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ANALYSIS OF NONLINEAR ULTRAACOUSTIC WAVE PROPERTIES IN GERMANIUM MONOCRYSTAL

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Abstract. The present paper investigates the properties of the second harmonics of monochromatic symmetrical normal waves. The analytical representations for nonharmonic distortion of normal waves with a free propagation direction in the plane of a cubic anisotropic monocrystal germanium layer have been obtained. The intensity of the second harmonics and the wave motion forms have been analyzed for nonelastic equivalent propagation directions.

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1. Introduction

In the present paper nonlinear effects, which appear when stationary stress waves spread in elastic mediums, have been investigated. One of the widely used conceptions of such a study is the determination of the so-called higher orders harmonics, which describe nonlinear, i.e., nonharmonic effects. This conception is effectively used to investigate nonlinear elastic waves with low intensity, and on its base a great number of fundamental scientific and applied results are obtained. The most important features of this conception can be found in papers [1-3]. The procedure we apply is based on the representation of the elastic wave displacements in a series in terms of the acoustic Mach number, which can be regarded as a small parameter.

We obtain appropriate equations for the different members of the series from the nonlinear equations of dynamics written in terms of displacements. The solutions to these equations are referred to as second, third and fourth order harmonics, respectively. By applying this approach we want to determine the second harmonics of monochromatic elastic waves, which belong to one of the wave motions mode for the waveguide considered, or the compound second harmonics of linear waves. The latter belong to two different waveguide modes. In the second case, the question about the normal waves' three-phonon interaction deserves special attention.

Works [1-9] are devoted to the solution methods and provide a number of results for the propagation of the second harmonics of elastic waves in anisotropic bodies. In these works a geometrically nonlinear model is selected and the effect of the propagation medium on the equations of motion is also clarified. The issues of how to obtain the solutions, how to describe the second harmonics and the basic physical-mechanical effects for the nonlinear wave phenomena have all been analyzed both theoretically and experimentally. In the above works the second harmonics of longitudinal and shift bulk waves in isotropic mediums, the second harmonics of compound monochromatic waves in crystalline mediums in a number of crystal systems have been obtained. The questions about three-phonon interaction of bulk elastic waves in anisotropic and isotropic mediums have also been considered. In paper [10] the second harmonics are found for Relay-type surface waves in an isotropic medium.

There are only a few works devoted to the problem of how to obtain and analyze the second harmonics of normal waves in waveguides of different geometry with cross-section dimension, restricted at least on one coordinate. For example, in paper [11] the analysis of nonharmonic effects for the propagation of flexure elastic waves in a thin isotropic lamina is considered.

2. **The model and basic equations of the wave process**

Indicial notations are employed in a Cartesian coordinate system throughout this paper. In accordance with the general rules of indicial notations summation over repeated indices is implied and subscripts preceded by a colon denote differentiation with respect to the corresponding coordinate. Latin indices range over the integers 1, 2 and 3.

Nonlinear elastic wave propagation has been investigated in an arbitrary direction in the plane of the waveguide. The volume V under consideration is given by

$$V = \{-\infty < x_1, x_2 < \infty, |x_3| \leq h\}\,, \tag{2.1}$$

where $x_1,\ x_2$ and x_3 are non-dimensional coordinates.

The body under consideration is homogenous and anisotropic. The problem is a dynamic one. Components ε_{ij} of the Lagrange deformation tensor in terms of displacements u_i are given by the equation

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{l,i}u_{l,j})\,. \tag{2.2}$$

It is assumed that the elastic potential has the form

$$U = \frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{6}c_{ijklmn}\varepsilon_{ij}\varepsilon_{kl}\varepsilon_{mn}, \tag{2.3}$$

in which c_{ijkl} and c_{ijklmn} are the tensors of elastic constants. The second Piola–Kirchoff stress tensor σ_{jq} can be divided into two parts:

$$\sigma_{jq} = \partial U/\partial(u_{j,q}) = \sigma_{jq}^{(l)} + \sigma_{jq}^{(n)}, \tag{2.4}$$

where

$$\sigma_{jq}^{(l)} = c_{jqik}u_{i,k}, \quad \sigma_{jq}^{(n)} = \frac{1}{2}c_{jqik}u_{l,i}u_{l,k} + c_{pqik}u_{j,p}u_{i,k} + \frac{1}{2}c_{jqiklm}u_{i,k}u_{l,m}. \tag{2.5}$$

The density and elastic properties of the O^h class monocrystal cubic system layer under consideration are characterized by the following second and third order nonzero elastic constants:

$$\begin{gathered}
\tilde{\rho} = \rho\rho_*; \quad \tilde{c}_{11} = \tilde{c}_{22} = \tilde{c}_{33} = c_{11}c_*; \\
\tilde{c}_{12} = \tilde{c}_{13} = \tilde{c}_{21} = \tilde{c}_{23} = \tilde{c}_{31} = \tilde{c}_{32} = c_{12}c_*; \\
\tilde{c}_{44} = \tilde{c}_{55} = \tilde{c}_{66} = c_{44}c_*; \quad \tilde{c}_{111} = \tilde{c}_{222} = \tilde{c}_{333} = c_{111}c_*; \\
\tilde{c}_{112} = \tilde{c}_{113} = \tilde{c}_{122} = \tilde{c}_{133} = \tilde{c}_{223} = \tilde{c}_{233} = c_{112}c_*; \\
\tilde{c}_{144} = \tilde{c}_{255} = \tilde{c}_{366} = c_{144}c_*; \quad \tilde{c}_{123} = c_{123}c_*; \quad \tilde{c}_{456} = c_{456}c_*; \\
\tilde{c}_{155} = \tilde{c}_{166} = \tilde{c}_{244} = \tilde{c}_{266} = \tilde{c}_{344} = \tilde{c}_{355} = c_{155}c_*,
\end{gathered} \tag{2.6}$$

where the values of the normalizing parameters are $c_* = 10^{10}$ Pa, $\rho_* = 10^3$ kg/m^3.

The elastic potential in quadratic and cubic terms of $u_{i,j}$ for the monocrystal layer has the form

$$\begin{gathered}
U = \frac{1}{2}c_{11}\sum_{k=1}^{3} u_{k,k}^2 + \frac{1}{2}c_{44}\sum_{k,l=1,\,k\neq l}^{3} u_{k,l}^2 + \\
+c_{44}\sum_{k,l=1,\,k<l}^{3} u_{k,l}u_{l,k} + c_{12}\sum_{k,l=1,\,k<l}^{3} u_{k,k}u_{l,l} + \\
+\frac{1}{2}\Delta_3\sum_{k,l=1,\,k\neq l}^{3} u_{k,k}u_{l,k}^2 + \frac{1}{2}\Delta_2\sum_{k,l=1,\,k\neq l}^{3} u_{k,k}u_{k,l}^2 + \\
+\Delta_6\sum_{k,l,m=1,\,k\neq l,m,\,l<m}^{3} u_{k,l}u_{k,m}(u_{l,m}+u_{m,l}) + \\
+\frac{1}{6}\Delta_1\sum_{k=1}^{3} u_{k,k}^3 + \frac{1}{2}\Delta_5\sum_{k,l=1,\,k\neq l}^{3} u_{k,k}u_{l,l}^2 + \\
+\frac{1}{2}\Delta_7\sum_{k,l,m=1,\,k\neq l,m,\,l\neq m}^{3} u_{k,k}u_{l,m}^2 + \\
+c_{144}\sum_{k,l,m=1,\,k\neq l,m,\,l<m}^{3} u_{k,k}u_{l,m}u_{m,l} + \\
+\Delta_4\sum_{k,l=1,\,k<l}^{3} u_{k,k}u_{k,l}u_{l,k} + \\
+\sum_{k,l,m=1,\,k\neq l,m,\,l\neq m}^{3} u_{k,l}u_{k,m}u_{m,l} + \\
+c_{456}\sum_{l,m=2,\,l\neq m}^{3} u_{1,l}u_{l,m}u_{m,1},
\end{gathered} \tag{2.7}$$

where

$$\begin{gathered}
\Delta_1 = 3\,c_{11} + c_{111}; \quad \Delta_2 = c_{12} + 2\,c_{44} + c_{155}; \quad \Delta_3 = c_{11} + c_{155}; \\
\Delta_4 = c_{44} + c_{155}; \quad \Delta_5 = c_{12} + c_{112}; \quad \Delta_6 = c_{44} + c_{456}; \quad \Delta_7 = c_{12} + c_{144}.
\end{gathered} \tag{2.8}$$

The equations of motion in terms of displacements are obtained from the equation of motion

$$\sigma_{ij,j} = \rho_0\ddot{u}_i \tag{2.9}$$

and have the form

$$\rho_0 \ddot{u}_j - \Delta_8(u_{l,lj} + u_{k,kj}) - c_{44}(u_{j,ll} + u_{j,kk}) - c_{11}u_{j,jj} =$$
$$= \Delta_1 u_{j,j}u_{j,jj} + \Delta_2(2u_{j,l}u_{j,lj} + 2u_{j,k}u_{j,kj} + u_{j,j}u_{j,ll} + u_{j,j}u_{j,kk}) +$$
$$+\Delta_3(u_{l,j}u_{l,jj} + u_{k,j}u_{k,jj} + u_{l,l}u_{j,ll} + u_{j,l}u_{l,ll} + u_{k,k}u_{j,kk} + u_{j,k}u_{k,kk}) +$$
$$+\Delta_4(2u_{l,j}u_{j,lj} + u_{j,l}u_{l,jj} + 2u_{k,j}u_{j,kj} + u_{j,k}u_{k,jj} + u_{l,j}u_{l,ll} + u_{k,j}u_{k,kk}) +$$
$$+\Delta_5(u_{l,l}u_{j,jj} + u_{k,k}u_{j,jj}) +$$
$$+\Delta_9(u_{k,l}u_{l,kj} + u_{l,k}u_{k,lj} + u_{k,j}u_{l,kl} + u_{l,j}u_{k,lk}) + \tag{2.10}$$
$$+\Delta_{10}(u_{k,k}u_{l,lj} + u_{l,l}u_{k,kj}) +$$
$$+\Delta_6(u_{k,j}u_{k,ll} + 2u_{k,l}u_{j,kl} + u_{j,k}u_{k,ll} + 2u_{l,k}u_{j,lk} + u_{j,l}u_{l,kk} + u_{l,j}u_{l,kk}) +$$
$$+\Delta_7(u_{k,k}u_{j,ll} + u_{l,l}u_{j,kk}) +$$
$$+(\Delta_4 + \Delta_5)(u_{l,l}u_{l,lj} + u_{j,j}u_{l,lj} + u_{j,j}u_{k,kj} + u_{k,k}u_{k,kj}) +$$
$$+(\Delta_6 + \Delta_7)(u_{l,k}u_{l,kj} + u_{k,l}u_{k,lj} + u_{j,k}\ u_{l,kl} + u_{j,l}u_{k,lk}) \quad (j = \overline{1,3}),$$

where

$$l = \begin{cases} 1, & j = 2,3; \\ 2, & j = 1; \end{cases} \qquad k = \begin{cases} 3, & j = 1,2; \\ 2, & j = 3; \end{cases} \tag{2.11}$$

$$\Delta_8 = c_{12} + c_{44}; \quad \Delta_9 = c_{144} + c_{456}; \quad \Delta_{10} = c_{123} + c_{144}. \tag{2.12}$$

Our main objective is to find the analytical representations for second harmonics of normal three-partial waves. More precisely we would like to determine what forms the linear three-partial waves' second harmonics have and to investigate the intensity levels of the second harmonics.

3. Analytical solution of a homogeneous problem for linear waves

After giving the elastic displacements u_j as a sum of the linear harmonic terms $u_j^{(l)}$ and its inharmonic distortion $u_j^{(n)}$ – the latter is proportional to the acoustic Mach number of the first degree – we can determine the expressions for $u_j^{(l)}$ and $u_j^{(n)}$ from the first and the second boundary value problems:

$$\begin{aligned} &\rho_0 \ddot{u}_j^{(l)} - c_{jsrk}u_{s,k}^{(l)} = 0, \\ &(c_{3srk}u_{r,k}^{(l)})_{x_3=\pm h} = 0; \end{aligned} \tag{3.1}$$

$$\rho_0\ddot{u}_j^{(n)} - c_{jdik}u_{i,dk}^{(n)} = c_{jdik}u_{l,dk}^{(l)}u_{l,i}^{(l)} + c_{pdik}(u_{j,dp}^{(l)}u_{i,k}^{(l)}+$$
$$+u_{i,dk}^{(l)}u_{j,p}^{(l)}) + c_{jdiklm}u_{i,dk}^{(l)}u_{l,m}^{(l)},$$
$$(c_{3dik}u_{i,k}^{(n)})_{x_3=\pm h} = -(\tfrac{1}{2}c_{3dik}u_{l,i}^{(l)}u_{l,k}^{(l)}+$$
$$+c_{pdik}u_{3,p}^{(l)}u_{i,k}^{(l)} + \tfrac{1}{2}c_{3diklm}u_{i,k}^{(l)}u_{l,m}^{(l)})_{x_3=\pm h}. \tag{3.2}$$

Partial displacement functions of the linear normal waves, which propagate in the waveguide plain in an arbitrary direction characterized by the angle φ and the vector $\mathbf{n}$, can be represented in a complex exponential form

$$u_j^{(l)}(x_1,x_2,x_3,t) = f_j(x_3)\exp\{-i(\omega t - k(n_1x_1 + n_2x_2))\} \quad (j=\overline{1,3}), \tag{3.3}$$

where

$f_j(x_3)$ is the complex amplitude function;

ω is the circular frequency of the wave;

k is a non-dimensional normalized wave number;

$n_1 = \cos\varphi$ and $n_2 = \sin\varphi$ are the components of the wave vector $\mathbf{n}$.

Equations for the amplitude functions $f_j(x_3)$ are obtained from (3.1):

$$\begin{cases} f_1''(x_3) + A_{11}f_1(x_3) + A_{12}f_2(x_3) + A_{13}f_3'(x_3) = 0, \\ A_{21}f_1(x_3) + f_2''(x_3) + A_{22}f_2(x_3) + A_{23}f_3'(x_3) = 0, \\ A_{31}f_1'(x_3) + A_{32}f_2'(x_3) + f_3''(x_3) + A_{33}f_3(x_3) = 0; \end{cases} \tag{3.4}$$

$$\begin{cases} (in_1f_3(x_3) + f_1'(x_3))_{x_3=\pm h} = 0, \\ (in_2f_3(x_3) + f_2'(x_3))_{x_3=\pm h} = 0, \\ (c_{12}i(n_1f_1(x_3) + n_2f_2(x_3)) + c_{11}f_3'(x_3))_{x_3=\pm h} = 0. \end{cases} \tag{3.5}$$

In the above equations A_{ij} are the elements of the Christoffel matrix for the cubic medium:

$$\begin{aligned} A_{11} &= (\Omega^2 - k^2(c_{11}n_1^2 + c_{44}n_2^2))/c_{44}, \\ A_{22} &= (\Omega^2 - k^2(c_{44}n_1^2 + c_{11}n_2^2))/c_{44}, \\ A_{33} &= (\Omega^2 - c_{44}k^2(n_1^2 + n_2^2))/c_{44}, \\ A_{12} &= A_{21} = -k^2n_1n_2\Delta_8/c_{44}, \\ A_{13} &= A_{31} = -ikn_1\Delta_8/c_{44}, \\ A_{23} &= A_{32} = -ikn_2\Delta_8/c_{44}. \end{aligned} \tag{3.6}$$

Here $\Omega^2 = \rho\omega^2 R_*^2/C_*$ ($C_* = h$ m) is the non-dimensional frequency parameter.

The characteristic equation for the equation system (3.4) takes the form

$$\begin{vmatrix} \lambda^2 + A_{11} & A_{12} & A_{13} \\ A_{12} & \lambda^2 + A_{22} & A_{23} \\ A_{13} & A_{23} & \lambda^2 + A_{33} \end{vmatrix} = 0\,. \tag{3.7}$$

We will assume such a material for the layer that the characteristic equation (3.7) has three different roots λ_1, λ_2, λ_3 with nonzero real parts. Then the solution to problem (3.4)-(3.5) can be represented as

$$f_j(x_3) = \sum_{m=1}^{3} \beta_{jm} \exp(\lambda_m x_3). \tag{3.8}$$

The relations between the coefficients $\beta_{j,m}$ ($j = 2, 3$) and β_{1m} follow from equation (3.7) and are

$$\beta_{jm} = \frac{Q_{jm}}{D_m} \beta_{1m} \quad (j = \overline{1,3}), \tag{3.9}$$

where

$$\begin{aligned} Q_{1m} &= D_m = (\lambda_m^2 + A_{22})(\lambda_m^2 + A_{33}) - A_{23}^2, \\ Q_{2m} &= A_{13}A_{23} - A_{12}(\lambda_m^2 + A_{33}), \\ Q_{3m} &= A_{12}A_{23} - A_{13}(\lambda_m^2 + A_{22}). \end{aligned} \tag{3.10}$$

Substitution of representation (3.8) into the boundary conditions (3.5) results in a system of linear algebraic equations for the constants β_{1m} ($m = \overline{1,3}$)

$$\mathbf{B} \cdot (\beta_{11},\, \beta_{12},\, \beta_{13})^T = \mathbf{0}. \tag{3.11}$$

The elements of the coefficient matrix $\mathbf{B}$ are

$$\begin{aligned} B_{1m} &= in_1 \exp(h\lambda_m)(Q_{3m}D_m^{-1} + \lambda_m); \\ B_{2m} &= in_2 \exp(h\lambda_m)(Q_{2m}D_m^{-1} + \lambda_m); \\ B_{3m} &= \exp(h\lambda_m)(ic_{12}(n_1 + n_2 Q_{2m}D_m^{-1}) + c_{11}\lambda_m Q_{3m}D_m^{-1}) \quad (m = \overline{1,3}). \end{aligned} \tag{3.12}$$

The coefficients β_{1m} can be expressed from equations (3.11) as

$$\beta_{1m} = \frac{G_m}{M} g_m \quad (m = \overline{1,3}), \tag{3.13}$$

where

$$\begin{aligned} G_1 &= M = B_{22}B_{33} - B_{23}B_{32}, \\ G_2 &= B_{23}B_{31} - B_{21}B_{33}, \\ G_3 &= B_{21}B_{32} - B_{22}B_{31}, \end{aligned} \tag{3.14}$$

and g_m are arbitrary integration constants.

Equation (3.11) has non-trivial solutions if

$$\det \mathbf{B} = 0\,. \tag{3.15}$$

Vanishing of the above determinant yields a transcendental equation for Ω and k, i.e., for the spectrum.

Finally, the complex displacement functions for the normal symmetrical linear waves which propagate in the waveguide plane in an arbitrary direction (n_1, n_2) and belong to the mode q, are

$$\begin{aligned} u^{(l)}_{jq}(x_1,x_2,x_3,t) = {} & [M_q^{-1} \textstyle\sum_{m=1}^3 D^{-1}_{mq} g_{mq} Q_{jmq} G_{mq} \beta_{jmq} \exp(\lambda_{mq} x_3)] \times \\ & \times \exp(-i(\omega t - k_q(n_1x_1 + n_2x_2))) \quad (j = \overline{1,3}). \end{aligned} \tag{3.16}$$

4. **Analytical solution of a heterogeneous problem for a nonharmonic distortion**

After determining the amplitude functions of linear waves, it becomes possible to obtain the analytical representations for the nonharmonic distortion $u^{(n)}_{jq}$ from (3.2):

$$\begin{aligned} \rho_0 \ddot{u}^{(n)}_{jq} - c_{jdik} u^{(n)}_{iq,dk} = {} & \textstyle\sum_{l,m=1}^3 \mu_{jlmq} \exp(-2i(\omega t - \\ & -k_q(n_1x_1+n_2x_2)) + (\lambda_{lq} + \lambda_{mq})x_3); \end{aligned} \tag{4.1}$$

$$\begin{aligned} (c_{3dik} u^{(n)}_{iq,k})_{x_3=\pm h} = {} & \textstyle\sum_{l,m=1}^3 \eta_{jlmq} \exp(-2i(\omega t - \\ & -k_q(n_1x_1+n_2x_2)) + (\lambda_{lq} + \lambda_{mq})x_3) \quad (j = \overline{1,3}). \end{aligned} \tag{4.2}$$

In these relations the constants μ_{jlmq} $(j = 1, 2)$ can be written as

$$\begin{aligned} \mu_{jlmq} = {} & a^{lmq}_{jljm} i k_q n_j [2k_q^2(n_j^2 \Delta_1 + 3n_k^2 \Delta_2) - (\lambda^2_{lq} + 4\lambda_{lq}\lambda_{mq} + \lambda^2_{mq})\Delta_2] + \\ & + a^{lmq}_{lmml} i k_q [2k_q^2 n_k (n_j^2 (2\Delta_4 + \Delta_5) + n_k^2 \Delta_3) - \\ & \qquad - n_k(\lambda^2_{lq}\Delta_6 + \lambda_{lq}\lambda_{mq}\ (3\Delta_6 + \Delta_7) + \lambda^2_{mq}\Delta_7)] + \\ & + a^{lmq}_{llmm} i k_q [2k_q^2 n_k (n_j^2 (2\Delta_4 + \Delta_5) + n_k^2 \Delta_3) - \\ & \qquad - n_k(\lambda^2_{lq}\Delta_7 + \lambda_{lq}\lambda_{mq}\ (3\Delta_6 + \Delta_7) + \lambda^2_{mq}\Delta_6)] + \\ & + a^{lmq}_{klkm} i k_q n_j [2k_q^2 (n_j^2 \Delta_3 + n_k^2 (2\Delta_4 + \Delta_5)) - \\ & \qquad - \lambda^2_{lq}\Delta_7 - 2\lambda_{lq}\lambda_{mq}(\Delta_6 + \Delta_7) - \lambda^2_{mq}\Delta_6] + \end{aligned}$$

$$
\begin{aligned}
&+a^{lmq}_{3l3m} i k_q n_j [2k_q^2(n_j^2\Delta_3 + n_k^2(2\Delta_6+\Delta_7)) - \\
&\qquad -\lambda_{lq}^2\Delta_4 - 2\lambda_{lq}\lambda_{mq}(\Delta_4+\Delta_5) - \lambda_{mq}^2\Delta_5] + \\
&+a^{lmq}_{jm3l} k_q^2[(n_j^2(\Delta_4+2\Delta_5) + n_k^2(\Delta_6+2\Delta_7))\lambda_{lq} + \\
&\qquad +3(n_j^2\Delta_4 + n_k^2\Delta_6)\lambda_{mq} - \lambda_{lq}\lambda_{mq}(\lambda_{lq}+\lambda_{mq})\Delta_3] + \\
&+a^{lmq}_{jl3m} k_q^2[3(n_j^2\Delta_4 + n_k^2\Delta_6)\lambda_{lq} + (n_j^2(\Delta_4+2\Delta_5) + \\
&\qquad +n_k^2(\Delta_6+2\Delta_7))\lambda_{mq} - \lambda_{lq}\lambda_{mq}(\lambda_{lq}+\lambda_{mq})\Delta_3] + \\
&\qquad +a^{lmq}_{km3l} k_q^2 n_1 n_2[\lambda_{lq}(\Delta_9+2\Delta_{10}) + 3\lambda_{mq}\Delta_9] + \\
&\qquad +a^{lmq}_{kl3m} k_q^2 n_1 n_2[3\lambda_{lq}\Delta_9 + \lambda_{mq}(\Delta_9+2\Delta_{10})]\ .
\end{aligned}
\tag{4.3}
$$

If $j = 3$ we have

$$
\begin{aligned}
\mu_{3lmq} = &\ 2a^{lmq}_{1m3l} i k_q n_1[k_q^2(n_1^2\Delta_3 + n_2^2(2\Delta_6+\Delta_7)) - \\
&\qquad -\lambda_{lq}^2\Delta_5 - \lambda_{lq}\lambda_{mq}(3\Delta_4+\Delta_5) - \lambda_{mq}^2\Delta_4] + \\
&+2a^{lmq}_{1l3m} i k_q n_1[k_q^2(n_1^2\Delta_3 + n_2^2(2\Delta_6+\Delta_7)) - \\
&\qquad -\lambda_{lq}^2\Delta_4 - \lambda_{lq}\lambda_{mq}(3\Delta_4+\Delta_5) - \lambda_{mq}^2\Delta_5] + \\
&+2a^{lmq}_{2m3l} i k_q n_2[k_q^2(n_2^2\Delta_3 + +n_1^2(2\Delta_6+\Delta_7)) - \\
&\qquad -\lambda_{lq}^2\Delta_5 - \lambda_{lq}\lambda_{mq}(3\Delta_4+\Delta_5) - \lambda_{mq}^2\Delta_4] + \\
&+2a^{lmq}_{2l3m} i k_q n_2[k_q^2(n_2^2\Delta_3 + n_1^2(2\Delta_6+\Delta_7)) - \\
&\qquad -\lambda_{lq}^2\Delta_4 - \lambda_{lq}\lambda_{mq}(3\Delta_4+\Delta_5) - \lambda_{mq}^2\Delta_5] + \\
&+a^{lmq}_{1l1m}(\lambda_{lq}+\lambda_{mq})[k_q^2(n_1^2(2\Delta_4+\Delta_5) + \\
&\qquad +n_2^2(2\Delta_6+\Delta_7)) - \lambda_{lq}\lambda_{mq}\Delta_3] + \\
&+a^{lmq}_{2l2m}(\lambda_{lq}+\lambda_{mq})[k_q^2(n_1^2(2\Delta_6+\Delta_7) + \\
&\qquad +n_2^2(2\Delta_4+\Delta_5)) - \lambda_{lq}\lambda_{mq}\Delta_3] + \\
&+(a^{lmq}_{1m2l} + a^{lmq}_{1l2m})(\lambda_{lq}+\lambda_{mq})k_q^2 n_1 n_2(2\Delta_9+\Delta_{10}) + \\
&+a^{lmq}_{3l3m}(\lambda_{lq}+\lambda_{mq})(3k_q^2(n_1^2+n_2^2)\Delta_2 - \lambda_{lq}\lambda_{mq}\Delta_1).
\end{aligned}
\tag{4.4}
$$

The constants η_{jlmq} $(j = 1, 2)$ in the boundary conditions (4.2) are

$$\begin{aligned}\eta_{jlmq} = (a^{lmq}_{jm3l} + a^{lmq}_{jl3m})[k_q^2(n_1 n_j \Delta_3 + n_k^2 \Delta_6) - \lambda_{lq}\lambda_{mq}\Delta_4] + \\ +(a^{lmq}_{km3l} + a^{lmq}_{kl3m})k_q^2 n_1(n_j \Delta_7 + n_2 \Delta_6) - \\ -(a^{lmq}_{jljm} n_1 \Delta_4 + a^{lmq}_{klkm} n_j \Delta_6 + a^{lmq}_{3l3m} n_j \Delta_2) i k_q (\lambda_{lq} + \lambda_{mq}) - \\ -a^{lmq}_{jllm} i k_q (c_{144} n_1 \lambda_{lq} + +c_{456} n_k \lambda_{mq}) - a^{lmq}_{jmml} i k_q (c_{456} n_k \lambda_{lq} + c_{144} n_1 \lambda_{mq}) \,.\end{aligned} \quad (4.5)$$

If $j = 3$ we have

$$\begin{aligned}\eta_{3lmq} = a^{lmq}_{1l1m}[k_q^2(n_1^2 \Delta_5 + n_2^2 \Delta_7) - \lambda_{lq}\lambda_{mq}\Delta_3] + \\ +a^{lmq}_{2l2m}[k_q^2 n_1^2(\Delta_5 + \Delta_7) - \lambda_{lq}\lambda_{mq}\Delta_3] + \\ +(a^{lmq}_{1m2l} + a^{lmq}_{1l2m})k_q^2 n_1(c_{123} n_1 + c_{144} n_2) + \\ +a^{lmq}_{3l3m}[k_q^2(n_1^2 + n_2^2)\Delta_2 - \lambda_{lq}\lambda_{mq}\Delta_1] - \\ -a^{lmq}_{1m3l} i k_q n_1(\lambda_{lq}\Delta_5 + \lambda_{mq}\Delta_4) - a^{lmq}_{1l3m} i k_q n_1(\lambda_{lq}\Delta_4 + \lambda_{mq}\Delta_5) - \\ -a^{lmq}_{2m3l} i \times k_q(n_1 \lambda_{lq}\Delta_5 + n_2 \lambda_{mq}\Delta_4) - a^{lmq}_{2l3m} i k_q(n_2 \lambda_{lq}\Delta_4 + n_1 \lambda_{mq}\Delta_5).\end{aligned} \quad (4.6)$$

In equations (4.3)-(4.5)

$$a^{lmq}_{dpsr} = -(2M_q^2 D_{lq} D_{mq})^{-1} g_{lq} g_{mq} G_{lq} G_{mq} Q_{dpq} Q_{srq} \beta_{dpq} \beta_{srq}; \quad (4.7)$$

$$n_k = \begin{cases} n_1, & j = 2; \\ n_2, & j = 1. \end{cases} \quad (4.8)$$

Problem (4.1) has the analytical solution of the following structure:

$$\begin{aligned}u^{(n)}_{jq} = [\textstyle\sum_{l,m=1}^{3} \gamma_{j1lmq} \exp((\lambda_{lq} + \lambda_{mq})x_3) + \\ + \textstyle\sum_{m=1}^{3} (\gamma_{j2mq} + \gamma_{j3mq} x_3) \exp(2\lambda_{mq} x_3) + \\ + \textstyle\sum_{m=1}^{3} (\gamma_{j4mq} + \gamma_{j5mq} x_1 + \gamma_{j6mq} x_2) \exp(2\lambda_{mq} x_3)] \times \\ \times \exp(-2i(\omega t - k_q(n_1 x_1 + n_2 x_2))).\end{aligned} \quad (4.9)$$

The coefficients γ_{j1lmq} are determined from the linear equation system

$$\mathbf{L}^{(l,m)} \cdot (\gamma_{11lmq}, \gamma_{21lmq}, \gamma_{31lmq})^T = (\mu_{1lmq}, \, \mu_{2lmq}, \, \mu_{3lmq})^T \quad (l \neq m), \quad (4.10)$$

where the elements of the matrix $\mathbf{L}^{(l,m)}$ are

$$
\begin{gathered}
L_{11q}^{(l,m)} = 4k_q^2(c_{11}n_1^2 + c_{44}n_2^2) - c_{44}(\lambda_{lq} + \lambda_{mq})^2 - 4\Omega^2; \\
L_{22q}^{(l,m)} = 4k_q^2(c_{44}n_1^2 + c_{11}n_2^2) - c_{44}(\lambda_{lq} + \lambda_{mq})^2 - 4\Omega^2; \\
L_{33q}^{(l,m)} = 4k_q^2 c_{44}(n_1^2 + n_2^2) - c_{11}(\lambda_{lq} + \lambda_{mq})^2 - 4\Omega^2; \\
L_{12q}^{(l,m)} = L_{21q}^{(l,m)} = 4k_q^2 n_1 n_2 \Delta_8; \\
L_{13q}^{(l,m)} = L_{31q}^{(l,m)} = -2ik_q n_1(\lambda_{lq} + \lambda_{mq})\Delta_8; \\
L_{23q}^{(l,m)} = L_{32q}^{(l,m)} = -2ik_q n_2(\lambda_{lq} + \lambda_{mq})\Delta_8.
\end{gathered}
\tag{4.11}
$$

The coefficients γ_{j2mlq} $(j = \overline{1,3})$ are obtained as

$$
\gamma_{j2mq} = \gamma_{12mq} Z_{jmq}^{(2)} (P_{mq}^{(2)})^{-1}, \tag{4.12}
$$

where

$$
\begin{gathered}
Z_{1mq}^{(2)} = P_{mq}^{(2)} = L_{22q}^{(m,m)} L_{33q}^{(m,m)} - (L_{23q}^{(m,m)})^2; \\
Z_{2mq}^{(2)} = [L_{33q}^{(m,m)}(\mu_{2mmq} + \chi_{1mq} - L_{12q}^{(m,m)}\gamma_{12mq}) - \\
-L_{23q}^{(m,m)}(\mu_{3mmq} + \chi_{2mq} - L_{13q}^{(m,m)}\gamma_{12mq})]\gamma_{12mq}^{-1}; \\
Z_{3mq}^{(2)} = [L_{22q}^{(m,m)}(\mu_{3mmq} + \chi_{2mq} - L_{13q}^{(m,m)}\gamma_{12mq}) - \\
-L_{23q}^{(m,m)}(\mu_{2mmq} + \chi_{1mq} - L_{12q}^{(m,m)}\gamma_{12mq})]\gamma_{12mq}^{-1}.
\end{gathered}
\tag{4.13}
$$

The coefficients $L_{srq}^{(m,m)}$ are defined in the same way as in equation (4.11); γ_{12mq} are arbitrary integration constants;

$$
\begin{gathered}
\chi_{1mq} = 2ik_q n_2 \gamma_{33mq}\Delta_8 + 4c_{44}\gamma_{23mq}\lambda_{mq}; \\
\chi_{2mq} = 2ik_q(n_1\gamma_{13mq} + n_2\gamma_{23mq})\Delta_8 + 4c_{11}\gamma_{33mq}\lambda_{mq}.
\end{gathered}
\tag{4.14}
$$

The coefficients γ_{j3mq} $(j = \overline{1,3})$ have the structure

$$
\gamma_{j3mq} = \gamma_{13mq} Z_{jmq}^{(3)} (P_{mq}^{(3)})^{-1}, \tag{4.15}
$$

where

$$
\begin{gathered}
Z_{1mq}^{(3)} = P_{mq}^{(3)} = Z_{1mq}^{(2)}; \\
Z_{2mq}^{(3)} = L_{13q}^{(m,m)} L_{23q}^{(m,m)} - L_{12q}^{(m,m)} L_{33q}^{(m,m)}; \\
Z_{3mq}^{(3)} = L_{12q}^{(m,m)} L_{23q}^{(m,m)} - L_{13q}^{(m,m)} L_{22q}^{(m,m)} ,
\end{gathered}
\tag{4.16}
$$

and γ_{13mq} are arbitrary integration constants. The coefficients γ_{j4mq} $(j = 2, 3)$ are

$$
\gamma_{j4mq} = \gamma_{14mq} Z_{jmq}^{(4)} (P_{mq}^{(4)})^{-1}, \tag{4.17}
$$

where

$$Z_{1mq}^{(4)} = P_{mq}^{(4)} = Z_{1mq}^{(2)};$$

$$\begin{aligned} Z_{2mq}^{(4)} &= [L_{33q}^{(m,m)}(\nu_{1mq} - L_{12q}^{(m,m)}\gamma_{14mq}) - \\ &-L_{23q}^{(m,m)}(\nu_{2mq} - L_{13q}^{(m,m)}\gamma_{14mq})]\gamma_{14mq}^{-1}; \\ Z_{3mq}^{(4)} &= [L_{22q}^{(m,m)}(\nu_{2mq} - L_{13q}^{(m,m)}\gamma_{14mq}) - \\ &-L_{23q}^{(m,m)}(\nu_{1mq} - L_{12q}^{(m,m)}\gamma_{14mq})]\gamma_{14mq}^{-1}, \end{aligned} \tag{4.18}$$

where the constants ν_{smq} $(s = 1, 2)$ have the structure

$$\begin{aligned} \nu_{1mq} &= 2ik_q(n_1\gamma_{16mq} + n_2\gamma_{15mq})\Delta_8 + \\ &+4ik_q(n_1\gamma_{25mq}c_{44} + n_2\gamma_{26mq}c_{11}) + 2\gamma_{36mq}\lambda_{mq}\Delta_8, \\ \nu_{2mq} &= 4ik_qc_{44}(n_1\gamma_{35mq} + n_2\gamma_{36mq}) + 2\lambda_{mq}(\gamma_{15mq} + \gamma_{26mq})\Delta_8. \end{aligned} \tag{4.19}$$

The representations for the coefficients γ_{jsmq} $(s = 5, 6)$ are

$$\begin{aligned} \gamma_{jsmq} &= \gamma_{1smq}Z_{jmq}^{(s)}(P_{mq}^{(s)})^{-1}, \\ Z_{1mq}^{(s)} &= P_{mq}^{(s)} = Z_{1mq}^{(2)}; \quad Z_{jmq}^{(s)} = Z_{1mq}^{(3)} \quad (j = 2, 3). \end{aligned} \tag{4.20}$$

The coefficients γ_{1s1q} $(s = 5, 6)$ are arbitrary integration constants. The coefficients γ_{1smq} $(s = 5, 6,\ m = 2, 3)$ assume the forms

$$\begin{aligned} \gamma_{1s2q} &= (d_{11q}d_{22q} - d_{12q}^2)^{-1}(d_{22q}\theta_{1sq} - d_{12q}\theta_{2sq}); \\ \gamma_{1s3q} &= (d_{11q}d_{22q} - d_{12q}^2)^{-1}(d_{11q}\theta_{2sq} - d_{21q}\theta_{1sq}), \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} \theta_{1sq} &= -i\exp(2h\lambda_{1q})(k_qn_1\gamma_{3s1q} + \lambda_{1q}\gamma_{1s1q}) - \\ &-ik_qn_1[\exp(2h\lambda_{2q})\gamma_{3s2q} + \exp(2h\lambda_{3q})\gamma_{3s3q}]; \\ \theta_{2sq} &= -c_{12}k_q\exp(2h\lambda_{1q})(n_1\gamma_{1s1q} + n_2\gamma_{2s1q}) - \\ &-c_{12}k_qn_2[\exp(2h\lambda_{2q})\gamma_{2s2q} + \exp(2h\lambda_{3q})\gamma_{3s3q}] + \\ &+ic_{11}[\lambda_{1q}\exp(2h\lambda_{1q})\gamma_{3s1q} + \lambda_{2q}\exp(2h\lambda_{2q})\gamma_{3s2q} + \\ &+\lambda_{3q}\exp(2h\lambda_{3q})\gamma_{3s3q}]; \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} d_{1rq} &= \lambda_{(r+1)q}\exp(2h\lambda_{(r+1)q}); \\ d_{2rq} &= c_{12}k_qn_1\exp(2h\lambda_{(r+1)q}) \quad (r = 1, 2). \end{aligned} \tag{4.23}$$

Finally, the constants γ_{14sq} $(s = \overline{1,3})$ are obtained from the linear equations system

$$\mathbf{H} \cdot (\gamma_{141q},\, \gamma_{142q},\, \gamma_{143q})^T = (\xi_{1q},\, \xi_{2q},\, \xi_{3q})^T, \tag{4.24}$$

where the matrix $\mathbf{H}$ is defined as

$$\begin{gathered} H_{1jq} = 2c_{44}\exp(2h\lambda_{jq})(ik_q n_1 Z_{3jq}^{(4)}(P_{jq}^{(4)})^{-1} + \lambda_{jq}); \\ H_{2jq} = 2c_{44}\exp(2h\lambda_{jq})(P_{jq}^{(4)})^{-1}(ik_q n_2 Z_{3jq}^{(4)} + \lambda_{jq} Z_{2jq}^{(4)}); \\ H_{3jq} = 2\exp(2h\lambda_{jq})(ik_q c_{12}(n_1 + Z_{2jq}^{(4)}(P_{jq}^{(4)})^{-1} n_2) + \\ + c_{11} Z_{3jq}^{(4)}(P_{jq}^{(4)})^{-1}) \quad (j = \overline{1,3}). \end{gathered} \tag{4.25}$$

The elements ξ_{jq} $(j = \overline{1,3})$ of the right side (4.24) are

$$\begin{gathered} \xi_{jq} = \textstyle\sum_{l,m=1}^{3}(\eta_{jlmq} - 2ic_{44}k_q n_j \gamma_{31lmq} - \\ - c_{44}(\lambda_{lq} + \lambda_{mq})\gamma_{j1lmq})\exp((\lambda_{lq} + \lambda_{mq})h) \quad (j = 1,2); \\ \xi_{3q} = \textstyle\sum_{l,m=1}^{3}(\eta_{3lmq} - 2ic_{12}(n_1\gamma_{11lmq} + n_2\gamma_{21lmq}) - \\ - c_{11}\gamma_{31lmq})(\lambda_{lq} + \lambda_{mq})\exp((\lambda_{lq} + \lambda_{mq})h). \end{gathered} \tag{4.26}$$

Finally, we have obtained closed analytical representations for the second harmonics of normal three-partial waves. These solutions allow us to carry out a detailed analysis of the nonlinear effects for the anisotropic waveguide considered.

5. **Numerical results**

Numerical computations have been made for the cubic system monocrystal germanium layer for waves propagating in the plain Ox_1x_2 along the nonelastoequivalent direction of the crystal, characterized by the angle $\varphi = 15°$.

The analysis of some nonlinear effects for waves which belong to two low linear modes with zero locking frequency has been performed.

For a germanium monocrystal the density and the second and third order nonzero normalized elastic constants have the following values:

$$\begin{gathered} \rho = 5,32; \quad c_{11} = 12,92; \quad c_{12} = 4,79; \quad c_{44} = 6,70; \\ c_{111} = -7,10; \quad c_{112} = -3,89; \quad c_{144} = -2,3; \\ c_{155} = -2,92; \quad c_{123} = -0,18; \quad c_{456} = -0,53. \end{gathered} \tag{5.1}$$

The evaluation of the correlation between the longitudinal and cross horizontal components in the second harmonics of monochromatic normal waves with different frequencies $\Omega_1 = \Omega_4 = 6.92$, $\Omega_2 = \Omega_5 = 9.23$, $\Omega_3 = \Omega_6 = 11.53$ has been obtained. The points j in Figure 1 correspond to the waves with frequencies Ω_j (they have

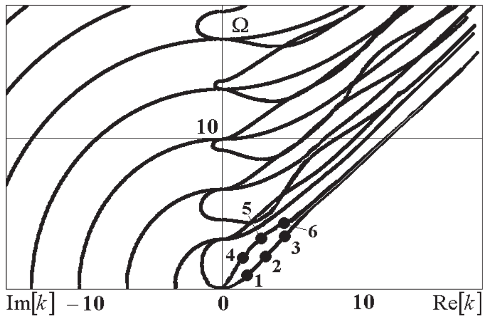

Figure 1. Linear waves spectrum for monocrystal germanium layer

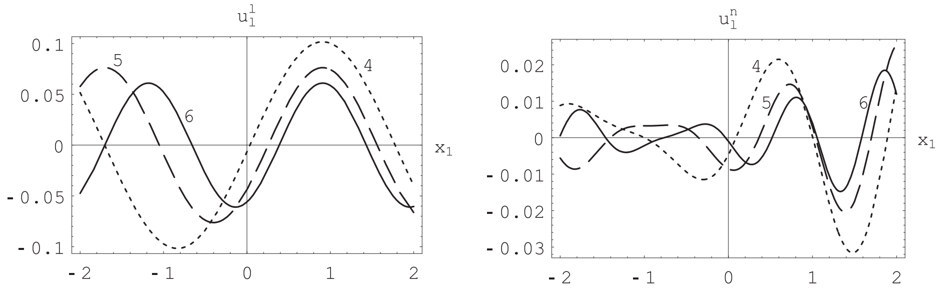

Figure 2. Displacements u_l distributions for $x_3 = 1/2$

been analyzed). These correlations are compared with the correlations between the longitudinal and cross horizontal components in the linear waves.

The longitudinal and cross horizontal components of the normal waves considered are calculated by using the formulas

$$u_l = u_1 \cos\varphi + u_2 \sin\varphi; \quad u_t = -u_1 \sin\varphi + u_2 \cos\varphi, \tag{5.2}$$

where u_1, u_2 are the displacements in linear waves or the second harmonics of linear waves; φ is the angle between the wave propagation direction in the middle waveguide plane and Ox_1 is a coordinate direction.

In Figure 2 the wave functions u_l^l, u_l^n for the waveguide section $\{|x_1| \leq 4h,\ x_2 = 0,\ x_3 = h\}$ and $h = 1/2$ and at time $t = 1$ are depicted. Computations have been made for those waves which belong to the linear spectrum second mode; the curves j correspond to the waves j in Figure 1. The analogous distributions for the waveguide section $\{|x_1| \leq 4h,\ x_2 = 0,\ x_3 = 0\}$ are presented in Figure 3. The values u_l^l, u_l^n are obtained as

$$u_l^l = Re[u_l^{(l)}/\Omega_*^2]; \quad u_l^n = 10^5 Re[u_l^{(n)}/\Omega_*^2]; \quad \Omega_*^2 = \rho\,\Omega^2/\rho_*. \tag{5.3}$$

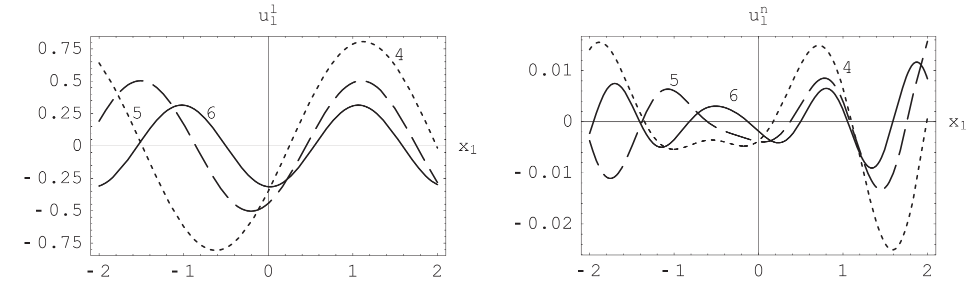

Figure 3. Displacements u_l distributions for $x_3 = 0$

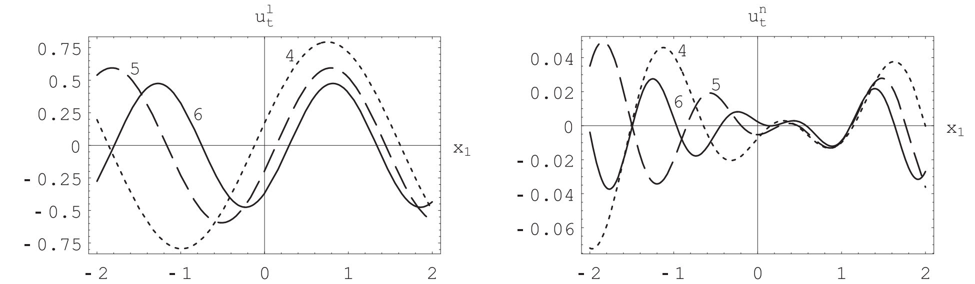

Figure 4. Displacements u_t distributions for $x_3 = 1/2$

In Figures 4 and 5 the wave functions u_t^l, u_t^n are shown for the different waveguide sections $x_3 = 1/2$ and $x_3 = 0$. Here

$$u_t^l = Re[u_l^{(l)}/\Omega_*^2]; \quad u_t^n = 10^5 Re[u_l^{(n)}/\Omega_*^2]; \quad \Omega_*^2 = \rho\,\Omega^2/\rho_*. \tag{5.4}$$

Both for linear waves and for their second harmonics the increasing of frequency leads to an increasing of the displacement maximum. For linear waves and for a non-harmonic distortion the increase in frequency and the changeability of the coordinate x_3 have little influence on u_l and u_t. In case of u_l the more intensive displacements appear in the waveguide area $x_3 = 0$ for linear waves, but the second harmonics are more vividly expressed on the layer surface while $x_3 = 1/2$. For u_t the displacements in linear waves have higher levels on the layer surface $x_3 = 1/2$, but the characteristics for the second harmonics are almost equal.

The graphs in Figure 6 show the distributions of the ratio u_l^n/u_t^n for the waves which correspond to points 1 and 4 in Figure 1, that is for waves with similar frequencies, but belonging to different linear spectrum modes. Computations have been made for the waveguide area $\{|x_1| \leq 2h,\ |x_2| \leq 2h,\ x_3 = 0\}$. It was found that in linear waves the first mode is the pseudotransverse mode, and the second is pseudolongitudinal. From the correlations obtained it is clear that in both cases the second harmonics are the pseudolongitudinal waves, that is the component u_l^n is dominant; for the case of

Ω_1 frequency the dependence is monotonic, and for Ω_4 frequency case the dependence is not continuous.

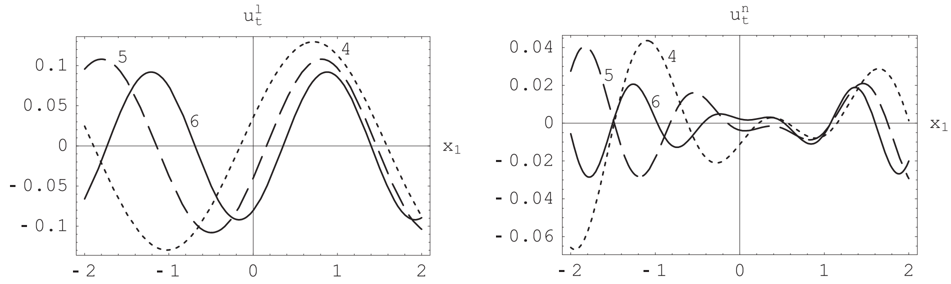

Figure 5. Displacements u_t distributions for $x_3 = 0$

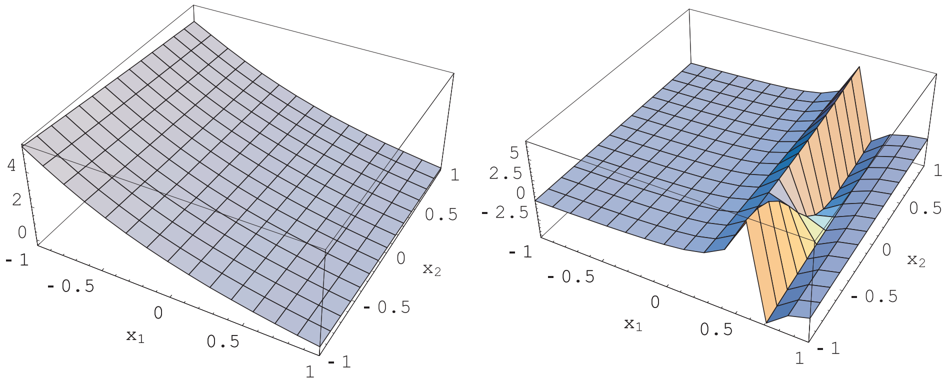

Figure 6. Distributions of $u_l^{(n)}/u_t^{(n)}$ for $x_3 = 0$

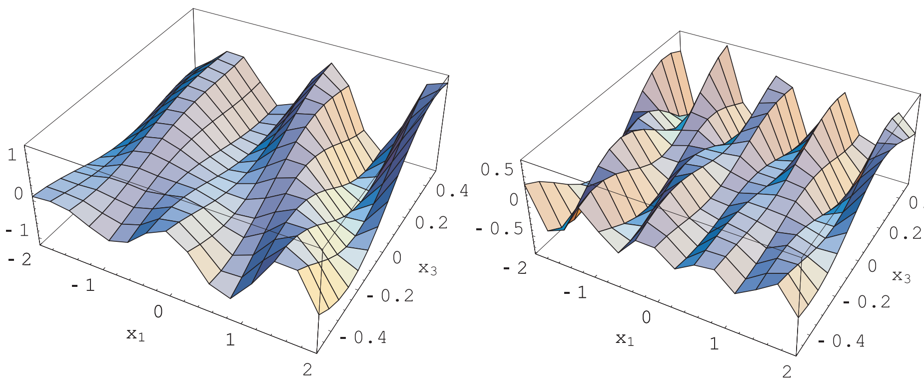

Figure 7. Frequency dependencies of $u_l^{(n)}$ for $x_1 = 0$

In Figure 7 the dependencies u_t^n on frequency for the waveguide section $\{|x_1| \leq 4h,\ x_2 = 0,\ |x_3| \leq h\}$ are shown. The first figure corresponds to frequency Ω_1, the second to frequency Ω_2. From the given data it follows that an increase in the first mode leads to a decrease in the maximum of u_t^n. The displacements themselves are almost constant.

6. Conclusions

The method presented in the paper allows us to analyze the nonlinear normal wave propagation in an arbitrary direction in the plane of anisotropic elastic layer waveguides. We have obtained and analyzed how the frequencies depend on the displacement characteristics, what the distributions for the amplitude characteristics of the linear normal waves are and what second harmonics they have. The data, obtained by this method, could be helpful while using a new class of nonlinear devices for signal information study.

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REFERENCES

1. Zarembo L. K., Krasilnicov V. A.: *Nonlinear phenomena in the propagation of elastic waves in solids.* Successes in Phys. Sciens., **102**(4), (1970), 549-584. (in Russian)
2. Truell R., Elbaum C., Chic B.: *Ultrasound Methods in the Physics of Solid Bodies.* Mir, Moscow, 1972. (in Russian)
3. Taker J., Rempton V.: *Hypersound in the Physics of Solid Bodies.* Mir, Moscow, 1975. (in Russian)
4. Goldberg Z. A., Grebneva P. V.: *Nonlinear interaction of longitudinal and two cross waves in isotropic solid.* Acoust. J., **18**(3), (1972), 386-390. (in Russian)
5. Yermilin K. K., Lyamov V. A., Pyatakov P. A.: *Canted interaction of bulk waves in crystals.* Solids Phys., **15**(11), (1973), 3226-3230. (in Russian)
6. Krasilnicov V. A., Lyamov V. E.: *Nonlinear interaction of elastic waves in solids and signal information study.* Acoust. J., **19**(5), (1973), 801-804. (in Russian)
7. Maugin G.: *Nonlinear waves in elastic crystals.* Oxford, University Press, London, 1999.
8. Parker D. F., Mayer A. P., Maradudin A. A.: *The projection method for nonlinear surface acoustic waves.* Wave Motion, **16**, (1992), 151-162.
9. Harvey A. P., Tupholme G. E.: *Propagation of anisotropic elastic and piezo-electric nonlinear surface acoustic waves.* Wave Motion, **16**, (1992), 125-135.
10. Lopen P.O.: *Second-harmonic generation in an elastic surface wave in α-quarts.* J. Appl. Phys., **39**(12), (1968), 5400-5404.
11. Sugimoto M., Majima Y.: *Nonlinear third harmonic resonance of flexural waves on a waveguide of thin elastic plate*, J. Acoust. Sos. Am., **69**(2), (1981), 410-415.