

Some Proofs on Statistical Magnitudes for Continuous Phenomena

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Abstract: In this work, the proofs concerning the continuity of the disequilibrium, Shannon information and statistical complexity in the space of distributions are presented. Also, some results on the existence of Shannon information for continuous systems are given.

Key-Words: Near-continuity, disequilibrium, Shannon information, statistical complexity

1 Introduction

The concepts of Shannon information (or entropy), S , and disequilibrium, D , for a continuous system are given by

$$S = -k \int_{-\infty}^{\infty} p(x) \log p(x) dx, \quad (1)$$

$$D = \int_{-\infty}^{\infty} p^2(x) dx, \quad (2)$$

where k is a positive constant (that is chosen to be equal to 1), x represents the continuum of the system states, and $p(x)$ stands for the normalized density function of all these states.

The statistical complexity C , the so-called LMC complexity [1], is defined as

$$C = H \cdot D, \quad (3)$$

where H gives account of the broadness of the distribution and D gives an idea of how much spiky is the distribution respect to equilibrium distribution, that in this case is the equiprobability. For our purpose, we take a version used in Ref. [2] as quantifier of H . This is the simple exponential Shannon entropy [3], that takes the form,

$$H = e^S, \quad (4)$$

that implies the positivity of H and C . These information theoretic indicators has been successfully applied in different contexts in order to unveil a complex behavior or structure, for instance, in gases out of equilibrium [4], in coupled map lattices [5] or in quantum systems [6], to cite some of them.

In this communication, it is our objective to prove rigorously that all these functionals, H , D and C , are

near-continuous, i.e. similar distributions take a similar value of information, disequilibrium and statistical complexity. Also, we present some exact results on the values that Shannon information can take for continuous systems.

2 Continuity of H , D and C

First, we remind some definitions that were used in Ref. [2].

Definition 1 . Let I be an interval in \mathbb{R} and $\delta > 0$. We will say that two density functions, namely f and g , are δ -neighboring functions on I if both are supported on I and the Lebesgue measure of the set of points of I such that $|f(x) - g(x)| \geq \delta$ is zero, i.e. if the essential supremum (ess sup) of $|f - g|$ verifies

$$\operatorname{ess\,sup}_{x \in I} |f(x) - g(x)| < \delta.$$

Note that this definition can be applied to non bounded density functions.

Definition 2 . We will say that a real map T defined on density functions is near-continuous on I if for any $\varepsilon > 0$ there exists $\delta(\varepsilon, I) > 0$ such that if f and g are two δ -neighboring functions on I then $|T(f) - T(g)| < \varepsilon$.

Theorem 3 . Disequilibrium D is near-continuous on \mathbb{R} .

Proof. Take $\varepsilon > 0$, $\delta < \frac{\varepsilon}{2}$. Let f, g be two δ -neighboring functions on \mathbb{R} . Then,

$$D(f) - D(g) = \int_{-\infty}^{\infty} (f^2(x) - g^2(x)) dx \quad (5)$$

$$= \int_{-\infty}^{\infty} (f(x) + g(x)) * (f(x) - g(x)) dx .$$

As f, g are δ -neighboring functions on \mathbb{R} we have that $|f(x) - g(x)| \leq \delta$ a.e. and so,

$$\begin{aligned} |D(f) - D(g)| &\leq \int_{-\infty}^{\infty} (f(x) + g(x)) * |f(x) - g(x)| dx \\ &\leq \delta \left[\int_{-\infty}^{\infty} f(x) dx + \int_{-\infty}^{\infty} g(x) dx \right] \\ &= 2\delta < \epsilon. \quad \clubsuit \end{aligned}$$

Note that if we consider only bounded density functions, then D is a continuous map between normed spaces in the topological sense.

In order to prove the near-continuity of the Shannon information, S , and for extension of its exponential, in this case, H , the following technical Lemma is needed.

Lemma 4 . *For all $\alpha > 1$ there exists $\delta_\alpha < 1$ such that for any pair of points x_1, x_2 with $0 < x_1, x_2 < \alpha$ and $|x_1 - x_2| < \delta < \delta_\alpha$ we have that*

$$|x_1 \log x_1 - x_2 \log x_2| < |\delta \log \delta| .$$

Proof. Given $\delta \in (0, 1)$ consider the function

$$s_\delta(x) = (x + \delta) \log(x + \delta) - x \log x,$$

which can be extended to $x = 0$ with continuity and is derivable in \mathbb{R}^+ with $s'_\delta(x) = \log \frac{x+\delta}{x}$. As $\delta > 0$, $x + \delta > x$ and so $s'_\delta > 0$. This means that s_δ is rising on \mathbb{R}^+ . Observe that as $\delta < 1$, $\log \delta < 0$ and so $s_\delta(0) < 0$. On the other hand, as s_δ is rising and $\alpha > 1$, we have that $0 < (1 + \delta) \log(1 + \delta) = s_\delta(1) < s_\delta(\alpha)$. This means that there is only one unique x_0 so that $s_\delta(x) < 0$ for any $0 \leq x < x_0$ and $s_\delta(x) > 0$ for the values of $x_0 < x \leq \alpha$.

The result of the Lemma, expressed in terms of the function s_δ , is equivalent to finding δ_α such that for any $0 < \delta < \delta_\alpha$ and $0 < x < \alpha$,

$$|s_\delta(x)| < |s_\delta(0)| .$$

Clearly, this inequality is verified for any $x < x_0$. So, we can suppose $x > x_0$. In this case the inequality can be written as

$$s_\delta(x) < -s_\delta(0) .$$

On the other hand, as s_δ is rising, it can be seen that the verification of the former inequality is equivalent to the verification of the next one:

$$s_\delta(\alpha) < -s_\delta(0) .$$

Let us consider the function $F_\alpha(\delta) = s_\delta(\alpha) + s_\delta(0)$. The problem has been reduced to finding a δ_α such that for all $0 < \delta < \delta_\alpha$, $F_\alpha(\delta) < 0$. Since $F'_\alpha(\delta) = 2 + \log(\delta(\alpha + \delta))$, we will take $\tilde{\delta}_\alpha$ as the only positive number such that $F'_\alpha(\tilde{\delta}_\alpha) = 0$, then

$$\tilde{\delta}_\alpha = \frac{1}{2e} \left(-\alpha e + \sqrt{(\alpha e)^2 + 4} \right) .$$

As F_α is decreasing on $[0, \tilde{\delta}_\alpha]$, rising on $[\tilde{\delta}_\alpha, \infty)$ and $F_\alpha < 0$, we can conclude that there exists a unique δ_α bigger than $\tilde{\delta}_\alpha$ such that for all $0 < \delta < \delta_\alpha$, $F_\alpha(\delta) < 0$. \clubsuit

Theorem 5 . *H is near-continuous on every compact $K \subset \mathbb{R}$.*

Proof. First, we will prove that S is near-continuous. Then, by continuity of the exponential operation, H will also be near-continuous.

Let K be a compact set in \mathbb{R} . Take $\varepsilon < 1$, $\delta > 0$ with $\delta < \min\{\delta_2, \frac{\varepsilon}{8}\}$, where δ_2 is the δ_α given in Lemma 4 for $\alpha = 2$, and $|\delta \log \delta| < \frac{\varepsilon}{2|K|}$, where $|K|$ stands for the Lebesgue measure of K .

Let f and g be two δ -neighboring functions on K and let us consider the sets

$$\begin{aligned} A &= \{x \in K; f(x) > g(x) \geq 1\}, \\ B &= \{x \in K; g(x) > f(x) \geq 1\}, \\ C &= \{x \in K; f(x) = g(x)\}, \\ D &= \{x \in K; f(x) \leq 2, g(x) \leq 2\}. \end{aligned}$$

As $\delta < 1$, $A \cup B \cup C \cup D = \mathbb{R}$. Defining $G = f \log f - g \log g$, we have

$$\begin{aligned} |S(f) - S(g)| &\leq \int_{-\infty}^{\infty} |G(x)| dx \quad (6) \\ &\leq \int_A |G(x)| dx + \int_B |G(x)| dx + \\ &\quad \int_C |G(x)| dx + \int_D |G(x)| dx . \end{aligned}$$

We will bound separately each one of the above integrals. Note that the third one is null and the first two ones behave in a similar manner.

Applying the Mean Value Theorem to the function $s(t) = t \log(t)$ we have that for any $x \in A$ there exists \tilde{x} such that $g(x) < \tilde{x} < f(x)$ and

$$\begin{aligned} G(x) &= [f(x) - g(x)]s'(\tilde{x}) \\ &= [f(x) - g(x)][1 + \log \tilde{x}] . \end{aligned}$$

As $x \in A$, $1 \leq g(x)$. Then, $1 \leq \tilde{x} < f(x)$ and so $\log \tilde{x} < \tilde{x} < f(x)$. From here we conclude that $1 + \log \tilde{x} < f(x) + g(x)$ and then the first integral can be bounded as

$$\begin{aligned} \int_A |G(x)| dx &\leq \int_A |f(x) - g(x)| [1 + \log \tilde{x}] dx \\ &\leq \delta \int_{\mathbb{R}} (f(x) + g(x)) dx = 2\delta < \frac{\varepsilon}{4}. \end{aligned}$$

Analogously,

$$\int_B |G(x)| dx < \frac{\varepsilon}{4}.$$

For the last integral we make use of the Lemma 4. As $\delta < \delta_2$, we have that

$$|G(x)| = |f \log f - g \log g| < |\delta \log \delta|.$$

And as δ has been chosen so that $|\delta \log \delta| < \frac{\varepsilon}{2|K|}$

$$\int_D |G(x)| dx \leq \frac{\varepsilon}{2}.$$

By putting together in (6) the inequalities obtained for each integral, then finally it is proved that S is near-continuous. Hence, by extension, H is also near-continuous and the Theorem 5 is proved. ♣

Corollary 6 . *The complexity C is near-continuous on compacts.*

Proof. It is immediate from the definition of C and Theorems 3 and 5. ♣

As it was observed in Ref. [2], let us remark the necessity to have compactness in the sets K in order to prove the near-continuity of H . The same can be said for C .

3 On the Existence of Entropy

Here, we proceed to show some results on the existence of the Shannon entropy S for continuous systems.

Proposition 7 . *Given a density function $p(x)$ bounded and compactly supported, its entropy $S(p)$ is finite.*

Proof. Let C be the upper bound of $p(x)$. As $0 < p \leq C$ on the support K of p , and the function $x \log x$ has range in $[-e^{-1}, \infty)$ for $x > 0$, then

$$-\frac{1}{e} \mu(K) \leq -S(p) \leq \max\{0, C \log C\} \mu(K).$$

So the integral of the entropy is convergent. ♣

Let us observe that if $p(x)$ is not bounded or has no bounded support, this proposition may not be truth. In this situation, it will be possible to have one of the following cases:

- (1) $S(p)$ does exist and is finite,
- (2) $S(p)$ does exist and is $+\infty$,
- (3) $S(p)$ does exist and is $-\infty$,
- (4) $S(p)$ does not exist.

All these cases are possible, as it can be seen with the following example. Take $p(x) = \frac{1}{x}$ supported on different sets in \mathbb{R}^+ . Without loss of generality it will be assumed that the support of the density function $p(x)$ is a set B consisting on a countable union of intervals in \mathbb{R}^+ .

The following notation: $B = B^+ \cup B^- = (\bigcup_{n \in \mathbb{N}} I_n) \cup (\bigcup_{n \in \mathbb{N}} J_n)$ is used, where

$$I_n = (\gamma_n, \gamma'_n) \quad \text{verifies}$$

$$1 \leq \gamma_0 \leq \gamma'_0 \leq \dots \leq \gamma_n \leq \gamma'_n \leq \dots$$

$$\text{and } J_n = (\eta_n, \eta'_n) \quad \text{verifies}$$

$$1 \geq \eta'_0 \geq \eta_0 \geq \dots \geq \eta'_n \geq \eta_n \geq \dots > 0.$$

Now we consider the projection of such intervals onto \mathbb{R} through the logarithmic function. Let us call $T_n = (\tau_n, \tau'_n) = \log I_n$ and $S_n = (\sigma_n, \sigma'_n) = \log J_n$ and so $\dots \leq \sigma_n \leq \sigma'_n \leq \dots \leq \sigma_0 \leq \sigma'_0 \leq 0 \leq \tau_0 \leq \tau'_0 \leq \dots \leq \tau_n \leq \tau'_n \leq \dots$. We will call $A = (\bigcup_{n \in \mathbb{N}} T_n) \cup (\bigcup_{n \in \mathbb{N}} S_n) = A^+ \cup A^-$, and introduce the notation $\delta_n = \tau'_n - \tau_n$ and $\rho_n = \sigma'_n - \sigma_n$.

It is easy to see that $p(x) = \frac{1}{x} \chi_B(x)$ is of density 1 if and only if $\sum_{n=0}^{\infty} (\delta_n + \rho_n) = 1$.

Let μ be the Lebesgue measure. Let us define

$$f(x) = \begin{cases} \mu\{y > x; y \in A\} & x > 0, \\ -\mu\{y < x; y \in A\} & x < 0, \end{cases}$$

and observe that

$$\lim_{x \rightarrow 0^+} f(x) - \lim_{x \rightarrow 0^-} f(x) = \mu(A^+) + \mu(A^-) = 1.$$

Theorem 8 . *Let the former $p(x) = \frac{1}{x} \chi_B(x)$ be a density function. Then,*

1. *If f is integrable Riemann on \mathbb{R} ($f \in \mathcal{R}(\mathbb{R})$) then $S(p) \in \mathbb{R}$. Besides, the finite value of $S(p)$ is given by*

$$S(p) = \int_{-\infty}^{\infty} f(x) dx = \int_{\mathbb{R}^-} f(x) dx + \int_{\mathbb{R}^+} f(x) dx.$$

2. *If the value of the integral of f in \mathbb{R}^- is finite and in \mathbb{R}^+ is infinite, i.e. $f \in \mathcal{R}(\mathbb{R}^-) \setminus \mathcal{R}(\mathbb{R}^+)$, then $S(p) = +\infty$.*
3. *If the value of the integral of f in \mathbb{R}^- is infinite and in \mathbb{R}^+ is finite, i.e. $f \in \mathcal{R}(\mathbb{R}^+) \setminus \mathcal{R}(\mathbb{R}^-)$, then $S(p) = -\infty$.*

4. If $f \notin \mathcal{R}(\mathbb{R}^-) \cup \mathcal{R}(\mathbb{R}^+)$ then $S(p)$ does not exist.

Proof. First, we see that

$$-\int_1^\infty p(x) \log p(x) dx = \int_0^\infty f(x) dx. \quad (7)$$

In fact,

$$\begin{aligned} \int_1^\infty p(x) \log p(x) dx &= -\int_{B^+} \frac{\log(x)}{x} dx \\ &= -\sum_{n=0}^{\infty} \int_{I_n} \frac{\log(x)}{x} dx \\ &= -\frac{1}{2} \sum_{n=0}^{\infty} \log^2(x)|_{I_n} \\ &= -\frac{1}{2} \sum_{n=0}^{\infty} (\tau_n'^2 - \tau_n^2) \\ &= -\frac{1}{2} \sum_{n=0}^{\infty} \delta_n (\tau_n' + \tau_n). \end{aligned}$$

By other side, analyzing the function f on \mathbb{R}^+ , we have that $\forall x \in [0, \tau_0]$, $f(x) = \mu(A^+)$, and that $\forall x \in [\tau_0', \tau_1]$, $f(x) = \mu(A^+) - \delta_0$. Also, $\forall x \in [\tau_0, \tau_0']$, $f(x) = \mu(A^+) - (x - \tau_0)$, and so on. It is also known that $\sum_{n \in \mathbb{N}} \delta_n = \mu(A^+)$. Hence,

$$\begin{aligned} \int_0^\infty f(x) dx &= \sum_{n=0}^{\infty} (\delta_n \tau_n + \frac{1}{2} \delta_n^2) \\ &= \sum_{n=0}^{\infty} \delta_n (\tau_n + \frac{1}{2} \delta_n) \\ &= \sum_{n=0}^{\infty} (\delta_n (\tau_n + \frac{1}{2} (\tau_n' - \tau_n))) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \delta_n (\tau_n' + \tau_n). \end{aligned}$$

This proves (7).

Similarly, it can be proved that

$$-\int_0^1 p(x) \log p(x) dx = \int_{-\infty}^0 f(x) dx. \quad (8)$$

The result of the theorem is obtained immediately from (7) and (8). ♣

4 Conclusion

Continuity is a desirable property for any good behaved magnitude, although it does not imply the triviality of its mathematical proof. Here, we have shown that entropy, disequilibrium and statistical complexity

are continuous magnitudes when applied to continuous phenomena. In this case, it has been put in evidence that the property of compactness for the support of the density distribution is crucial. Also, some exact results on the possible values that entropy can take have been presented.

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