LONG STRANGE SEGMENTS, RUIN PROBABILITIES AND THE EFFECT OF MEMORY ON MOVING AVERAGE PROCESSES

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ABSTRACT. We obtain the rate of growth of long strange segments and the rate of decay of infinite horizon ruin probabilities for a class of infinite moving average processes with exponentially light tails. The rates are computed explicitly. We show that the rates are very similar to those of an i.i.d. process as long as moving average coefficients decay fast enough. If they do not, then the rates are significantly different. This demonstrates the change in the length of memory in a moving average process associated with certain changes in the rate of decay of the coefficients.

1. INTRODUCTION

How does the length of memory in a stationary stochastic process affect the behavior of important characteristics of the process such as the rate of increase of the *long strange segments* and the rate of decay of the *ruin probabilities*? From a different point of view: can one use such important characteristics of a stationary process to tell whether or not the process has long memory. In this paper such questions are discussed for a class of $\mathbb{R}^d$-valued infinite moving average processes with exponentially light tails. These are processes of the form

$$(1.1) \qquad X_n = \sum_{i \in \mathbb{Z}} \phi_i Z_{n-i}, \ n \in \mathbb{Z},$$

where $(Z_i, i \in \mathbb{Z})$ are i.i.d., centered, random vectors taking values in $\mathbb{R}^d$. We assume existence of some exponential moments, i.e.

there exists $\epsilon > 0$ such that $\Lambda(t) := \log E\left[e^{tZ_0}\right] < \infty$ for all $t \in \mathbb{R}^d$ with $|t| < \epsilon$.

Such a process, also known as a linear process (see Brockwell and Davis (1991)), is well defined if the coefficients are square summable:

$$(1.2) \qquad \sum_{i=-\infty}^{\infty} \phi_i^2 < \infty.$$

If the stronger condition of absolute summability of the coefficients holds, namely

$$(1.3) \qquad \sum_{i \in \mathbb{Z}} |\phi_i| < \infty,$$

then it is often said that the process has short memory. This is mainly because the covariances of the process are summable in this case, and a process with absolutely summable covariances is often considered to have short memory, see e.g. Samorodnitsky (2006). What about other characteristics of a process, that are often more informative than covariances?

In a recent article Ghosh and Samorodnitsky (2009) gave a complete picture of functional large, moderate and huge deviations for the moving average process and discussed the effect of memory on them. In this paper we follow up by obtaining the rate of growth of long strange segments and the rate of decay of the ruin probabilities for the moving average processes. We consider two cases: one where the coefficients of the process are absolutely summable, i.e. (1.3) holds, and the other when (1.3) fails and the coefficients are balanced regularly varying. We show that the rates are significantly different in these two cases. We view these results as showing the effect of memory as well as indicating that the processes with absolutely summable coefficients can be legitimately called short memory processes, while the alternative family of processes can be legitimately viewed as a family of long memory processes.

We now define precisely that characteristics of a process that we will study in this paper. Suppose that $(X_n, n \in \mathbb{Z})$ is a zero mean $\mathbb{R}^d$−valued, stationary and ergodic stochastic process. Given any measurable set $A \subset \mathbb{R}^d$, the lengths of the long strange segments are random variables, defined as

$$R_n(A) := \sup \left\{ j - i : 0 \le i < j \le n, \frac{S_j - S_i}{j - i} \in A \right\},$$

where $S_k = X_1 + \cdots + X_k$ are the partial sums. That is, $R_n(A)$ is the maximum length of a segment from the first n observations whose average is in A. To understand the justification for the name long strange segments, consider any set A bounded away from the origin (that is $0 \notin \bar{A}$, where $\bar{A}$ is the closure of A.) Since the process is ergodic, we would not expect the average value of the process over a long time segment to be in A, and it is strange if that happens. If we use the process to model a system, then the long strange segments are the time intervals where the system runs at a different "rate" than anticipated, and it is of obvious interest to know how long such strange intervals could be.

The easiest way to see the connection between the long strange segments and large deviations is by defining

$$T_n(A) := \inf \left\{ l : \text{ there exists } k, 0 \le k \le l - n, \frac{S_l - S_k}{l - k} \in A \right\};$$

$T_n(A)$ is the minimum number of observations required to have a segment of length at least n, whose average is in the set A. It is elementary to check that there is a duality relation between the rate of growth of T_n and the rate of growth of R_n. Furthermore, for any sequence (X_n) of random vectors,

(1.4) $$- \limsup_{n \to \infty} \frac{1}{n} \log P\left[S_n/n \in A \right] \le \liminf_{n \to \infty} \frac{1}{n} \log T_n(A), \text{ P-a.s.}$$

and, if (X_n) are i.i.d., then also

(1.5) $$- \liminf_{n \to \infty} \frac{1}{n} \log P\left[S_n/n \in A \right] \ge \limsup_{n \to \infty} \frac{1}{n} \log T_n(A), \text{ P-a.s.};$$

see e.g. Theorem 3.2.1 in Dembo and Zeitouni (1998). In Section 2 we exploit the connection between a general version of long strange segments and large deviations

to establish the rate of growth of the long strange segments for the two classes of moving average processes we are considering. We will observe a marked change (or a phase transition) in the rate of growth when switching from one family of moving averages to the other.

The relations of the form (1.4) and (1.5) are referred to as the Erdös-Rényi law; Erdös and Rényi (1970) proved asymptotics for longest head runs in i.i.d. coin tosses. See Gordon et al. (1986), Arratia et al. (1990), Novak (1992), Gantert (1998) and Vaggelatou (2003) and the references therein for versions on this result under various Markov chain settings.

We mention at this point that a different case of this problem was considered in Mansfield et al. (2001) and Rachev and Samorodnitsky (2001), where the assumption of certain finite exponential moments was replaced by the assumption of balanced regular varying tails with exponent $-\beta < -1$. These papers consider linear processes as in (1.1) in dimension $d = 1$. In particular, Mansfield et al. (2001) showed that if (1.3) holds, then for any $y > 0$ and $x > 0$

$$(1.6) \qquad P\big(a_n^{-1}R_n((y,\infty)) \leq x\big) \to \exp(-C_s y^{-\beta} x^{-\beta})$$

where (a_n) is a sequence that does not depend on the moving average coefficients, and it is regular varying at infinity with index β^{-1} (see Resnick (1987) or Bingham et al. (1987) for details on regular variation). On the other hand, $C_s > 0$ is a constant, which may depend on the moving average coefficients. This rate of growth a_n of the long strange segments is the same as in the i.i.d. case, that results when choosing $\phi_0 = 1$ and $\phi_i = 0$ for all $i \neq 0$. In the subsequent paper Rachev and Samorodnitsky (2001) considered the case when (1.3) fails to hold, but the coefficients (ϕ_i) are balanced regular varying at infinity with exponent $-\alpha$, satisfying $\max\left\{\frac{1}{\beta}, \frac{1}{2}\right\} < \alpha \leq 1$. This means that there is a nonnegative function ψ with

$$(1.7) \qquad \psi \in RV_{-\alpha}, \text{ such that } \frac{\phi_n}{\psi(n)} \to p, \ \frac{\phi_{-n}}{\psi(n)} \to 1 - p, \text{ as } n \to \infty$$

for some $0 \leq p \leq 1$. Under this assumption, for any $y > 0$ and $x > 0$,

$$(1.8) \qquad P\big(b_n^{-1}R_n((y,\infty)) \leq x\big) \to \exp(C_l y^{-\beta} x^{-\beta\alpha}),$$

for some sequence $(b_n) \in RV_{(\alpha\beta)^{-1}}$. Therefore, the long strange segments now grow at the higher rate (b_n). This phase transition be taken as the evidence of long range dependence in the moving average process under the regular variation (1.7) of the coefficients. A similar phenomenon can be observed in Section 2 of the present paper.

The second topic that we consider in this article is that of the *ruin probabilities*. If (Y_n) is an $\mathbb{R}^d$-valued stochastic process, and A a measurable set in $\mathbb{R}^d$, an infinite horizon ruin probability is a probability of the type

$$(1.9) \qquad \rho(u; A) = \rho(u) = P\left[Y_n \in uA, \text{ for some } n \geq 1\right].$$

The name "ruin probability" derives from the one-dimensional case with $A = (1, \infty)$: if we interpret Y_n as the total losses incurred by a company until time n, and u is the initial capital of the firm, then the event in (1.9) is the event that the company eventually goes bankrupt. Probabilities of the type are of interest in queuing theory as well; see e.g. Asmussen (2003).

In the context of moving average processes, we will define

$$(1.10) \qquad Y_n = \sum_{i=1}^{n} X_i - a_n \mu,$$

for some $\mu \in \mathbb{R}^d$, a sequence (a_n) increasing to ∞, with (X_n) the infinite moving average process (1.1). The classical Cramér-Lundberg Theory (see e.g. Section XIII.5 in Asmussen (2003)) says that, in dimension $d = 1$, if (X_n) are i.i.d., and (a_n) is a linear sequence then (under an additional condition) there exist positive constants c and θ such that

$$(1.11) \qquad \rho(u) \sim ce^{-\theta u} \qquad \text{as } u \to \infty.$$

This result was later extended by Gerber (1982) to the situation where (X_n) an ARMA(p, q) process satisfying certain assumptions, including that of bounded innovations, and Promislow (1991) has a further extension to certain infinite moving average processes while removing the assumption of the boundedness of the innovations. In all these cases (1.3), which we regard as a short memory case is assumed to hold (in fact, much stronger assumptions are needed).

A weaker version of the estimate (1.11) is the logarithmic scale estimate

$$(1.12) \qquad \lim_{u \to \infty} \frac{1}{u} \log \rho(u) = -\theta \,.$$

Such results were derived in Nyrhinen (1994, 1995) in a fairly great generality in the one dimensional case. When specified to the moving average case, in order to give a non-trivial limit, these results require, once again, absolute summability of the coefficients.

There have been other recent studies of ruin probabilities for certain stationary increment processes with long memory. The papers Hüsler and Piterbarg (2004) and Hüsler and Piterbarg (2008) analyzed the (continuous time) ruin where the increment process was a version of the fractional Gaussian noise. Further, Barbe and McCormick (2008) also obtained a logarithmic form of ruin asymptotics, as in (1.12), under the assumption that the increment process is the classical Fractional ARIMA process or belongs to a class of related processes.

In this paper we solve the logarithmic scale ruin problem (1.12) when the increment process (X_n) in (1.10) is the infinite moving average process. We present a fairly complete picture. Namely, we prove results both in the short memory case (when (1.3) holds), and in the long memory case, under the assumption of balanced regularly varying coefficients. We allow a very broad class of drift sequences (a_n). Ruin probabilities are also related to large deviations, but not as directly as the long strange segments. We use a combination of multiple techniques, but the large deviation principle for the moving average process proved in Ghosh and Samorodnitsky (2009) still plays an important role. The techniques we use here can modified for more other, and more general, classes of stationary processes but we do not make any such attempt in this paper. We present the results and their proofs in Section 3 and in the process we clearly demonstrate the effect of memory in the process (X_n) on the rate of the decay of the ruin probability $\rho(u)$. The Appendix contains a multivariate extension of the estimates in Nyrhinen (1994) that are not restricted to moving average processes.

2. Long Strange Segments

Let $(X_n, n \in \mathbb{Z})$ be a $\mathbb{R}^d$-valued, centered stationary infinite moving average process (1.1) defined on a probability space $(\Omega, \mathcal{F}, P)$, and let (S_n) be its partial sum process. In this section we discuss the rate of growth of a general version of the length of the long strange segments, which we define as follows. For a sequence $\underline{a} = (a_n)$ increasing to infinity and a measurable set $A \subset \mathbb{R}^d$, we define

$$(2.1) \qquad R_m(A; \underline{a}) := \sup \left\{ n : \frac{S_l - S_{l-n}}{a_n} \in A \text{ for some } l = n, \ldots, m \right\}$$

and the "dual characteristic"

$$(2.2) \qquad T_r(A; \underline{a}) := \inf \left\{ l : \text{ there exists } k, 0 \le k \le l - r, \frac{S_l - S_k}{a_{l-k}} \in A \right\}.$$

Notice that $\{R_m(A; \underline{a}) \ge r\}$ if and only if $\{T_r(A; \underline{a}) \le m\}$. We will often refer to $R_m(A; \underline{a})$ as R_m and to $T_r(A; \underline{a})$ as T_r, as long as the set A and the sequence (a_n) under consideration are obvious.

The assumptions and results below use the following notion of balanced regular variation on $\mathbb{R}^d$.

A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *balanced regular varying* with exponent $\beta > 0$, if there exists a non-negative bounded function ζ_f defined on the unit sphere on $\mathbb{R}^d$ and a function $\tau_f : (0, \infty) \to (0, \infty)$ satisfying

$$(2.3) \qquad \lim_{t \to \infty} \frac{\tau_f(tx)}{\tau_f(t)} = x^\beta$$

for all $x > 0$ (i.e. τ_f is regularly varying with exponent β) and such that for any $(\lambda_t) \subset \mathbb{R}^d$ with $|\lambda_t| = 1$ for all t, converging to λ,

$$(2.4) \qquad \lim_{t \to \infty} \frac{f(t\lambda_t)}{\tau_f(t)} = \zeta_f(\lambda).$$

The subscript f will typically be omitted if doing so is unlikely to cause confusion.

Next, we state the specific assumptions on the moving average process, the normalizing sequence (a_n) in (2.1) and (2.2), the resulting large deviations rate sequence (b_n), and the noise variables. We will consider two different situations, corresponding to what we view as a short memory moving average, when the coefficients in (1.1) decay fast, and a long memory moving average, when the coefficients in (1.1) decay slowly. The Assumptions 2.1 and 2.2 below correspond, roughly, to Assumptions 2.1 and 2.3 in Ghosh and Samorodnitsky (2009), respectively.

We start with the assumption describing the short memory case. Throughout this paper we use $\Lambda(\cdot)$ to denote the log-moment generating function of the i.i.d. innovations (Z_i):

$$\Lambda(t) := \log E\left[e^{tZ_0} \right],$$

and by $\mathcal{F}_\Lambda \subset \mathbb{R}^d$ we denote the set where $\Lambda(\cdot)$ is finite:

$$\mathcal{F}_\Lambda = \{ t : \Lambda(t) < \infty \}.$$

Furthermore, for any set A, A° and $\bar{A}$ denote the interior and closure of A, respectively.

Assumption 2.1. *All the scenarios below assume that*

$$(2.5) \qquad \sum_{i \in \mathbb{Z}} |\phi_i| < \infty \text{ and } \sum_{i \in \mathbb{Z}} \phi_i = 1.$$

S1. $a_n = n, 0 \in \mathcal{F}_\Lambda^\circ$ and $b_n = n$.

S2. $a_n = n, \mathcal{F}_\Lambda = \mathbb{R}^d$ and $b_n = n$.

S3. $a_n/\sqrt{n \log n} \to \infty$, $a_n/n \to 0$, $0 \in \mathcal{F}_\Lambda^\circ$ and (b_n) an increasing positive sequence such that $b_n \sim a_n^2/n$ as $n \to \infty$.

S4. $a_n/n \to \infty$, $\Lambda(\cdot)$ is balanced regular varying with exponent $\beta > 1$ and (b_n) an increasing positive sequence such that $b_n \sim n\tau(c_n)$, where

$$(2.6) \qquad c_n = \sup\{x : \tau(x)/x \le a_n/n\}.$$

The next assumption describes the long memory case.

Assumption 2.2. *All the scenarios assume that the coefficients (ϕ_i) are balanced regular varying with exponent $-\alpha, 1/2 < \alpha \le 1$ and $\sum_{i=-\infty}^\infty |\phi_i| = \infty$. Specifically, we assume that (1.7) holds for α in this range. Let $\Psi_n := \sum_{1 \le i \le n} \psi(i)$.*

R1. $a_n = n\Psi_n, 0 \in \mathcal{F}_\Lambda^\circ$ and $b_n = n$.

R2. $a_n = n\Psi_n, \mathcal{F}_\Lambda = \mathbb{R}^d$ and $b_n = n$.

R3. $a_n/(\sqrt{n \log n}\Psi_n) \to \infty, a_n/(n\Psi_n) \to 0, 0 \in \mathcal{F}_\Lambda^\circ$ and (b_n) is an increasing positive sequence such that $b_n \sim a_n^2/(n\Psi_n^2)$ as $n \to \infty$.

R4. $a_n/(n\Psi_n) \to \infty$, $\Lambda(\cdot)$ is balanced regular varying with exponent $\beta > 1$ and (b_n) is an increasing positive sequence such that $b_n \sim n\tau(\Psi_n c_n)$, where

$$(2.7) \qquad c_n = \sup\{x : \tau(\Psi_n x)/x \le a_n/n\}.$$

Let $\mu_n(\cdot) \equiv \mu_n(\cdot; \underline{a})$ denote the law of $a_n^{-1} S_n$. We quote the "marginal version" of the functional results in Ghosh and Samorodnitsky (2009); in certain cases these have been known even earlier. The sequence (μ_n) satisfies the large deviation principle on $\mathbb{R}^d$:

(2.8)

$$-\inf_{x \in A^\circ} I_l(x) \le \liminf_{n \to \infty} \frac{1}{b_n} \log \mu_n(A; \underline{a}) \le \limsup_{n \to \infty} \frac{1}{b_n} \log \mu_n(A; \underline{a}) \le -\inf_{x \in \bar{A}} I_u(x)$$

with a good lower function I_l and a good upper function I_u given by

$$(2.9) \qquad \begin{aligned} I_l = \Lambda^*, \ I_u = \Lambda^\sharp \quad &\text{under the assumption } S1 \\ I_l = I_u = \Lambda^* \quad &\text{under the assumption } S2 \\ I_l = I_u = \left(G_\Sigma\right)^* \quad &\text{under the assumption } S3 \\ I_l = I_u = \left(\Lambda^h\right)^* \quad &\text{under the assumption } S4 \\ I_l = \left(\Lambda_\alpha\right)^*, \ I_u = \Lambda_\alpha^\sharp \quad &\text{under the assumption } R1 \\ I_l = I_u = \left(\Lambda_\alpha\right)^* \quad &\text{under the assumption } R2 \\ I_l = I_u = \left((G_\Sigma)_\alpha\right)^* \quad &\text{under the assumption } R3 \\ I_l = I_u = \left((\Lambda^h)_\alpha\right)^* \quad &\text{under the assumption } R4 \end{aligned}$$

Here, for a convex function $f : \mathbb{R}^d \to (-\infty, \infty]$, we denote by f^* its Legendre transform $f^*(x) = \sup_{\lambda \in \mathbb{R}^d}\{\lambda \cdot x - f(\lambda)\}$, $x \in \mathbb{R}^d$. Further, under the assumption $S1$, $\Lambda^\sharp(x) = \sup_{\lambda \in \Pi}\{\lambda \cdot x - \Lambda(\lambda)\}$, with

$$(2.10) \qquad \Pi = \Big\{\lambda \in \mathbb{R}^d : \text{ for some } N_\lambda, \sup_{n \ge N_\lambda, i \in \mathbb{Z}} \Lambda(\lambda\phi_{i,n}) < \infty\Big\},$$

where $\phi_{i,n} = \phi_{i+1} + \cdots + \phi_{i+n}$, is a partial sum of the moving average coefficients. Further, under the assumptions $S3$ and $R3$, G_Σ is the log-moment generating function of a zero mean Gaussian random vector in $\mathbb{R}^d$ with the same variance-covariance matrix as that of Z_0. Next, under the assumptions $S4$ and $R4$, $\Lambda^h(\lambda) = \zeta_\Lambda(\lambda/\|\lambda\|)\|\lambda\|^\beta$. Under the assumptions $R1$-$R4$, for a nonnegative measurable function f on $\mathbb{R}^d$ we define

$$(2.11) \quad f_\alpha(\lambda) = \int_{-\infty}^\infty f\left(\lambda(1-\alpha)\int_x^{x+1} |y|^{-\alpha}\big(p\mathbf{1}(y \geq 0) + q\mathbf{1}(y < 0)\big)\,dy\right)dx$$

if $1/2 < \alpha < 1$ and $f_1 = f$. Finally, under the assumption $R1$, we define $\Lambda_\alpha^\sharp(x) = \sup_{\lambda \in \Pi_\alpha}\{\lambda \cdot x - \Lambda_\alpha(\lambda)\}$, with Π_α given by

$$(2.12) \quad \Pi_\alpha := \left\{\lambda : (p \wedge q)\lambda \in \mathcal{F}_\Lambda^\circ, \text{ and for some } N_\lambda, \sup_{n \geq N_\lambda,\, i \in \mathbb{Z}} \Lambda\left(\frac{\lambda\phi_{i,n}}{\Psi_n}\right) < \infty\right\}$$

for $1/2 < \alpha < 1$, while for $\alpha = 1$, we define

$$(2.13) \quad \Pi_1 := \left\{\lambda : \lambda \in \mathcal{F}_\Lambda^\circ, \text{ and for some } N_\lambda, \sup_{n \geq N_\lambda,\, i \in \mathbb{Z}} \Lambda\left(\frac{\lambda\phi_{i,n}}{\Psi_n}\right) < \infty\right\}.$$

We are now ready to state the main result of this section. The following theorem considers the various cases in Assumptions 2.1 and 2.2 and gives us the rate of growth of the lengths of the long strange segments in each of the cases. For a set A in $\mathbb{R}^d$ and $\eta > 0$ we denote

$$(2.14) \qquad\qquad A(\eta) := \big\{x : d(x, A^c) > \eta\big\},$$

where $d(x, A^c)$ is the distance from the point x to the complement A^c.

Theorem 2.3. *If any one of $S1$-$S4$ or $R1$-$R4$ hold, then for any Borel set $A \subset \mathbb{R}^d$,*

$$(2.15) \qquad I_* \leq \liminf_{r\to\infty} \frac{\log T_r(A;\underline{a})}{b_r} \leq \limsup_{r\to\infty} \frac{\log T_r(A;\underline{a})}{b_r} \leq I^*$$

and

$$(2.16) \qquad\qquad \frac{1}{I^*} \leq \liminf_{m\to\infty} \frac{b_{R_m}}{\log m} \leq \limsup_{m\to\infty} \frac{b_{R_m}}{\log m} \leq \frac{1}{I_*}$$

with probability 1, where, under the assumptions $S2, S3, S4, R2, R3$ and $R4$,

$$I_* = \inf_{x\in\bar{A}} I_u(x) \quad and \quad I^* = \inf_{x\in A^\circ} I_l(x),$$

with I_l and I_u as in (2.9). Under the assumption $S1$, I_* is defined in the same way, while I^* is defined now as follows. Let $\lambda^* = \sup\{\lambda : \lambda \in \Pi\} > 0$. Then

$$I^* = \inf_{\eta\in\Theta} \inf_{x\in A(\eta)} I_l(x),$$

where $\Theta = \{\eta > 0 : \eta > (\lambda^*)^{-1}\inf_{x\in A(\eta)} I_l(x)\}$. Finally, under the assumption $R1$, I_* is defined in the same way, and with $\lambda_\alpha^* = \sup\{\lambda : \lambda \in \Pi_\alpha\} > 0$, and $\Theta_\alpha = \{\eta > 0 : \eta > (\lambda_\alpha^*)^{-1}\inf_{x\in A(\eta)} I_l(x)\}$, one sets

$$I^* = \inf_{\eta\in\Theta_\alpha} \inf_{x\in A(\eta)} I_l(x).$$

Remark 2.4. In certain cases it turns out that $I_* = I^*$ in Theorem 2.3, and then its conclusions may be strengthened. For example, under the assumptions S2, S3, S4, R2, R3 or R4, suppose that for some Borel set A,

$$\inf_{x \in A^\circ} I_l(x) = \inf_{x \in \bar{A}} I_u(x) = I \text{ (say).}$$

Then, with probability 1,

$$(2.17) \qquad \qquad \lim_{r \to \infty} \frac{\log T_r}{b_r} = I$$

and

$$(2.18) \qquad \qquad \lim_{m \to \infty} \frac{b_{R_m}}{\log m} = \frac{1}{I} \, .$$

Because of the large deviation principle for the sequence (μ_n), the sequence (b_n) is the "right" normalization to use in the Theorem 2.3. In particular, if, for instance, the set A is bounded away from the origin (which we recall to be the mean of the moving average process), then the quantity I_* is strictly positive. Under further additional assumptions on the set A the quantity I^* will be finite, and then (2.15) and (2.16) give us precise information on the order of magnitude of long strange segments.

Notice that under the "usual" normalization $a_n = n$, Theorem 2.3 says that R_m grows like $\log m$ in the short memory case (i.e. under the assumption $S1$); see also Theorem 3.2.1 in Dembo and Zeitouni (1998). On the other hand, in the long memory case, it is easy to see that the case $a_n = n$ falls into the assumption $R3$, and then the length R_m of the long strange segments grows at the rate $\Theta(\log m)$, where Θ is regularly varying at infinity with exponent $1/(2\alpha - 1)$. Therefore, long strange segments are much longer in the long memory case than in the short memory case. In fact, to get long strange segments with length of order $\log m$ in the long memory case one needs to use a stronger normalization $a_n = n\Psi_n$ (the assumptions $R1$ and $R2$). This phase transition property is directly inherited from the similar phenomenon for large deviations; see Ghosh and Samorodnitsky (2009).

To emphasize more generally the difference between the length of the long strange segments in the two cases we summarize in the table below the corresponding statements of Theorem 2.3 for (a_n) being a regularly varying sequence with exponent $\omega \geq 1/2$ of regular variation. We will implicitly assume that the appropriate assumptions of the theorem hold in each case, and that the limits I_* and I^* are positive and finite. The general statement is that, with probability 1, R_m is of the order $\Theta(\log m)$, where Θ is regularly varying at infinity with some exponent θ. We describe θ as a function of ω in all cases. The value $\theta = \infty$ corresponds to R_m growing faster than any power of $\log m$. In all cases the long strange segments are much longer in the long memory case than in the short memory case. Recall that $-\alpha$ is the exponent of regular variation of the coefficients in Assumption 2.2, and β is the exponent of regular variation of Λ in assumptions $S4$ and $R4$. Notice that the long range dependent case in the first row of the table does not correspond to any assumption we have made. The fact that $\theta = \infty$ in this case follows as one of the extreme cases of the second row in the table.

Proof of Theorem 2.3. The duality relation $\{R_m(A; \underline{a}) \geq r\} = \{T_r(A; \underline{a}) \leq m\}$ and monotonicity of the sequence (b_n) imply that the statements (2.15) and (2.16) are equivalent. We will, therefore, concentrate on proving (2.15). The proof of the

TABLE 1. The effect of memory on the rate of growth of Long Strange Segments of a Moving Average Process

Range of ω	Assumptions	Short memory	Long memory
$\frac{1}{2} \leq \omega \leq \frac{3}{2} - \alpha$	$S3$	$\theta = \frac{1}{2\omega - 1}$	$\theta = \infty$
$\frac{3}{2} - \alpha \leq \omega \leq 1$	$S1, S2, S3, R3$	$\theta = \frac{1}{2\omega - 1}$	$\theta = \frac{1}{2\omega + 2\alpha - 3}$
$1 \leq \omega \leq 2 - \alpha$	$S4, R1, R2, R3$	$\theta = \frac{\beta - 1}{\beta\omega - 1}$	$\theta = \frac{1}{2\omega + 2\alpha - 3}$
$\omega \geq 2 - \alpha$	$S4, R4$	$\theta = \frac{\beta - 1}{\beta\omega - 1}$	$\theta = \frac{\beta - 1}{\beta(\omega + \alpha - 1) - 1}$

lower bound is standard, and does not rely on the fact that the underlying process is a moving average; see Theorem 3.2.1 in Dembo and Zeitouni (1998). We include an argument for completeness. Note that for every $r, m \geq 1$

$$P\big(T_r(A; \underline{a}) \leq m\big) \leq m \sum_{n=r}^{\infty} \mu_n(A; \underline{a}).$$

If $I_* = 0$, there is nothing to prove. Suppose that $0 < I_* < \infty$. Choose $0 < \varepsilon < I_*$. By the definition of I_* and the large deviation principle (2.8), we know that there is $c = c_\varepsilon \in (0, \infty)$ such that $\mu_n(A; \underline{a}) \leq ce^{-b_n(I_* - \varepsilon/2)}$ for all $n \geq 1$. Choosing $m = \lfloor e^{b_r(I_* - \varepsilon)} \rfloor$ gives us

$$\sum_{r=1}^{\infty} P(T_r \leq e^{b_r(I_* - \varepsilon)}) \leq \sum_{r=1}^{\infty} e^{b_r(I_* - \varepsilon)} \sum_{n=r}^{\infty} ce^{-b_n(I_* - \varepsilon/2)}$$

$$\leq c' \sum_{r=1}^{\infty} e^{-b_r \varepsilon/2} < \infty$$

for some positive constant c' (depending on ε). Using the first Borel-Cantelli lemma and letting $\varepsilon \downarrow 0$ established the lower bound in (2.15). When $I_* = \infty$, we take any $\varepsilon > 0$ and observe that by the definition of I_* there is $c = c_\varepsilon \in (0, \infty)$ such that $\mu_n(A; \underline{a}) \leq ce^{-2b_n/\varepsilon}$ for all $n \geq 1$. Choose now $m = \lfloor e^{b_r/\varepsilon} \rfloor$ and proceed as above to conclude that

$$\sum_{r=1}^{\infty} P(T_r \leq e^{b_r/\varepsilon}) < \infty,$$

after which one uses, once again, the first Borel-Cantelli lemma and lets $\varepsilon \downarrow 0$ to obtain the lower bound in (2.15).

For the upper bound in (2.15), we only need to consider the case $I^* < \infty$. In that case the set A has nonempty interior. Define two new probability measures by

$$\mu'_n(\cdot) := P\Big(\frac{1}{a_n} \sum_{|i| \leq n^2} \phi_{i,n} Z_i \in \cdot\Big) \text{ and } \mu''_n(\cdot) := P\Big(\frac{1}{a_n} \sum_{|i| > n^2} \phi_{i,n} Z_i \in \cdot\Big),$$

where, as before, $\phi_{i,n} = \phi_{i+1} + \cdots + \phi_{i+n}$.

For any sequence (k_n) of integers, with $k_n/n \to \infty$, and any $\lambda > 0$ under the assumptions $S2, S3, S4, R2, R3$ and $R4$, any $\lambda \in \Pi$ under the assumption $S1$, or any $\lambda \in \Pi_\alpha$ under the assumption $R1$,

$$(2.19) \qquad \lim_{n \to \infty} \frac{1}{b_n} \sum_{i=-k_n}^{k_n} \Lambda\Big(\frac{b_n}{a_n} \lambda \phi_{i,n}\Big) = \lim_{n \to \infty} \frac{1}{b_n} \sum_{i=-\infty}^{\infty} \Lambda\Big(\frac{b_n}{a_n} \lambda \phi_{i,n}\Big);$$

see Remark 3.7 in Ghosh and Samorodnitsky (2009). This means that the sequence (μ'_n) satisfies the LDP with speed b_n and same upper rate functions I_u given in (2.9) as the sequence (μ_n). The fact that the same is true for the lower rate functions in (2.9) follows from the argument in theorems 2.2 and 2.4 in Ghosh and Samorodnitsky (2009).

For fixed integers r, q, and $l = 1, \ldots, \lfloor q/(2r^2 + 1) \rfloor$, define

$$B_l := \frac{1}{a_r} \sum_{i=1+(l-1)(2r^2+1)}^{r+(l-1)(2r^2+1)} X_i,$$

and

$$B'_l := \frac{1}{a_r} \sum_{j=-r^2}^{r^2} \phi_{j,r} Z_{-j+(l-1)(2r^2+1)}.$$

Since the B'_l are independent, for any r and q we have,

$$
\begin{aligned}
&P\big[T_r > q\big] \\
\leq\ & P\left[B_l \notin A, l = 1, \ldots, \left\lfloor \frac{q}{2r^2+1} \right\rfloor\right] \\
\leq\ & P\left[B'_l \notin A(\eta), l = 1, \ldots, \left\lfloor \frac{q}{2r^2+1} \right\rfloor\right] + \sum_{l=1}^{\lfloor q/(2r^2+1)\rfloor} P\Big[|B_l - B'_l| > \eta\Big] \\
=\ & \Big(1 - \mu'_r\big(A(\eta)\big)\Big)^{\lfloor q/(2r^2+1)\rfloor} + \sum_{l=1}^{\lfloor q/(2r^2+1)\rfloor} P\Big[|B_l - B'_l| > \eta\Big] \\
\leq\ & \exp\Big(-\frac{q}{2r^2+1}\mu'_r\big(A(\eta)\big)\Big) + \frac{q}{2r^2+1}\mu''_r\big(\{x : |x| > \eta\}\big).
\end{aligned}
$$

By the definition of I^* and the large deviation principle (2.8), for any $\varepsilon > 0$ there is $c = c_\varepsilon \in (0, \infty)$ such that for all $\eta > 0$ small enough, $\mu_n(A(\eta)) \geq ce^{-b_n(I^*+\varepsilon/2)}$ for all n large than some n_ε. Therefore, fixing $\varepsilon > 0$ and using the bound above with $q = e^{b_r(I^*+\varepsilon)}$, we see that for some $C = C_\varepsilon \in (0, \infty)$, for all $\eta > 0$ small enough,

$$
\begin{aligned}
\sum_{r=1}^{\infty} \exp\Big(-\frac{e^{b_r(I^*+\epsilon)}}{2r^2+1}\mu'_r\big(A(\eta)\big)\Big) \ \leq\ & C\sum_{r=1}^{\infty} \exp\Big(-c\frac{e^{b_r(I^*+\epsilon)}}{2r^2+1}e^{-b_r(I^*+\epsilon/2)}\Big) \\
(2.20) \qquad =\ & C\sum_{r=1}^{\infty} \exp\Big(-c\frac{e^{b_r(\epsilon/2)}}{2r^2+1}\Big) < \infty.
\end{aligned}
$$

Suppose first that we are under the assumptions S2, S3, S4, R2, R3 or R4. Fixing $\varepsilon > 0$ and choosing $\eta > 0$ small enough for the above to holds, we see that

$$
\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{b_n} \log \mu''_n\big(\{x : |x| > \eta\}\big) \\
\leq\ & \limsup_{n\to\infty} \frac{1}{b_n} \log\left(e^{-b_n\lambda\eta} E\Big[\exp\Big\{\lambda\frac{b_n}{a_n}\sum_{|i|>n^2}\phi_{i,n}Z_i\Big\}\Big]\right) \\
=\ & -\lambda\eta + \limsup_{n\to\infty} \frac{1}{b_n}\sum_{|i|>n^2}\Lambda\Big(\frac{b_n}{a_n}\lambda\phi_{i,n}\Big) = -\lambda\eta,
\end{aligned}
$$

with the last equality following from (2.19). Choosing now $\lambda > (I^* + \epsilon)/\eta$ (which is possible under the current assumptions no matter how small $\eta > 0$ is), we obtain

$$(2.21) \qquad \sum_{r=1}^{\infty} \frac{e^{b_r(I^* + \epsilon)}}{2r^2 + 1} \mu_r''\big(\{x : |x| > \eta\}\big) < \infty.$$

Combining (2.20) and (2.21) we have $\sum_{r=1}^{\infty} P\Big[T_r > e^{b_r(I^* + \epsilon)}\Big] < \infty$, so that using the first Borel-Cantelli lemma gives and letting $\varepsilon \downarrow 0$ proves the upper bound in (2.15). The cases of the assumptions $S1$ and $R1$ are the same, except now λ cannot be taken to be arbitrarily large, which restricts the feasible values of $\eta > 0$. This completes the proof. $\qquad\square$

3. Ruin Probabilities

This section discusses the rate of decay ruin probability for a moving average process $(X_n, n \in \mathbb{Z})$ in (1.1). We study the probability of ruin in infinite time, defined as

$$(3.1) \qquad \rho(u; A; \underline{a}; \mu) = \rho(u) = P\left[Y_n \in uA \text{ for some } n \geq 1\right]$$

where (Y_n) is given by (1.10) for some $\mu \in \mathbb{R}^d$ and a sequence $\underline{a} = (a_n)$ increasing to ∞, and $A \subset \mathbb{R}^d$ is a Borel set. A related notion is the *time of ruin* defined by

$$T(u) = \inf\{n : Y_n \in uA\}.$$

Clearly, $\rho(u) = P[T(u) < \infty]$. We will study the asymptotic behavior of $\rho(u)$ as u increases to infinity.

Our main results are in the following theorems, roughly corresponding to assumptions 2.1 and 2.2 of the previous section. We start with the short memory regimes.

Theorem 3.1. *If $S1$ holds, then*

$$-\inf_{t \in \mathcal{F}} r(t)\big[t\nabla\Lambda(t) - \Lambda(t)\big] \leq \liminf_{u \to \infty} \frac{1}{u} \log \rho(u)$$

$$\leq \limsup_{u \to \infty} \frac{1}{u} \log \rho(u) \leq -\sup_{t \in \mathcal{D}} \inf_{\gamma \in A} t\gamma,$$

where

$$\mathcal{D} = \Big\{t \in \mathbb{R}^d : \inf_{\gamma \in A} t\gamma > 0, \sup_{n \geq 1}\Big[\sum_{i \in \mathbb{Z}} \Lambda\left(t\phi_{i,n}\right) - nt\mu\Big] < \infty\Big\},$$

$$\mathcal{F} = \big\{t \in \Pi^\circ : r\left(\nabla\Lambda(t) - \mu\right) \in A^\circ \text{ for some } \rho > 0\big\},$$

and $r(t) = \inf\{r > 0 : r\left(\nabla\Lambda(t) - \mu\right) \in A^\circ\}$.

Remark 3.2. In certain cases Theorem 3.1 provides a precise and explicit statement. Suppose for simplicity that $\Lambda(t) < \infty$ for all t, and that the random variable μZ is unbounded. Then there exists a unique $w > 0$ such that

$$\Lambda(w\mu) = w\|\mu\|^2.$$

Assume that $r\big(\nabla\Lambda(w\mu) - \mu\big) \in A^\circ$ for some $r > 0$, and let

$$\gamma_* = r(w\mu)\big(\nabla\Lambda(w\mu) - \mu\big) \in \overline{(A^\circ)}.$$

Then the lower bound in Theorem 3.1 gives us

$$\liminf_{u \to \infty} \frac{1}{u} \log \rho(u) \geq -w\gamma_*\mu.$$

If we assume, additionally, that $\inf_{\gamma \in A} \mu\gamma > 0$, then it follows that $a\mu \in \mathcal{D}$ for any $0 < a < w$, and a further assumption $\gamma_* \in \operatorname{argmin}\{\mu\gamma : \gamma \in A\}$ will allow us to conclude from the upper bound Theorem 3.1 that

$$\limsup_{u\to\infty} \frac{1}{u} \log \rho(u) \leq -w\gamma_*\mu .$$

Therefore,

$$(3.2) \qquad \lim_{u\to\infty} \frac{1}{u} \log \rho(u) = -w\gamma_*\mu .$$

All of the assumptions are easily seen to be satisfied in the one-dimensional case with $\mu > 0$ and $A = (1, \infty)$.

For the next two theorems we introduce the following condition on the set A.

Condition 3.3. *We say that a set $A \in \mathbb{R}^d$ satisfies Condition $\mathcal{A}$ if*

- *there is $t \in \mathbb{R}^d$ such that $t\mu > 0$ and $\inf_{\gamma \in A} t\gamma > 0$;*
- *for any $x \in A$ and $\rho > 0$, $x + \rho\mu \in A$ and $(1 + \rho)x \in A$.*

Theorem 3.4. *Suppose that the set A satisfies Condition $\mathcal{A}$ (Condition 3.3). If S3 holds, and $(a_n) \in RV_\omega$ for some $1/2 < \omega \leq 1$, then*

$$- \inf_{c>0} \left[c^{-(2w-1)/w} \inf_{\gamma \in A^\circ} \left(\frac{1}{2}(\mu + c\gamma)'\Sigma^{-1}(\mu + c\gamma) \right) \right] \leq \liminf_{u\to\infty} \frac{1}{b_{a\leftarrow(u)}} \log \rho(u)$$

$$\leq \limsup_{u\to\infty} \frac{1}{b_{a\leftarrow(u)}} \log \rho(u) \leq - \inf_{c>0} \left[c^{-(2w-1)/w} \inf_{\gamma \in A} \left(\frac{1}{2}(\mu + c\gamma)'\Sigma^{-1}(\mu + c\gamma) \right) \right],$$

where the inverse of (a_n) is defined by $a^\leftarrow(u) = \inf\{n \geq 1 : a_n \geq u\}$, $u > 0$.

Remark 3.5. Again, in certain cases the statement of Theorem 3.4 takes a very explicit form. Suppose, for example, that

$$(3.3) \qquad \text{there is } \gamma_0 \in \overline{(A^\circ)} \text{ such that}$$

$$\gamma'\Sigma^{-1}\gamma \geq \gamma_0'\Sigma^{-1}\gamma_0 \text{ and } \mu'\Sigma^{-1}(\gamma - \gamma_0) \geq 0 \text{ for all } \gamma \in A.$$

This would be, for instance, the situation in the one-dimensional case with $\mu > 0$ and $A = (1, \infty)$. Under this assumption, for every $c > 0$,

$$\inf_{\gamma \in A} \left((\mu + c\gamma)'\Sigma^{-1}(\mu + c\gamma) \right) = \inf_{\gamma \in A^\circ} \left((\mu + c\gamma)'\Sigma^{-1}(\mu + c\gamma) \right)$$

$$= \left((\mu + c\gamma_0)'\Sigma^{-1}(\mu + c\gamma_0) \right),$$

and so optimizing over $c > 0$ we obtain

$$(3.4) \qquad \lim_{u\to\infty} \frac{1}{b_{a\leftarrow(u)}} \log \rho(u) = -\frac{1}{2} c_0^{-(2w-1)/w} \left((\mu + c_0\gamma_0)'\Sigma^{-1}(\mu + c_0\gamma_0) \right),$$

where

$$c_0 = \frac{\sqrt{(2w-1)\left[\left(\mu'\Sigma^{-1}\mu\right)\left(\gamma_0'\Sigma^{-1}\gamma_0\right) - \left(\mu'\Sigma^{-1}\gamma_0\right)^2 \right] + w^2\left(\mu'\Sigma^{-1}\gamma_0\right)^2}}{\left(\gamma_0'\Sigma^{-1}\gamma_0\right)}$$

$$-(1-w)\frac{\left(\mu'\Sigma^{-1}\gamma_0\right)}{\left(\gamma_0'\Sigma^{-1}\gamma_0\right)}.$$

Theorem 3.6. *Suppose that the set A satisfies Condition $\mathcal{A}$ (Condition 3.3). If S4 holds, and $(a_n) \in RV_\omega$ for some $\omega \geq 1$, then*

$$- \inf_{c>0} \left[c^{-\nu/w} \inf_{\gamma \in A^\circ} \left(\Lambda^h\right)^*(\mu + c\gamma) \right] \leq \liminf_{u \to \infty} \frac{1}{b_{a^\leftarrow(u)}} \log \rho(u)$$

$$\leq \limsup_{u \to \infty} \frac{1}{b_{a^\leftarrow(u)}} \log \rho(u) \leq \inf_{c>0} \left[c^{-\nu/w} \inf_{\gamma \in \bar{A}} \left(\Lambda^h\right)^*(\mu + c\gamma) \right],$$

where

$$\nu = 1 + (\omega - 1)\frac{\beta}{\beta - 1}.$$

Remark 3.7. Once again, in certain cases the statement of Theorem 3.6 takes a very explicit form. Let us suppose, for example, that

(3.5) there is $\gamma_0 \in \overline{(A^\circ)}$ such that

$$\|\gamma\| \geq \|\gamma_0\| \quad \text{and} \quad \mu'(\gamma - \gamma_0) \geq 0 \quad \text{for all } \gamma \in \bar{A}.$$

Suppose, further, that for some $a > 0$ the function Λ satisfies

(3.6) $\zeta_\Lambda(\lambda) = a$ for any unit vector λ such that

$$\lambda\mu > 0 \quad \text{or} \quad \lambda\gamma > 0 \text{ for some } \gamma \in A.$$

Again, this would be the the situation in the one-dimensional case with $\mu > 0$ and $A = (1, \infty)$. Under the assumption (3.6),

$$\left(\Lambda^h\right)^*(\mu + c\gamma) = K_\beta \|\mu + c\gamma\|^{\beta/(\beta-1)}$$

for any $c > 0$ and $\gamma \in \bar{A}$, with

$$K_\beta = (\beta - 1)\left(a\beta^\beta\right)^{1/(1-\beta)}.$$

This, together with the assumption (3.5), implies that, for any $c > 0$,

$$\inf_{\gamma \in A^\circ} \left(\Lambda^h\right)^*(\mu + c\gamma) = \inf_{\gamma \in \bar{A}} \left(\Lambda^h\right)^*(\mu + c\gamma) = K_\beta \|\mu + c\gamma_0\|^{\beta/(\beta-1)}.$$

Optimizing over $c > 0$ we obtain

(3.7) $$\lim_{u \to \infty} \frac{1}{b_{a^\leftarrow(u)}} \log \rho(u) = -K_\beta c_0^{-\nu/w} \|\mu + c_0\gamma_0\|^{\beta/(\beta-1)},$$

where

$$c_0 = \frac{\sqrt{4(\beta w - 1)\left(\|\mu\|^2\|\gamma_0\|^2 - (\mu\gamma_0)^2\right) + \beta^2 w^2 (\mu\gamma_0)^2} + (\beta w - 2)\mu\gamma_0}{2\|\gamma_0\|^2}.$$

We now turn to the asymptotic behavior of the ruin probabilities in the long memory regimes. In all 3 theorems we assume that the set A satisfies Condition $\mathcal{A}$.

Theorem 3.8. *Suppose that the set A satisfies Condition $\mathcal{A}$ (Condition 3.3). If R2 holds, then*

$$- \inf_{c>0} c^{\frac{1}{\alpha-2}} \inf_{\gamma \in A^\circ} \left(\Lambda_\alpha\right)^*(\mu + c\gamma) \leq \liminf_{u \to \infty} \frac{1}{a^\leftarrow(u)} \log \rho(u)$$

$$\leq \quad \limsup_{u\to\infty} \frac{1}{a^{\leftarrow}(u)}\log\rho(u)$$

$$\leq \quad \begin{cases} \inf\limits_{t\in G}\sup\limits_{u>0}\left\{-u^{\alpha-1}\inf_{\gamma\in A}t\gamma + u\Big(\Lambda_\alpha(t)-t\mu\Big)\right\} & \text{if } \alpha < 1 \\ -\inf_{c>0}c^{-1}\inf_{\gamma\in\bar{A}}\Lambda^*\big(\mu+c\gamma\big) & \text{if } \alpha = 1. \end{cases}$$

where

$$G = \{t\in\mathbb{R}^d : t\mu > 0,\ \inf_{\gamma\in A}t\gamma > 0\ \ and\ \ \Lambda_\alpha(t)-\mu t < 0\},$$

and $\Lambda_\alpha(\cdot)$ is defined in (2.11).

Observe that the set G in the above theorem is not empty because of Condition $\mathcal{A}$ and the fact that $|\Lambda_\alpha(t)| \leq c|t|^2$ for t in a neighborhood of the origin.

To state the next two theorems we introduce the notation

$$(3.8)\quad C_{\alpha,\beta} = \begin{cases} (1-\alpha)^\beta\int\limits_{-\infty}^{\infty}\Big(\int\limits_{x}^{x+1}|y|^{-\alpha}\big(pI_{[y\geq0]}+qI_{[y<0]}\big)dy\Big)^\beta dx & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \end{cases}$$

for $1/2 < \alpha \leq 1$ and $\beta > 1$.

Theorem 3.9. *Suppose that the set A satisfies Condition $\mathcal{A}$ (Condition 3.3). If R3 holds, and $(a_n)\in RV_\omega$ for some $3/2 - \alpha < \omega \leq 2 - \alpha$, then*

$$-\frac{1}{C_{\alpha,2}}\inf_{c>0}\left[c^{-2+(3-2\alpha)/w}\inf_{\gamma\in A^\circ}\left(\frac{1}{2}(\mu+c\gamma)'\Sigma^{-1}(\mu+c\gamma)\right)\right] \leq \liminf_{u\to\infty}\frac{1}{b_{a^{\leftarrow}(u)}}\log\rho(u)$$

$$\leq \quad \limsup_{u\to\infty}\frac{1}{b_{a^{\leftarrow}(u)}}\log\rho(u)$$

$$\leq \quad \begin{cases} -K_{\alpha,\omega}\sup\limits_{t\in G}\Big[\big(t\mu-\tfrac{1}{2}C_{\alpha,2}\,t'\Sigma t\big)^{-1+(3-2\alpha)/\omega}\big(\inf_{\gamma\in A}t\gamma\big)^{2-(3-2\alpha)/\omega}\Big] & \text{if } \alpha < 1 \\ -\frac{1}{C_{\alpha,2}}\inf_{c>0}\Big[c^{-2+(3-2\alpha)/w}\inf_{\gamma\in A}\big(\tfrac{1}{2}(\mu+c\gamma)'\Sigma^{-1}(\mu+c\gamma)\big)\Big] & \text{if } \alpha = 1 \end{cases},$$

where

$$G = \{t\in\mathbb{R}^d : t\mu > 0,\ \inf_{\gamma\in A}t\gamma > 0\ \ and\ \ \frac{1}{2}C_{\alpha,2}\,t'\Sigma t - \mu t < 0\},$$

and

$$K_{\alpha,\omega} = \frac{w\big(3-2\alpha-\omega\big)^{1-(3-2\alpha)/\omega}}{\big(2(\alpha+\omega)-3\big)^{2-(3-2\alpha)/\omega}}.$$

Remark 3.10. It is easy to check that in the one-dimensional case with $\mu > 0$, $A = (1,\infty)$ and $\Sigma = \sigma^2$, the statement of the theorem gives the explicit limit

$$\lim_{u\to\infty}\frac{1}{b_{a^{\leftarrow}(u)}}\log\rho(u) = -\frac{\big(2(\omega+\alpha)-3\big)^{\frac{3-2(\omega+\alpha)}{\omega}}}{\big(3-2\alpha\big)^{\frac{3-2\alpha}{\omega}}}\frac{2}{\sigma^2 C_{\alpha,2}}\omega^2\mu^{\frac{3-2\alpha}{\omega}}.$$

One can check that under certain assumptions similar explicit expressions can be obtained in the multivariate case as well.

Theorem 3.11. *Suppose that the set A satisfies Condition $\mathcal{A}$ (Condition 3.3). If R4 holds, and $(a_n)\in RV_\omega$ for some $\omega \geq 2-\alpha$, $\omega \neq \beta(1-\alpha)+1$, then*

$$-\inf_{c>0}\left[c^{-(\beta(\omega+\alpha-1)-1)/\omega(\beta-1)}\frac{\inf_{\gamma\in A^\circ}\big(\Lambda^h\big)^*(\mu+c\gamma)}{\big(C_{\alpha,\beta}\big)^{1/(\beta-1)}}\right]$$

$$\leq \liminf_{u\to\infty} \frac{1}{b_{a\leftarrow(u)}} \log \rho(u) \leq \limsup_{u\to\infty} \frac{1}{b_{a\leftarrow(u)}} \log \rho(u)$$

$$\leq \begin{cases} -K^{(1)}_{\alpha,\beta,\omega} \sup_{t\in G^{(1)}} \left[\dfrac{\left(t\mu - C_{\alpha,\beta}\,\Lambda^h(t)\right)^{(\beta(1-\alpha)+1-\omega)/\omega(\beta-1)}}{\left(\inf_{\gamma\in A} t\gamma\right)^{(1-\beta(\omega+\alpha-1))/\omega(\beta-1)}} \right] & \text{if } \omega < \beta(1-\alpha)+1 \\[4mm] -\sup_{t\in G^{(2)}} \left[\inf_{\gamma\in A} t\gamma - K^{(2)}_{\alpha,\beta,\omega} \dfrac{\left(C_{\alpha,\beta}\,\Lambda^h(t)\right)^{\omega/(\omega-1-\beta(1-\alpha))}}{(t\mu)^{(1+\beta(1-\alpha))/(\omega-1-\beta(1-\alpha))}} \right] & \text{if } \omega > \beta(1-\alpha)+1 \end{cases}$$

if $\alpha < 1$, and

$$\leq -\inf_{c>0} \left[c^{-(\beta\omega-1)/\omega(\beta-1)} \inf_{\gamma\in\bar{A}} \left(\Lambda^h\right)^*(\mu + c\gamma) \right]$$

if $\alpha = 1$. Here

$$G^{(1)} = \{t \in \mathbb{R}^d : t\mu > 0, \ \inf_{\gamma\in A} t\gamma > 0 \ \text{ and } \ C_{\alpha,\beta}\,\Lambda^h(t) - \mu t < 0\},$$

$$G^{(2)} = \left\{t \in \mathbb{R}^d : t\mu > 0, \ \inf_{\gamma\in A} t\gamma > K^{(2)}_{\alpha,\beta,\omega} \frac{\left(C_{\alpha,\beta}\,\Lambda^h(t)\right)^{\omega/(\omega-1-\beta(1-\alpha))}}{(t\mu)^{(1+\beta(1-\alpha))/(\omega-1-\beta(1-\alpha))}} \right\},$$

and

$$K^{(1)}_{\alpha,\beta,\omega} = \frac{\omega(\beta-1)\left(\beta(1-\alpha)+1-\omega\right)^{-(\beta(1-\alpha)+1-\omega)/\omega(\beta-1)}}{\left(\beta(\omega+\alpha-1)-1\right)^{(\beta(\omega+\alpha-1)-1)/\omega(\beta-1)}},$$

$$K^{(2)}_{\alpha,\beta,\omega} = \frac{\left(\omega-1-\beta(1-\alpha)\right)\left(1+\beta(1-\alpha)\right)^{(1+\beta(1-\alpha))/(\omega-1-\beta(1-\alpha))}}{\omega^{\omega/(\omega-1-\beta(1-\alpha))}}.$$

Remark 3.12. Once again, the sets $G^{(1)}$ and $G^{(2)}$ in the theorem are not empty. In the one-dimensional case with $\mu > 0$, $A = (1, \infty)$ and $\Lambda^h(t) = \xi_+ t^\beta$ for $t > 0$, the statement of the theorem gives the explicit limit

$$\lim_{u\to\infty} \frac{1}{b_{a\leftarrow(u)}} \log \rho(u)$$

$$= -\frac{\left(\beta(\omega+\alpha-1)-1\right)^{\frac{1-\beta(\omega+\alpha-1)}{\omega(\beta-1)}}}{\left(1+\beta(1-\alpha)\right)^{\frac{1+\beta(1-\alpha)}{\omega(\beta-1)}}} (\beta-1)\left(\frac{\omega^\beta}{\xi_+ C_{\alpha,\beta}}\right)^{\frac{1}{\beta-1}} \mu^{\frac{1+\beta(1-\alpha)}{\omega(\beta-1)}}.$$

Remark 3.13. As in the previous section, we clearly see how long range dependent variables (X_n) (the "claim sizes") influence the behavior of the ruin probability. Assume that the relevant upper bounds are finite and the relevant lower bounds are positive. In the classical case of a linear sequence (a_n), in the short memory case (i.e. under the assumption $S1$), we have

$$\log \rho(u) \approx -c_S u \quad \text{as } u \to \infty$$

for $c_S > 0$, as in Cramér's theorem. On the other hand, in the long memory case the linear sequence falls into the assumption $R3$, and then we have, instead,

$$\log \rho(u) \approx -c_L \frac{u}{\Psi_u^2} \quad \text{as } u \to \infty.$$

for $c_L > 0$, and the right hand side above is in $RV_{2\alpha-1}$, yielding a much larger ruin probability.

To further illustrate the effect of memory of a moving average process on ruin probabilities we present Table 2, that presents the order of magnitude of $-\log \rho(u)$ for large u which we view in the form $-\log \rho(u) \approx -c u^\theta$ for $c > 0$. The tables

presents dependence of θ on the exponent ω of regular variation of the sequence (a_n) in both short and long memory cases. The value $\theta = 0$ corresponds to the case when $-\log \rho(u)$ grows slower than any positive power of u. Notice that, for the same value of ω, the value of θ is always smaller in the long memory case than in the short memory case, so that the ruin probability is much larger in the former case than in the latter case.

TABLE 2. The effect of memory on the rate of decay of ruin probability when the claims process is a Moving Average.

Range of ω	Short range dependent	Long range dependent
$\frac{1}{2} \leq \omega \leq \frac{3}{2} - \alpha$	$\theta = \frac{2\omega-1}{\omega}$	$\theta = 0$
$\frac{3}{2} - \alpha \leq \omega \leq 1$	$\theta = \frac{2\omega-1}{\omega}$	$\theta = \frac{2\omega+2\alpha-3}{\omega}$
$1 < \omega < 2 - \alpha$	$\theta = \frac{\beta\omega-1}{\omega(\beta-1)}$	$\theta = \frac{2\omega+2\alpha-3}{\omega}$
$\omega \geq 2 - \alpha$	$\theta = \frac{\beta\omega-1}{\omega(\beta-1)}$	$\theta = \frac{\beta(\omega+\alpha-1)-1}{\omega(\beta-1)}$

Proof of Theorem 3.1. Notice that for the moving average process

$$\log E \exp\left(t\big(S_n - n\mu\big) \right) = \sum_{i \in \mathbb{Z}} \Lambda \left(t\phi_{i,n} \right) - nt\mu.$$

The upper bound in part (i) follows immediately from part (i) of Theorem 4.1.

For the lower bound we apply part (ii) of Theorem 4.1. By Lemma 3.5 (i) in Ghosh and Samorodnitsky (2009), $\Pi^\circ \subseteq \mathcal{E}$, and for every $t \in \Pi^\circ$, $g(t) = \Lambda(t) - t\mu$. The lower bound of part (i) of the present theorem follows. $\square$

Proof of Theorem 3.4. We start with the (easier) lower bound. We use the assumption of regular variation of (a_n) as follows. First of all, $b_n = a_n^2/n$ is regularly varying with exponent $2\omega - 1$. Next, for any $c > 0$,

$$\frac{a^{\leftarrow}(ca_n)}{nc^{1/\omega}} = c^{-1/\omega} \frac{a^{\leftarrow}(a_n)}{n} \frac{a^{\leftarrow}(ca_n)}{a^{\leftarrow}(a_n)} \to 1$$

as $n \to \infty$, see e.g. Theorem 1.5.12 in Bingham et al. (1987). Therefore, by the regular variation of (a_n) and (b_n),

$$
\begin{aligned}
\liminf_{u \to \infty} \frac{1}{b_{a^{\leftarrow}(u)}} \log \rho(u) &= \liminf_{n \to \infty} \frac{1}{b_{a^{\leftarrow}(ca_n)}} \log P\big[T(ca_n) < \infty\big] \\
&\geq \liminf_{n \to \infty} \frac{b_n}{b_{nc^{1/\omega}}} \frac{1}{b_n} \log P\left[\frac{S_n}{a_n} \in \mu + cA\right] \\
&\geq -c^{-(2\omega-1)/\omega} \inf_{\gamma \in A^\circ} \left(\frac{1}{2}(\mu + c\gamma)' \Sigma^{-1}(\mu + c\gamma) \right)
\end{aligned}
$$

(3.9)

by the large deviation principle; see (2.9). Now the lower bound of part (ii) follows by optimizing over $c > 0$.

Next we concentrate on the upper bound. We start with showing that

$$(3.10) \qquad \lim_{M \to \infty} \limsup_{n \to \infty} \frac{1}{b_n} \log P\big[nM < T(a_n) < \infty\big] = -\infty.$$

To see this choose $t \in \mathbb{R}^d$ as in Condition $\mathcal{A}$ and $\epsilon > 0$ such that $J(t) - t\mu + \epsilon < 0$, where $J(t) = \frac{1}{2} t \cdot \Sigma t$. For all n,

$$
\begin{aligned}
P\big[nM < T(a_n) < \infty\big] &= \sum_{k=nM+1}^{\infty} P\big[T(a_n) = k\big] \\
&\leq \sum_{k=nM+1}^{\infty} P\big[S_k - a_k\mu \in a_n A\big] \\
&\leq \sum_{k=nM+1}^{\infty} P\big[tS_k - a_k t\mu > a_n \inf_{\gamma \in A} t\gamma\big].
\end{aligned}
$$

Using Lemma 3.5(ii) in Ghosh and Samorodnitsky (2009) we know that for all n large enough,

$$
\frac{1}{b_n} \log E\Big[\exp\Big(t\frac{b_n}{a_n} S_n \Big) \Big] \leq J(t) + \epsilon.
$$

Therefore, applying an exponential Markov inequality we see that for all M large enough,

$$
\begin{aligned}
P\big[nM < T(a_n) < \infty\big] &\leq \sum_{k=nM+1}^{\infty} \exp\Big\{ -\frac{a_n b_k}{a_k} \inf_{\gamma \in A} t\gamma + b_k\Big(J(t) - \mu t + \epsilon\Big) \Big\} \\
&\leq \sum_{k=nM+1}^{\infty} \exp\Big\{ b_k\Big(J(t) - \mu t + \epsilon\Big) \Big\}.
\end{aligned}
$$

The assumption of regular variation of the sequence (a_n) implies that the sequence $(b_n) \in RV_\nu$ with $\nu = 2\omega - 1$. Therefore, by Theorem 4.12.10 in Bingham et al. (1987)

$$
\log \sum_{k=nM+1}^{\infty} \exp\Big\{ b_k\Big(J(t) - \mu t + \epsilon\Big) \Big\} \sim b_{nM}\Big(J(t) - \mu t + \epsilon\Big)
$$

as $n \to \infty$, and so

$$
\limsup_{n \to \infty} \frac{1}{b_n} \log P\big[nM < T(a_n) < \infty\big] \leq M^\nu \Big(J(t) - \mu t + \epsilon\Big).
$$

Now (3.10) follows by letting $M \to \infty$. A similar argument also shows that for any $N \geq 1$,

$$
\lim_{n \to \infty} \frac{1}{b_n} \log P\big[T(a_n) \leq N\big] = -\infty,
$$

and so in order to prove the upper bound in part (ii) of the theorem, it suffices to show that

$$
\begin{aligned}
(3.11) \qquad &\limsup_{M \to \infty} \limsup_{N \to \infty} \limsup_{n \to \infty} \frac{1}{b_n} \log P\big[N < T(a_n) \leq nM\big] \\
&\leq -\inf_{c>0}\Big[c^{-(2w-1)/w} \inf_{\gamma \in A} \Big(\frac{1}{2}(\mu + c\gamma)'\Sigma^{-1}(\mu + c\gamma) \Big) \Big].
\end{aligned}
$$

Notice that

$$P\big[N < T(a_n) \le nM\big]$$

$$= P\big[S_k - a_k\mu \in a_n A \text{ for some } N < k \le nM\big]$$

$$= P\left[S_{[nMt]} \in a_{nMt}\mu + a_n A \text{ for some } \frac{N}{nM} < t \le 1\right]$$

$$= P\left[Y_{nM}(t) \in \frac{a_{nMt}}{a_{nM}}\mu + \frac{a_n}{a_{nM}}A \text{ for some } \frac{N}{nM} < t \le 1\right].$$

Let $0 < \delta < 1$. By the Potter bounds, for all $N \ge 1$ large enough we have

$$\frac{a_x}{a_y} > (1-\delta)\left(\frac{x}{y}\right)^{\omega+\delta} \qquad \text{for all } N < x \le y.$$

For such N and any $n > N$ we have by the second part of Condition $\mathcal{A}$,

$$P\big[N < T(a_n) \le nM\big]$$

$$\le P\left[Y_{nM}(t) \in (1-\delta)\left(t^{\omega+\delta}\mu + M^{-(\omega+\delta)}A\right) \text{ for some } \frac{N}{nM} < t \le 1\right]$$

$$\le P\left[Y_{nM}(t) \in (1-\delta)\left(t^{\omega+\delta}\mu + M^{-(\omega+\delta)}A\right) \text{ for some } 0 \le t \le 1\right]$$

$$= P\big[Y_{nM} \in B\big],$$

where

$$B = \left\{f \in \mathcal{BV} : f(t) \in (1-\delta)\left(t^{\omega+\delta}\mu + M^{-(\omega+\delta)}A\right) \text{ for some } 0 \le t \le 1\right\}.$$

Applying the functional large deviation principle in Theorem 2.2 in Ghosh and Samorodnitsky (2009) we obtain

$$\limsup_{n\to\infty} \frac{1}{b_{nM}} P\big[N < T(a_n) \le nM\big] \le - \inf_{f \in \bar{B}} I(f),$$

where the closure of B is taken in the uniform topology, and

$$I(f) = \begin{cases} \int_0^1 I_l\big(f'(t)\big)\,dt & \text{if } f \in \mathcal{AC}, f(0) = 0 \\ \infty & \text{otherwise.} \end{cases}$$

Clearly,

$$\bar{B} = \left\{f \in \mathcal{BV} : f(t) \in (1-\delta)\left(t^{\omega+\delta}\mu + M^{-(\omega+\delta)}\bar{A}\right) \text{ for some } 0 \le t \le 1\right\},$$

and so

$$(3.12) \quad \limsup_{n\to\infty} \frac{1}{b_{nM}} P\big[N < T(a_n) \le nM\big] \le - \inf_{y \in \bar{A}} \inf_{0 \le t_0 \le 1} \inf_{f \in G_{y,t_0}} \int_0^1 I_l\big(f'(t)\big)\,dt,$$

where

$$G_{y,t_0} = \left\{f \in \mathcal{AC} : f(t_0) = (1-\delta)\left(t_0^{\omega+\delta}\mu + M^{-(\omega+\delta)}y\right)\right\}.$$

Next, we notice that for every $f \in G_{y,t_0}$ we have by the definition of the rate function I_l in (2.8) and convexity,

$$\int_0^1 I_l\big(f'(t)\big)dt = \int_0^1 \frac{1}{2}f'(t)'\Sigma^{-1}f'(t)\,dt$$

$$\geq \int_0^{t_0} \frac{1}{2}f'(t)'\Sigma^{-1}f'(t)\,dt$$

$$\geq \frac{1}{2t_0}\left(\int_0^{t_0} f'(t)\,dt\right)'\Sigma^{-1}\left(\int_0^{t_0} f'(t)\,dt\right)$$

$$= \frac{1}{2t_0}f(t_0)\Sigma^{-1}f(t_0)$$

$$= \frac{1}{2t_0}(1-\delta)^2\left[\left(t_0^{\omega+\delta}\mu + M^{-(\omega+\delta)}y\right)'\Sigma^{-1}\left(t_0^{\omega+\delta}\mu + M^{-(\omega+\delta)}y\right)\right].$$

Introducing the variable $c = (t_0 M)^{-(\omega+\delta)}$, we obtain

$$\limsup_{n\to\infty} \frac{1}{b_{nM}}P\big[N < T(a_n) \leq nM\big]$$

$$\leq -\inf_{c\geq M^{-(\omega+\delta)}} \inf_{y\in\bar{A}} M^{1-2(\omega+\delta)}c^{1/(\omega+\delta)-2}(1-\delta)^2\frac{1}{2}(\mu+cy)'\Sigma^{-1}(\mu+cy),$$

and so for every $0 < \delta < 1$,

$$\limsup_{n\to\infty} \frac{1}{b_n}P\big[N < T(a_n) \leq nM\big]$$

$$\leq -M^{-2\delta}(1-\delta)^2 \inf_{c\geq M^{-(\omega+\delta)}} c^{1/(\omega+\delta)-2} \inf_{y\in\bar{A}} \frac{1}{2}(\mu+cy)'\Sigma^{-1}(\mu+cy).$$

Letting $\delta \to 0$, and noticing that the closure of A plays no role in the right hand side above, we obtain (3.11) and, hence, conclude the proof. $\qquad\square$

Proof of Theorem 3.6. The proof of this theorem is very similar to that of Theorem 3.4. Note that now (b_n) is a regularly varying sequence with exponent ν. We establish the lower bound of this part of the theorem in the same was as in Theorem 3.4, except that we are using a different rate in the large deviation principle, as given in (2.9).

For the upper bound, we also proceed as in the proof of the upper bound in Theorem 3.4, but now we use Lemma 3.5(iii) and the appropriate part of Theorem 2.2 in Ghosh and Samorodnitsky (2009). This gives us (3.12), but this time the rate function I_l scales according to

$$I_l(ax) = a^{\beta/(\beta-1)}I_l(x), \ a > 0, x \in \mathbb{R}^d.$$

Therefore, for every $f \in G_{y,t_0}$

$$\int_0^1 I_l\big(f'(t)\big)dt = \int_0^1 \big(\Lambda^h\big)^*\big(f'(t)\big)\,dt$$

$$\geq \frac{1}{t_0^{1/(\beta-1)}}\big(\Lambda^h\big)^*\big(f(t_0)\big)$$

$$= \frac{1}{t_0^{1/(\beta-1)}} (1-\delta)^{\beta/(\beta-1)} \left(\Lambda^h\right)^* \left(t_0^{\omega+\delta}\mu + M^{-(\omega+\delta)}y\right).$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{b_{nM}} P\big[N < T(a_n) \leq nM\big]$$

$$\leq - \inf_{c \geq M^{-(\omega+\delta)}} \inf_{y\in\bar{A}} M^{(1-\beta(\omega+\delta))/(\beta-1)} c^{(1/(\omega+\delta)-\beta)/(\beta-1)} (1-\delta)^{\beta/(\beta-1)} \left(\Lambda^h\right)^* (\mu+cy),$$

and so for every $0 < \delta < 1$,

$$\limsup_{n\to\infty} \frac{1}{b_n} P\big[N < T(a_n) \leq nM\big]$$

$$\leq -M^{-\beta\delta/(\beta-1)} (1-\delta)^{\beta/(\beta-1)} \inf_{c \geq M^{-(\omega+\delta)}} c^{(1/(\omega+\delta)-\beta)/(\beta-1)} \inf_{y\in\bar{A}} \left(\Lambda^h\right)^* (\mu+cy).$$

Now we let $\delta \to 0$ and complete the proof. $\square$

Proof of Theorem 3.8. The lower bound is obtained as in (3.9), with $b_n = n$ and $\omega = 2 - \alpha$, using the appropriate part of the large deviation principle in (2.8) and (2.9).

The proof of the upper bound for $\alpha = 1$ proceeds, once again, similarly to that of Theorem 3.4. Let $J(t) = \Lambda(t)$. By the assumption of zero mean we know that, for some $c > 0$, $J(t) \leq c\|t\|^2$ for all t in a neighborhood of the origin. Therefore, we can still select $t \in \mathbb{R}^d$ as in Condition $\mathcal{A}$ and $\epsilon > 0$ such that $J(t) - t\mu + \epsilon < 0$, and we conclude that (3.10) still holds. Furthermore, using part (ii) of Theorem 2.4 in Ghosh and Samorodnitsky (2009), we conclude that (3.12) holds as well. Note that for every $f \in G_{y,t_0}$ by the convexity of the function Λ^*,

(3.13)
$$\int_0^1 I_l\big(f'(t)\big)dt = \int_0^1 \Lambda^*\big(f'(t)\big)\,dt$$

$$\geq t_0\Lambda^*\big(t_0^{-1}f(t_0)\big) = t_0\Lambda^*\left(t_0^{-1}(1-\delta)\big(t_0^{1+\delta}\mu + M^{-(1+\delta)}y\big)\right).$$

The same argument as in the proof of the upper bound in Theorem 3.4 shows that for any fixed $0 < \theta < 1$,

$$\inf_{y\in\bar{A}} \inf_{\theta \leq t_0 \leq 1} \inf_{f\in G_{y,t_0}} \int_0^1 I_l\big(f'(t)\big)dt \geq M^{-1} \inf_{c>0} c^{-1} \inf_{\gamma\in\bar{A}} \Lambda^*\big(\mu + c\gamma\big).$$

On the other hand, under the assumptions of the theorem, Λ^* grows super-linearly fast as the norm of its argument increases. Therefore, it follows from (3.13) that

$$\liminf_{\theta\to 0} \inf_{y\in\bar{A}} \inf_{0 < t_0 < \theta} \inf_{f\in G_{y,t_0}} \int_0^1 I_l\big(f'(t)\big)dt = \infty.$$

This proves the upper bound in the case $\alpha = 1$.

Next we consider the case $\alpha < 1$. Fix $t \in G$, and choose $0 < \epsilon < t\mu - \Lambda_\alpha(t)$. We start with recalling that, by Lemma 3.6(i) in Ghosh and Samorodnitsky (2009),

$$\frac{1}{k} \log E\big[e^{tS_k/\Psi_k}\big] \leq \Lambda_\alpha(t) + \epsilon$$

for all k large enough, say, $k \geq N$. In particular, $\sup_{k \geq 1} E\left[e^{tS_k/\Psi_k - kt\mu}\right] < \infty$. Let $\delta > 0$. Notice that

$$
\begin{aligned}
\limsup_{n \to \infty} \frac{1}{n} \log P\big[T(a_n) \leq n\delta\big] \quad &\leq \quad \limsup_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{[n\delta]} P\big[tS_k - a_k t\mu > a_n \inf_{\gamma \in A} t\gamma\big] \\
&\leq \quad \limsup_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{[n\delta]} e^{-n\Psi_n/\Psi_k \inf_{\gamma \in A} t\gamma} E\big[e^{tS_k/\Psi_k - kt\mu}\big] \\
(3.14) \qquad &\leq \quad -\inf_{\gamma \in A} t\gamma \limsup_{n \to \infty} \frac{\Psi_n}{\Psi_{[n\delta]}} = -\delta^{\alpha-1} \inf_{\gamma \in A} t\gamma .
\end{aligned}
$$

Next, for $n \geq N/\delta$, the same argument gives us

$$
\begin{aligned}
P\big[n\delta < T(a_n) < \infty\big] \quad &\leq \quad \sum_{k=[n\delta]+1}^{\infty} P\Big[\frac{tS_k}{\Psi_k} > kt\mu + \frac{n\Psi_n}{\Psi_k} \inf_{\gamma \in A} t\gamma\Big] \\
&\leq \quad \sum_{k=[n\delta]+1}^{\infty} \exp\Big\{-\frac{n\Psi_n}{\Psi_k} \inf_{\gamma \in A} t\gamma + k\big(\Lambda_\alpha(t) - t\mu + \epsilon\big)\Big\} .
\end{aligned}
$$

We break up the sum into pieces. By the monotonicity of the sequence $\big(\Psi_n\big)$ and the choice of ϵ we have for $i \geq 1$,

$$
\begin{aligned}
&\sum_{k=i[n\delta]+1}^{(i+1)[n\delta]} \exp\Big\{-\frac{n\Psi_n}{\Psi_k} \inf_{\gamma \in A} t\gamma + k\big(\Lambda_\alpha(t) - t\mu + \epsilon\big)\Big\} \\
&\leq \quad n\delta \exp\Big\{-\frac{n\Psi_n}{\Psi_{(i+1)[n\delta]}} \inf_{\gamma \in A} t\gamma + (i[n\delta]+1)\big(\Lambda_\alpha(t) - t\mu + \epsilon\big)\Big\} .
\end{aligned}
$$

Let $0 < \eta < 1 - \alpha$. By the Potter bounds (see Proposition 0.8 in Resnick (1987)) there exists $N_1 \geq 1$ such that for every $n \geq N_1$ we have both

$$
\frac{\Psi_n}{\Psi_{(i+1)[n\delta]}} \geq x_{i,\delta}(\eta) := (1-\eta) \min\Big\{\big((i+1)\delta\big)^{\alpha-1-\eta}, \big((i+1)\delta\big)^{\alpha-1+\eta}\Big\}
$$

and $(i[n\delta]+1)/n \geq i\delta(1-\eta)$. We conclude that for $n > \max\{N/\delta, N_1\}$ and $i \geq 1$,

$$
\begin{aligned}
&\sum_{k=i[n\delta]+1}^{(i+1)[n\delta]} \exp\Big\{-\frac{n\Psi_n}{\Psi_k} \inf_{\gamma \in A} t\gamma + k\big(\Lambda_\alpha(t) - \mu t + \epsilon\big)\Big\} \\
&\leq \quad n\delta \exp\Big\{-n\Big(x_{i,\delta}(\eta) \inf_{\gamma \in A} t\gamma - i\delta(1-\eta)\big(\Lambda_\alpha(t) - \mu t + \epsilon\big)\Big)\Big\} .
\end{aligned}
$$

Denoting $y_i = x_{i,\delta}(\eta) \inf_{\gamma \in A} t\gamma - i\delta(1-\eta)\big(\Lambda_\alpha(t) - \mu t + \epsilon\big)$ and $y^* = \min_{i \geq 1} y_i$, we see that $y^* > 0$ and that $y^* = y_{i^*}$ for some $i^* \geq 1$. Therefore, for every $n > \max\{N/\delta, N_1\}$ we have

$$
P\big[n\delta < T(a_n) < \infty\big] \leq n\delta \exp\big\{-ny^*\big\} \sum_{i=1}^{\infty} \exp\big\{-n(y_i - y^*)\big\}
$$

and, therefore,

$$
(3.15) \qquad \limsup_{n \to \infty} \frac{1}{n} \log P\big[n\delta < T(a_n) < \infty\big] \leq -y^* .
$$

Combining (3.14) and (3.15) we obtain

$$\limsup_{u\to\infty} \frac{1}{a^{\leftarrow}(u)} \log\rho(u) = \limsup_{n\to\infty} \frac{1}{n}\log P\big[T(a_n) < \infty\big] \le \max\big\{-t^*\delta^{\alpha-1}, -y^*\big\}.$$

Letting ϵ and η decrease to 0, we conclude that

$$\limsup_{u\to\infty} \frac{1}{a^{\leftarrow}(u)}\log\rho(u) \le -\min_{i\ge1}\Big(\big((i+1)\delta\big)^{\alpha-1}\inf_{\gamma\in A} t\gamma - i\delta\big(\Lambda_\alpha(t) - \mu t\big)\Big)$$

$$\le -\inf_{u>0}\Big\{u^{\alpha-1}\inf_{\gamma\in A}t\gamma - u\big(\Lambda_\alpha(t) - t\mu\big)\Big\} + \delta\big(\Lambda_\alpha(t) - t\mu\big).$$

Letting, finally, $\delta \to 0$ and optimizing over $t \in G$ completes the proof.   □

Proof of Theorem 3.9. The lower bound in the theorem is established in the same way as the lower bound in Theorem 3.4, using the fact that in the present theorem, the sequence (b_n) is regularly varying with exponent $\nu = 2(\omega + \alpha) - 3$, the large deviation principle (2.9), and the fact that $(G_\Sigma)_\alpha = C_{\alpha,2}G_\Sigma$.

For the upper bound, we consider, once again, the cases $\alpha < 1$ and $\alpha = 1$ separately. In the case $\alpha < 1$ we notice that the sequence (a_n/b_n) is regularly varying with the exponent

$$\omega - \nu = 3 - 2\alpha - \omega \ge 1 - \alpha > 0.$$

Therefore, the argument used in the proof of the upper bound in the case $\alpha < 1$ in Theorem 3.8 applies in this case as well, resulting in

$$\limsup_{u\to\infty}\frac{1}{b_{a^{\leftarrow}(u)}}\log\rho(u) \le -\sup_{t\in G}\inf_{u>0}\Big\{u^{-(\omega-\nu)}\inf_{\gamma\in A}t\gamma - u^\nu\big(\tfrac{1}{2}C_{\alpha,2}\,t'\Sigma t - t\mu\big)\Big\}.$$

The infimum over u is achieved at

$$u = \left(\frac{\nu}{\omega-\nu}\frac{t\mu - \frac{1}{2}C_{\alpha,2}\,t'\Sigma t}{\inf_{\gamma\in A}t\gamma}\right)^{-1/\omega},$$

and the upper bound in the case $\alpha < 1$ is obtained by substitution.

The argument in the case $\alpha = 1$ is the same as the argument of the corresponding case in Theorem 3.8.   □

Proof of Theorem 3.11. The lower bound in the theorem is, once again, established in the same way as the lower bound in Theorem 3.4, using the fact that in the present theorem, the sequence (b_n) is regularly varying with exponent $\nu = \big(\beta(w + \alpha - 1) - 1\big)/(\beta - 1)$, the large deviation principle (2.9), and the fact that $(\Delta^h)_\alpha = C_{\alpha,\beta}\Delta^h$.

We prove now the upper bound. Suppose first that $\alpha < 1$ and $\omega < \beta(1 - \alpha)$. In this case $\omega - \nu > 0$ and we use, once again, the argument of the proof of the upper bound in the case $\alpha < 1$ in Theorem 3.8. This gives us this time

$$\limsup_{u\to\infty}\frac{1}{b_{a^{\leftarrow}(u)}}\log\rho(u) \le -\sup_{t\in G^{(1)}}\inf_{u>0}\Big\{u^{-(\omega-\nu)}\inf_{\gamma\in A}t\gamma - u^\nu\big(C_{\alpha,\beta}\,\Lambda^h(t) - t\mu\big)\Big\}.$$

The infimum over u is achieved at

$$u = \left(\frac{\nu}{\omega-\nu}\frac{t\mu - C_{\alpha,\beta}\,\Lambda^h(t)}{\inf_{\gamma\in A}t\gamma}\right)^{-1/\omega},$$

and the required upper bound follows by substitution.

Next, we suppose that $\alpha < 1$ and $\omega > \beta(1 - \alpha)$. The proof is similar to that of the proof of the upper bound in the case $\alpha < 1$ in Theorem 3.8, but relies on Lemma 3.14 below in addition to Lemma 3.6 in Ghosh and Samorodnitsky (2009).

For $t \in \mathbb{R}^d$ and $u > 0$ let $J_u(t) = u^{1+(1-\alpha)\beta} C_{\alpha,\beta} \Lambda^h(t)$. Let $0 < \delta < 1$, and note that by Lemma 3.14, for any $t \in \mathbb{R}^d$ as in Condition 3.3,

$$
\limsup_{n\to\infty} \frac{1}{b_n} \log P\big[T(a_n) \le n\delta\big]
$$

$$
\le \quad \limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{k=1}^{[n\delta]} P\big[tS_k > a_n \inf_{\gamma \in A} t\gamma\big]
$$

$$
\le \quad \limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{k=1}^{[n\delta]} e^{-b_n \inf_{\gamma \in A} t\gamma} E\Big[\exp\{\frac{b_n}{a_n} tS_k\}\Big]
$$

$$
\le \quad - \inf_{\gamma \in A} t\gamma + \limsup_{n\to\infty} \frac{1}{b_n} \log\Big\{n\delta \sup_{k \le n\delta} E\Big[\exp\{\frac{b_n}{a_n} tS_k\}\Big]\Big\} = - \inf_{\gamma \in A} t\gamma + J_\delta(t).
$$

Since $J_\delta(t) \to 0$ as $\delta \to 0$ for every t, we see that

$$
\lim_{\delta \to 0} \limsup_{n\to\infty} \frac{1}{b_n} \log P\big[T(a_n) \le n\delta\big] \le - \inf_{\gamma \in A} t\gamma .
$$

Since we may replace t by ct for any $c > 0$ without violating the restrictions imposed by Condition 3.3, we let $c \to \infty$ to conclude that

$$
(3.16) \qquad \lim_{\delta \to 0} \limsup_{n\to\infty} \frac{1}{b_n} \log P\big[T(a_n) \le n\delta\big] = -\infty .
$$

Further, using Lemma 3.6 in Ghosh and Samorodnitsky (2009) the argument used to prove (3.10) applies, and gives us

$$
(3.17) \qquad \lim_{\delta \to 0} \limsup_{n\to\infty} \frac{1}{b_n} \log P\big[n\delta^{-1} \le T(a_n) < \infty\big] = -\infty .
$$

Next, fix $t \in G^{(2)}$. This means that we can choose $0 < \epsilon < 1$ so small that $J_u(t) - u^\omega t\mu - \inf_{\gamma \in A} t\gamma + \epsilon < 0$ for all $u > 0$. For $0 < \delta < 1$ we have, as before,

$$
\limsup_{n\to\infty} \frac{1}{b_n} \log P\big[n\delta < T(a_n) < n\delta^{-1}\big]
$$

$$
\le \quad \limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{k=[n\delta]+1}^{[n\delta^{-1}]} P\big[tS_k - a_k t\mu > a_n \inf_{\gamma \in A} t\gamma\big]
$$

$$
\le \quad \limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{k=[n\delta]+1}^{[n\delta^{-1}]} \exp\Big\{ - b_n\Big(\inf_{\gamma \in A} t\gamma + \frac{a_k}{a_n} t\mu\Big)\Big\} E\Big[\exp\{\frac{b_n}{a_n} tS_k\}\Big].
$$

Let $0 < \eta < 1$. By the Potter bounds there exists $N_1 \ge 1$ such that for $k, l \ge N_1$

$$
\frac{a_k}{a_l} \ge a_{k,l}(\eta) := (1 - \eta) \min\Big\{ \Big(\frac{k}{l}\Big)^{\omega - \eta}, \Big(\frac{k}{l}\Big)^{\omega + \eta}\Big\}
$$

and for every $n \geq N_1$, $[n\delta]/n \geq (1 - \eta)\delta$. For every $n > N_1/\delta$ and $i \geq 1$,

$$\sum_{k=i[n\delta]+1}^{(i+1)[n\delta]} \exp\left\{ - b_n \left(\inf_{\gamma \in A} t\gamma + \frac{a_k}{a_n} t\mu \right) \right\} E\left[\exp\{\frac{b_n}{a_n} tS_k\} \right]$$

$$\leq \sum_{k=i[n\delta]+1}^{(i+1)[n\delta]} \exp\left\{ - b_n \left(\inf_{\gamma \in A} t\gamma + a_{k,n}(\eta)t\mu \right) \right\} E\left[\exp\{\frac{b_n}{a_n} tS_k\} \right]$$

$$\leq n\delta \exp\left\{ - b_n \left(\inf_{\gamma \in A} t\gamma + a_{i[n\delta],n}(\eta)t\mu \right) + \sup_{k \leq (i+1)[n\delta]} \log E\left[\exp\{\frac{b_n}{a_n} tS_k\} \right] \right\}.$$

By the choice of n, we known that for every $i \geq 1$, $a([in\delta], n)(\eta) \geq (1 - \eta)^{\omega + \eta + 1} a(i\delta, 1)(\eta)$. Furthermore, by Lemma 3.14, we can choose N_2 so large that for all $n \geq N_2$, all $i = 1, 2, \ldots, \delta^{-2} + 1$,

$$\sup_{k \leq (i+1)[n\delta]} \log E\left[\exp\{\frac{b_n}{a_n} tS_k\} \right] \leq b_n \left(J_{(i+1)\delta}(t) + \epsilon \right).$$

Therefore, for all $n \geq \max(N_1/\delta, N_2)$ and i as above,

$$\sum_{k=i[n\delta]+1}^{(i+1)[n\delta]} \exp\left\{ - b_n \left(\inf_{\gamma \in A} t\gamma + \frac{a_k}{a_n} t\mu \right) \right\} E\left[\exp\{\frac{b_n}{a_n} tS_k\} \right]$$

$$\leq n\delta \exp\left\{ - b_n \left(\inf_{\gamma \in A} t\gamma + (1 - \eta)^{\omega + \eta + 1} a(i\delta, 1)(\eta)t\mu \right) + b_n \left(J_{(i+1)\delta}(t) + \epsilon \right) \right\}.$$

We proceed as in the proof of the upper bound in the case $\alpha < 1$ in Theorem 3.8. Setting

$$y_i = + \inf_{\gamma \in A} t\gamma + (1 - \eta)^{\omega + \eta + 1} a(i\delta, 1)(\eta)t\mu - J_{(i+1)\delta}(t) - \epsilon$$

and $y^* = \min_i y_{i \geq 1}$, we proceed as in the above prove and conclude that

$$(3.18) \qquad \limsup_{n \to \infty} \frac{1}{b_n} \log P\left[n\delta < T(a_n) < \infty \right] \leq -y^*.$$

Combining (3.18), (3.16) and (3.17), and letting first $\delta \to 0$, and then $\eta \to 0$ and $\epsilon \to 0$, we obtain

$$\limsup_{u \to \infty} \frac{1}{b_{a^{\leftarrow}(u)}} \log \rho(u) \leq \sup_{u > 0} \left\{ - \inf_{\gamma \in A} t\gamma - u^{\omega} t\mu + u^{1 + \beta(1-\alpha)} C_{\alpha,\beta} \Lambda^h(t) \right\}.$$

The supremum is attained at

$$u = \left(\frac{1 + (1 - \alpha)\beta}{\omega t\mu} C_{\alpha,\beta} \Lambda^h(t) \right)^{\frac{1}{\omega - (1 + (1-\alpha)\beta)}},$$

and the required upper bound is obtained by substitution and optimizing over t.

Finally, in the case $\alpha = 1$ the upper bound of the present theorem can be obtained in the same way as in Theorem 3.4. $\qquad \square$

This section is concluded by a lemma needed for the proof of Theorem 3.11.

Lemma 3.14. *Under the assumption R4 with $\alpha < 1$, for any $\theta > 0$ and $t \in \mathbb{R}^d$*

$$\lim_{n \to \infty} \frac{1}{b_n} \sup_{k \leq \theta n} \log E\left[\exp\left\{ \frac{b_n}{a_n} tS_k \right\} \right] \leq u^{1 + (1-\alpha)\beta} C_{\alpha,\beta} \Lambda^h(t),$$

where $C_{\alpha,\beta}$ is given by (3.8).

Proof. Observe that since the coefficients satisfy (1.7), there is $N \geq 1$ such that $\phi_{i,n} > 0$ for all $i \in \mathbb{Z}$ and $n \geq N$. Using the fact that $\Lambda(t)$ is increasing along each ray emanating from the origin, we see that, if $n \geq N/\theta$,

$$
\begin{aligned}
\sup_{N \leq k \leq \theta n} \log E\Big[\exp\Big\{\frac{b_n}{a_n} t S_k\Big\}\Big] &= \sup_{N \leq k \leq \theta n} \sum_{i \in \mathbb{Z}} \Lambda\Big(t\frac{b_n}{a_n}\phi_{i,k}\Big) \\
&\leq \sup_{N \leq k \leq \theta n} \sum_{i \in \mathbb{Z}} \Lambda\Big(t\frac{b_n}{a_n}|\phi|_{i,k}\Big) \\
&= \sum_{i \in \mathbb{Z}} \Lambda\Big(t\frac{b_n}{a_n}|\phi|_{i,[\theta n]}\Big),
\end{aligned}
$$

where $|\phi|_{i,n} = |\phi_{i+1}| + \cdots + |\phi_{i+n}|$. Clearly, the sequence $(|\phi_i|)$ is also balanced regular varying and satisfies

$$
\frac{|\phi_n|}{\psi(n)} \to p \text{ and } \frac{|\phi_{-n}|}{\psi(n)} \to q \qquad \text{as } n \to \infty.
$$

With a minor modification of the proof of Lemma 3.6 in Ghosh and Samorodnitsky (2009) we obtain, for any $t \in \mathbb{R}^d$ and $\theta > 0$,

$$
\lim_{n \to \infty} \frac{1}{b_n} \sum_{i \in \mathbb{Z}} \Lambda\Big(t\frac{b_n}{a_n}|\phi|_{i,[un]}\Big) = u^{1+(1-\alpha)\beta} C_{\alpha,\beta}\Lambda^h(t).
$$

Since it is also easy to see that

$$
\lim_{n \to \infty} \frac{1}{b_n} \sup_{k \leq N} \log E\Big[\exp\Big(\frac{b_n}{a_n} t S_k\Big)\Big] = 0,
$$

the proof is complete. $\qquad\square$

4. Appendix

In this section we state certain straightforward multivariate analogs of the ruin probability estimates of Nyrhinen (1994). For completeness we provide the argument.

Let $\big(Y_n,\, n \geq 1\big)$ be an $\mathbb{R}^d$-valued stochastic process. For $n = 1, 2, \ldots$ and $t \in \mathbb{R}^d$ define $g_n(t) = n^{-1} \log E e^{t Y_n}$ and

$$
(4.1) \qquad g(t) = \limsup_{n \to \infty} g_n(t),\ t \in \mathbb{R}^d
$$

(these functions may take the value $+\infty$). Let $A \subset \mathbb{R}^d$ be a Borel set, and define

$$
(4.2) \qquad \mathcal{C} = \big\{t \in \mathbb{R}^d : \inf_{\gamma \in A} t\gamma > 0\big\}, \quad \mathcal{D} = \big\{t \in \mathcal{C} : \sup_{n \geq 1} E e^{t Y_n} < \infty\big\},
$$

and

$$
(4.3) \qquad \mathcal{E} = \big\{t \in \mathbb{R}^d : g \text{ is finite in a neighborhood of } t, \text{ exists as a limit at } t,
$$

and is differentiable at $t\big\}, \quad \mathcal{F} = \big\{t \in \mathcal{E} : \rho\, \nabla g(t) \in A^\circ \text{ for some } \rho > 0\big\}.$

Theorem 4.1. *(i) Suppose that there is $t_0 \in \mathcal{C}$ such that $g(t_0) < 0$. Then*

$$
\limsup_{u \to \infty} \frac{1}{u} \log P\big(Y_n \in uA \ \text{for some } n = 1, 2, \ldots\big) \leq -\sup_{t \in \mathcal{D}} \inf_{\gamma \in A} t\gamma.
$$

(ii) For $t \in \mathcal{F}$, let $\eta(t) = \inf\{\eta > 0 : \eta \nabla g(t) \in A^\circ\}$. Then

$$\liminf_{u \to \infty} \frac{1}{u} \log P\big(Y_n \in uA \ \text{ for some } n = 1, 2, \ldots\big) \geq \sup_{t \in \mathcal{F}} \eta(t)\big[g(t) - t \nabla g(t)\big].$$

Proof. (i) For $n = 1, 2, \ldots$ let $t \in \mathbb{R}^d$ be such that $g_n(t) < \infty$. Let Z_n be an $\mathbb{R}^d$-valued random vector such that

$$P(Z_n \in B) = e^{-n g_n(t)} E\Big[e^{tY_n} \mathbf{1}\big(Y_n \in nB\big)\Big], \ B \subseteq \mathbb{R}^d \text{ a Borel set.}$$

Then

$$(4.4)\ \ P(Y_n \in uA) = e^{n g_n(t)} E\Big[e^{-nt Z_n} \mathbf{1}\Big(Z_n \in un^{-1}A\Big)\Big] \leq \exp\Big\{n g_n(t) - u \inf_{\gamma \in A} t\gamma\Big\}.$$

Fix $M = 1, 2, \ldots$. Using (4.4) for $n \leq Mu$ and $t \in \mathcal{D}$ gives us

$$\sum_{n \leq Mu} P(Y_n \in uA) \leq (Mu) \sup_{n \geq 1} E^{tY_n} \exp\Big\{-u \inf_{\gamma \in A} t\gamma\Big\}.$$

Taking a limit and optimizing over $t \in \mathcal{D}$ we obtain

$$(4.5) \qquad \limsup_{u \to \infty} u^{-1} \log \left(\sum_{n \leq Mu} P(Y_n \in uA) \right) \leq -\sup_{t \in \mathcal{D}} \inf_{\gamma \in A} t\gamma.$$

Next, using (4.4) for $n > Mu$ and t_0 in the statement of the theorem (which is possible for u large enough) gives us for large u

$$\sum_{n > Mu} P(Y_n \in uA) \leq \sum_{n > Mu} e^{-\alpha n} \leq C e^{-\alpha Mu},$$

where $\alpha \in \big(g(t_0), 0\big)$ and $C > 0$ a constant. Therefore,

$$(4.6) \qquad \limsup_{u \to \infty} u^{-1} \log \left(\sum_{n > Mu} P(Y_n \in uA) \right) \leq -\alpha M.$$

Combining (4.5) with (4.6) and letting $M \to \infty$ we obtain the statement of part (i) of the theorem.

For part (ii), let $t \in \mathcal{F}$, and let $\eta > 0$ be such that $\eta \nabla g(t) \in A^\circ$. Choose $\varepsilon > 0$ so that the open ball $B\big(\eta \nabla g(t), \varepsilon\big)$ lies completely within A. Then for u large enough,

$$P(Y_{[u\eta]} \in uA) \geq P\Big(Y_{[u\eta]} \in uB\big(\eta \nabla g(t), \varepsilon\big)\Big) \geq P\left(\frac{Y_{[u\eta]}}{[u\eta]} \in B\big(\nabla g(t), \varepsilon/(2\eta)\big)\right).$$

On the other hand, for any $t \in \mathcal{E}$ and $\varepsilon > 0$, for all n large enough so that $g_n(t) < \infty$, we have

$$P\Big(Y_n \in nB\big(\nabla g(t), \varepsilon\big)\Big) = e^{n g_n(t)} E\Big[e^{-nt Z_n} \mathbf{1}\Big(Z_n \in B\big(\nabla g(t), \varepsilon\big)\Big)\Big]$$

$$\geq \exp\big\{n g_n(t) - nt\nabla g(t) - n\varepsilon\|t\|\big\} P\Big(Z_n \in B\big(\nabla g(t), \varepsilon\big)\Big),$$

so that

$$\liminf_{n \to \infty} n^{-1} \log P\Big(Y_n \in nB\big(\nabla g(t), \varepsilon\big)\Big)$$

$$\geq g(t) - t\nabla g(t) - \varepsilon\|t\| + \liminf_{n \to \infty} n^{-1} \log P\Big(Z_n \in B\big(\nabla g(t), \varepsilon\big)\Big)$$

$$= g(t) - t\nabla g(t) - \varepsilon\|t\|,$$

since, as is shown below, the last lower limit is equal to zero. Therefore, for any $t \in \mathcal{F}$, $\eta > 0$ as above and $\varepsilon > 0$ small enough,

$$\liminf_{u \to \infty} \frac{1}{u} \log P\big(Y_n \in uA \text{ for some } n = 1, 2, \ldots\big)$$

$$\geq \liminf_{u \to \infty} \frac{1}{u} \log P(Y_{[u\eta]} \in uA) \geq \eta\big[g(t) - t\nabla g(t) - \varepsilon\|t\|\big].$$

Letting $\varepsilon \to 0$, $\eta \to \eta(t)$, and optimizing over $t \in \mathcal{F}$, we obtain the claim of part (ii) of the theorem.

The proof of the theorem will be finished once we show that for every $t \in \mathcal{E}$ and $\varepsilon > 0$, $P\big(Z_n \in B\big(\nabla g(t), \varepsilon\big)\big) \to 1$ as $n \to \infty$. To this end, let e_i be the ith coordinate unit vector in $\mathbb{R}^d$, $i = 1, \ldots, d$. Then

$$P\Big(Z_n \notin B\big(\nabla g(t), \varepsilon\big)\Big) \leq \sum_{i=1}^{d} P\left(Z_n e_i \geq \frac{\partial g}{\partial y_i}(t) + \frac{\varepsilon}{d}\right)$$

$$+ \sum_{i=1}^{d} P\left(Z_n e_i \leq \frac{\partial g}{\partial y_i}(t) - \frac{\varepsilon}{d}\right).$$

Fix $i = 1, \ldots, d$, and choose $r > 0$ so small that $g(t + re_i) < \infty$. Then $g_n(t + re_i) < \infty$ for all n large enough, and for such n we have

$$P\left(Z_n e_i \geq \frac{\partial g}{\partial y_i}(t) + \frac{\varepsilon}{d}\right) = e^{-ng_n(t)} E\left[\mathbf{1}\Big(Y_n e_i \geq n\frac{\partial g}{\partial y_i}(t) + n\frac{\varepsilon}{d}\Big) e^{tY_n}\right]$$

$$\leq \exp\Big\{-ng_n(t) - rne_i\nabla g(t) - rn\varepsilon/d\Big\} E\left[\mathbf{1}\Big(Y_n e_i \geq n\frac{\partial g}{\partial y_i}(t) + n\frac{\varepsilon}{d}\Big) e^{(t+re_i)Y_n}\right]$$

$$\leq \exp\Big\{n\Big(g_n(t + re_i) - g_n(t) - re_i\nabla g(t) - r\varepsilon/d\Big)\Big\}.$$

Therefore,

$$\limsup_{n \to \infty} \frac{1}{n} \log P\left(Z_n e_i \geq \frac{\partial g}{\partial y_i}(t) + \frac{\varepsilon}{d}\right) \leq g(t + re_i) - g(t) - re_i\nabla g(t) - r\varepsilon/d.$$

Since

$$g(t + re_i) - g(t) - re_i\nabla g(t) = o(r) \text{ as } r \downarrow 0,$$

this expression is negative for r small enough, and so

$$P\left(Z_n e_i \geq \frac{\partial g}{\partial y_i}(t) + \frac{\varepsilon}{d}\right) \to 0$$

as $n \to \infty$ for every $i = 1, \ldots, d$. A similar argument gives us

$$P\left(Z_n e_i \leq \frac{\partial g}{\partial y_i}(t) - \frac{\varepsilon}{d}\right) \to 0$$

as $n \to \infty$ for every $i = 1, \ldots, d$ and the proof of the theorem is complete. $\qquad\square$

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