{pmatrix} -48l^2\tau - 6gl\tau^3 - 9g^2\tau^5 & 18g^2\tau^4 + 32l^2 + 228gl\tau^2 \end{pmatrix}\begin{pmatrix} P \\ D \end{pmatrix} -$$
$$18m^2g^2\tau^3(8l + g\tau^2) > 0, \tag{3.10}$$

while $a_{3,4,5} > 0$ and $H_{1,4} > 0$ are already satisfied. In this case, the stability chart in Figure 4 is bordered by the vertical line $a_0 = 0$, again, and by the parabola $H_3 = 0$.

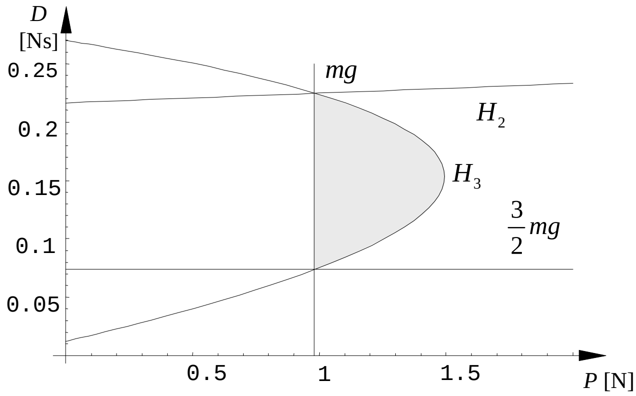


Figure 3. Stability domain (shaded) with reflex deviation for $m = 0.1$ [kg], $l = 0.3$ [m], $\tau = 0.05$ [s] with $n = 2$

The structure of the stability chart is the same as in the case of the $2^{nd}$ degree approximation of the discrete delay case in Figure 3 without deviation in the reflex delay.

Note that apart from the similar structure of the stability chart in the two cases, a condition appears for the time delay with $a_2>0$ in the latter case. This condition is not represented in Figure 4.

## 4. **Vibration frequencies**

In both stability charts, saddle-node bifurcation occurs when the proportional control gain $P$ decreases below the weight $mg$ of the beam, and the upper vertical position becomes unstable. When the proportional gain $P$ increases and the corresponding parameter point crosses the $H_2 = 0$ or the $H_3 = 0$ stability limit, Hopf bifurcation occurs. This means, that self-excited vibrations arise around the upper vertical position with frequency

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{a_1}{a_3}} \qquad [\mathrm{Hz}] \,, \tag{4.1}$$

where the coefficients of the corresponding characteristic polynomials (3.6) and (3.9) should be calculated at the corresponding stability limit. When the reflex delay deviation is zero, this frequency varies between

$$0 < f < \frac{\sqrt{2}}{2\pi}\frac{1}{\tau}\sqrt{\frac{l-3g\tau^2}{l+3g\tau^2}} \,. \tag{4.2}$$

The formula is much more complicated for the non-zero reflex delay deviation case, but it provides a similar range of frequencies.

## 5. **Critical reflex delay**

The stability domains in Figures 3 and 4 shrink as the characteristic time delay $\tau$ is increased either with zero or with non-zero reflex deviation characterized by 1/n. The critical time delay, where the stability domain disappears, can easily be calculated.

In the zero-deviation case, the stability conditions (3.6) lead to an empty set when

$$a_0 = 0, a_1 = 0, a_2 = 0 \;\;\Rightarrow\;\; \tau_{\mathrm{cr},n\to\infty} = \sqrt{\frac{l}{3g}} \,. \tag{5.1}$$

Similarly, the critical characteristic time delay is obtained from the stability conditions (3.9):

$$a_0 = 0, a_1 = 0, H_2 = 0 \;\;\Rightarrow\;\; \tau_{\mathrm{cr},n=2} = \sqrt{\frac{2l}{3g}} \,. \tag{5.2}$$

## 6. Conclusions

Some direct conclusions can be derived from the above linear theory of balancing an inverted pendulum. First of all, the stability charts clearly express the requirement of a combined application of proportional and differential controllers, that is, both the angle and angular velocity signals must be measured. Stabilization is impossible with one of the angle or the angular velocity signals only, when either $P$ or $D$ is zero. Although the self-balancing of the human body is a much more complicated process, but the balancing organ, the so-called labyrinth, in the inner ear of human beings also provides these signals. The corresponding two parts of the labyrinth are called static and dynamic receptors (see Figure 5), which provide a signal about the spatial inclination and angular velocity of the head.

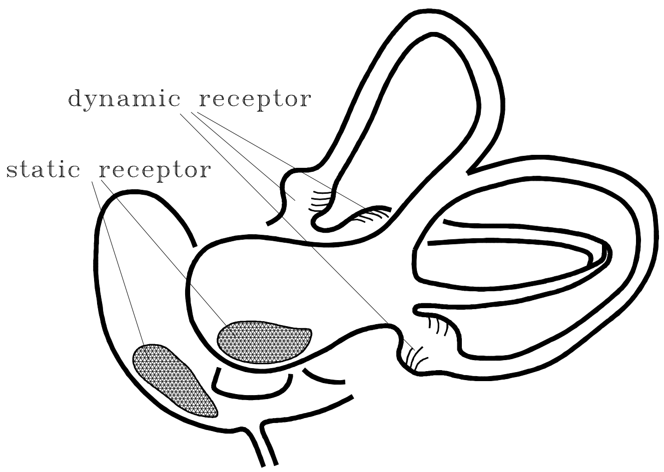


Figure 5. The labyrinth

The vibration frequencies given by formula (4.2) at the loss of stability also provide realistic numerical results. For the parameters given in Figure 3, the frequency range

$$0 < f < 3.5 \quad \text{[Hz]}$$

fits to the experimental observations well [7].

In [6], the critical time delay was determined for the infinite dimensional model (2.6) exactly, without using the small time delay approximation in (3.6). Still, the result is exactly the same as in (5.1). For a 0.3 [m] long stick, this formula means a 0.1 [s] critical reflex delay. This is just about the mean value of the reflex delay from the eye through the brain and the arm to the fingers. After some practice, anybody can balance a stick of length 30 [cm], but we cannot do this with shorter ones. On the other hand, we are able to balance a sweep of length 1.5 [m] even on our toe, in spite of the fact that the time delay is much more in that direction, it is in the range of 0.7 [s]. When the deviation of the reflex delay is non-zero, the critical value of the characteristic delay is larger by a factor of $\sqrt{2}$ in accordance with formula (5.2).

Finally, some comments are given on the observation that balancing is never perfect, small amplitude stochastic oscillations persist even in cases when the balancing should be exponentially stable. This vibration is called micro-chaos [8,9,10]. The mechanism of this vibration is as follows. There is a certain threshold of our senses: below a certain angle, for example, our senses are not able to make a distinction between the tiny angle values and zero. Consequently, the control is switched off without input signals, and the stick starts falling from the close vicinity of the unstable upper position till the angle signal reaches the perception threshold, where the human control is switched on, again, and starts pushing back the pendulum close to the upper vertical position. Then, below the threshold level, again, the stick is left without control, and so on. This process leads to a small amplitude chaotic oscillation.

Similar micro-chaotic oscillation is observed in the walking of drunken people. While an increasing amount of alcohol first decreases, later increases the threshold level of perception, any small amount of alcohol consumed increases the reflex delay. The result is instability and a small amplitude random oscillation with low-frequency components at about 1 [Hz] in its continuous spectrum.

## REFERENCES


1. Moss, F. and Milton, JG.: Medical technology - Balancing the unbalanced. *Nature*, **425**, (2003), 911-912.
2. Stépán, G.: Micro-chaos in digitally controlled mechanical systems, in Nonlinearity and Chaos in Engineering Dynamics (Eds.: J.M.T. Thompson and S.R. Bishop), John Wiley & Sons, Chichester, 1994,. 143-154.
3. Bretz, K. and Kaske, R. J.: Some parameters of multiloop biofeedback control of posture. *Proceedings of XII Int. Symposium on Biomechanics in Sport*, (Siófok, 1994), 124-128.
4. Stépán, G. and Kollár, L.: Balancing with reflex delay. *Mathematical and Computer Modelling*, **31**, (2000), 199-205.
5. Kuo, B. C.: *Digital Control Systems*. SRL Publishing Company, Champaign, Illinois, 1977.
6. Stépán, G.: *Retarded Dynamical Systems*. Longman, Harlow UK, 1989.
7. Kawazoe, Y.: Manual control and computer control of an inverted pendulum on a cart, *Proc. 1st Int. Conf. on Motion and Vibration Control*, (Yokohama, 1992), 930-935.
8. Haller, G. and Stépán, G.: Micro-chaos in digital control. *Journal of Nonlinear Science*, **6**, (1996), 415–448.
9. Enikov, E. and Stépán, G.: Micro-chaotic motion of digitally controlled machines. *Journal of Vibration and Control*, **4**, (1998), 427-443.
10. Chen, G. and Dong, X.: *From Order to Chaos: Perspectives, Methodologies, and Applications*. World Scientific, Singapore, 1997.