PARAMETER ESTIMATIONS FOR SPDES WITH MULTIPLICATIVE FRACTIONAL NOISE

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Abstract. We study parameter estimation problem for diagonalizable parabolic stochastic partial differential equations driven by a multiplicative fractional noise with any Hurst parameter $H \in (0,1)$. Two classes of estimates are investigated: traditional maximum likelihood type estimates, and a new class called closed-form exact estimates. Finally several examples are discussed, including statistical inference for stochastic heat equation driven by a fractional Brownian motion.

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1. Introduction

Parameter estimation problem for stochastic partial differential equation has been of great interest in the past decade, and besides being a challenging theoretical problem, it finds its roots and motivations from various applied problems: fluid dynamics [12, 30], biology [9, 10], finance [1, 2, 8], meteorology [5] etc. At general level the problem is to find or estimate the model parameter ϑ (could be a vector) based on observations of the underlying process u_ϑ which is assumed to be a solution of a stochastic evolution equation in finite or infinite dimensional space. We will follow traditional continuous time approach and assume that the solution $u_\vartheta(t)$ is observed continuously in time $t \in [0,T]$. From statistical point of view, we suppose that there exists a family of probability measures $\mathbf{P}_\vartheta$ that depends on parameter $\vartheta \in \Theta \subset \mathbb{R}^n$, and each $\mathbf{P}_\vartheta$ is the distribution of a random element. Assuming that a realization of one random element corresponds to a particular value ϑ_0, the goal is to estimate this parameter from given observations. One approach is to select parameter ϑ that *most likely* produces the observations. This method assumes that the problem is regular or absolutely continuous, which means that there exists a reference probability measure $\mathbf{Q}$ such that all measures $\mathbf{P}_\vartheta,\ \vartheta \in \Theta$, are absolutely continuous with respect to $\mathbf{Q}$. Then Radon-Nikodym derivative $d\mathbf{P}_\vartheta/d\mathbf{Q}$, also called the likelihood ratio, exists, and *the Maximum Likelihood Estimator* (MLE) $\widehat{\vartheta}$ of the parameter of interest is computed by maximizing the likelihood ratio with respect to ϑ. Usually $\widehat{\vartheta} \neq \vartheta$ and the problem is to study the convergence of MLE to the true parameter as more information arrives (for example as time passes or by decreasing the amplitude of noise). If the measures $\mathbf{P}_\vartheta$ are singular for different parameters ϑ, then the model is called singular, and usually the parameter can be found exactly, at least theoretically. While all regular models are to

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some extend the same, each singular model requires individual approach. For example, estimating the drift coefficient for finite-dimensional stochastic differential equations is typically a regular problem, and the parameter can be estimated by means of MLEs, while estimating the diffusion (volatility) coefficient is a singular problem and one can find the diffusion coefficient exactly through quadratic variation of the underlying process. For some finite-dimensional systems, estimating the "drift coefficient" is also a singular problem, and as shown in Khasminskii et al [18] the estimators have nothing to do with MLEs. Generally speaking statistical inference for finite-dimensional diffusions has been studied widely, and there are established necessary and sufficient conditions for absolute continuity of corresponding measures (see, for example [23], [21] and references therein). Some of these results have been extended to infinite dimensional systems in particular to parabolic Stochastic Partial Differential Equations (SPDE). It turns out that in many cases the estimation of drift coefficient for SPDEs is a singular problem, and as general theory suggests one can find the parameter "exactly". One of the first fundamental result in this area that explorers this singularity is due to Huebner, Rozovskii, and Khasminskii [17]. The idea is to approximate the original singular problem by a sequence of regular problems for which MLEs exist. The approximation was done by considering Galerkin-type of projections of the solution on a finite-dimensional space where the estimation problem becomes regular, and it was proved that as dimension of the projection increases the corresponding MLE will converge to the true parameter. In [15, 16, 25, 26], the problem was extended to a general parabolic SPDE driven by additive noise and the convergence of the estimators was given in terms of the order of the corresponding differential operators. For recent developments and other types of inference problems in SPDEs see a survey paper by Lototsky [24] and references therein. Statistical inference for SPDEs driven by multiplicative noise is a more challenging problem. First and only attempt to study equations with multiplicative noise is given in [6], by considering Wiener (not fractional) type noise without spatial correlation structure. Besides MLE type estimates, a completely new class of *exact estimates* were found due essentially to the very singular nature of the problem.

The aim of this note is to study parameter estimation problem for stochastic parabolic equations driven by a *multiplicative fractional noise* with following dynamics

$$u(t) = u(0) + \int_0^t (\mathcal{A}_0 + \theta\mathcal{A}_1)u(s)ds + \int_0^t \mathcal{M}u(s)dW^H(s), \tag{1}$$

where $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{M}$ are some known linear operators, W^H is a fractional Brownian motion with a Hurst parameter $H \in (0,1)$, and θ is a real parameter belonging to a bounded set $\Theta \subset \mathbb{R}$. For now, assume that the stochastic integral with respect to fractional Brownian Motion W^H is well-defined, while the exact meaning will be specified in Section 2.1. The main goal is to estimate the parameter θ based on the observations of the underlying process $u(t),\ t \in [0,T]$. Similar problem for SPDEs driven by additive space-time fractional noise was investigated in [7, 31, 27]. Estimation of drift coefficient for finite-dimensional fractional Ornstein-Uhlenbeck and similar processes has been investigated by Tudor and Viens [32] for $H \in (0,1)$, Kleptsyna and Le Breton [19] for $H \in [1/2,1)$, by developing Girsanov type theorems and finding MLEs. Berzin and Leon [3] estimate simultaneously both drift and diffusion coefficients. Least square estimates for drift coefficients were established by Hu and Nualart [13], and MLE type estimates for discretely observed process by Hu, Weilin and Weiguo [14]. For a general theory, including Girsanov Theorem and some results on statistical inference, for finite dimensional diffusions driven by fractional noise see also the monograph by Mishura [28].

In this paper we continue to explore the impact of the noise in infinite-dimensional evolution equations and its implications on statistical inference. Besides its theoretical roots, this problem is also motivated by increasing demand in modeling various phenomena by SPDEs driven by fractional noise [5, 11]. We assume that the solution of (1) is observed at every $t \in [0, T]$, and hence each Fourier coefficient $u_k(t) = (u_k(t), h_k)_H$ is observable for every $t \in [0, T]$, where H is a Hilbert space in which the solution leaves and $h_k, k \geq 1$, is a CONS in H. All results are stated in terms of Fourier coefficients u_k. In the first part of Section 2 we set up the problem and establish the existence and uniqueness of the solution of the corresponding SPDE. In Subsection 2.2 we introduce the main notations and find the MLE for fractional Geometrical Brownian Motion (which is not covered explicitly in any other sources, at our best knowledge). In Section 3 we study the estimates of drift coefficient θ of equation (1) based on MLE of the corresponding Fourier coefficients. We establish sufficient conditions on operators $\mathcal{A}_1$, $\mathcal{A}_1$ and $\mathcal{M}$, that guarantee efficiency and asymptotic normality of the estimates and some of their versions. Section 4 is dedicated to investigation of a new type of estimates called *closed-form exact* estimates, similar to those studied in [6]. We show that θ can be found `exactly` by knowing just several (usually two) Fourier coefficients. Moreover, by the same technics we found an exact estimate of the Hurst parameter H too, in both regimes, θ known and unknown. Of course there are many other methods of finding the Hurst parameter, but it is out of scope of this publication to apply them to our equation. Some of the results follow from simple algebraic evaluations, but the very existence of such estimates is amazing and gives a better understanding of the nature of the problem's singularity. Also, we want to mention that, despite of memory property of the fractional Brownian Motion which is spilled over the solution too, the exact estimates are based only on observations at time zero and some future time T. In contrast, the MLEs require observation of the whole trajectory $u(t),\ t \in [0, T]$. We conclude the paper with two examples which are of interest along: stochastic heat equation with parameter θ next to Laplace operator, and a general second order parabolic SPDE with θ next to a lower order operator.

While we assume that data is sampled continuously in time, in practice usually this is not the case. For the MLEs derived in Section 3 the problem is reduced to approximate some integrals of a deterministic function with respect to the solution u and eventually to the fractional Brownian motion. However, the Exact Estimates from Section 4 depend only on the values of the solution at initial time $t = 0$ and some future time $t = T$, and thus do not depend on how the solution is observed in time.

2. Preliminary results

2.1. **The equation and existence of the solution.** Let $\mathbf{H}$ be a separable Hilbert space with the inner product $(\cdot, \cdot)_0$ and the corresponding norm $\| \cdot \|_0$. Let Λ be a densely-defined linear operator on $\mathbf{H}$ with the following property: there exists a positive number c such that $\|\Lambda u\|_0 \geq c\|u\|_0$ for every u from the domain of Λ. Then the operator powers $\Lambda^\gamma,\ \gamma \in \mathbb{R}$, are well defined and generate the spaces $\mathbf{H}^\gamma$: for $\gamma > 0$, $\mathbf{H}^\gamma$ is the domain of Λ^γ; $\mathbf{H}^0 = \mathbf{H}$; for $\gamma < 0$, $\mathbf{H}^\gamma$ is the completion of $\mathbf{H}$

with respect to the norm $\|\cdot\|_\gamma := \|\Lambda \cdot\|_0$ (see for instance Krein at al. [20]). By construction, the collection of spaces $\{\mathbf{H}^\gamma,\ \gamma\in\mathbb{R}\}$ has the following properties:

- $\Lambda^\gamma(\mathbf{H}^r) = \mathbf{H}^{r-\gamma}$ for every $\gamma, r\in\mathbb{R}$;
- For $\gamma_1<\gamma_2$ the space $\mathbf{H}^{\gamma_2}$ is densely and continuously embedded into $\mathbf{H}^{\gamma_1}$: $\mathbf{H}^{\gamma_2}\subset\mathbf{H}^{\gamma_1}$ and there exists a positive number c_{12} such that $\|u\|_{\gamma_1}\le c_{12}\|u\|_{\gamma_2}$ for all $u\in\mathbf{H}^{\gamma_2}$;
- for every $\gamma\in\mathbb{R}$ and $m>0$, the space $\mathbf{H}^{\gamma-m}$ is the dual of $\mathbf{H}^{\gamma+m}$ relative to the inner product in $\mathbf{H}^\gamma$, with duality $\langle\cdot,\cdot\rangle_{\gamma,m}$ given by

$$\langle u_1,u_2\rangle_{\gamma,m} = (\Lambda^{\gamma-m}u_1, \Lambda^{\gamma+m}u_2)_0,\ \text{where } u_1\in\mathbf{H}^{\gamma-m},\ u_2\in\mathbf{H}^{\gamma+m}.$$

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ be a stochastic basis with usual assumptions.

Definition 1. A fractional Brownian motion with a Hurst parameter $H\in(0,1)$ is a Gaussian process W^H with zero mean and covariance

$$\mathbb{E}W^H(t)W^H(s) = \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}),\quad t,s\ge 0.$$

Consider the following evolution equation

$$\begin{cases} du(t) = [(\mathcal{A}_0+\theta\mathcal{A}_1)u(t)+f(t)]dt + (\mathcal{M}u(t)+g(t))dW^H(t), & 0<t<T,\\ u(0)=u_0\,, \end{cases} \tag{2}$$

where $\mathcal{A}_0,\mathcal{A}_1,\mathcal{M}$ are linear operators in $\mathbf{H}$, f and g_k are adapted $\mathbf{H}$-valued processes, $u_0\in\mathbf{H}$, W^H is a fractional Brownian Motion with Hurst parameter $H\in(0,1)$, and θ is a scalar parameter bellowing to an open set $\Theta\subset\mathbb{R}$.

Definition 2. Equation (2) is called diagonalizable if the operators $\mathcal{A}_0,\ \mathcal{A}_1$ and $\mathcal{M}$ have point spectrum and a common system of eigenfunctions $\{h_j,\ j\ge1\}$.

Denote by ρ_k, ν_k, and μ_k the eigenvalues of the operators $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{M}$:

$$\mathcal{A}_0h_k=\rho_kh_k,\quad \mathcal{A}_1h_k=\nu_kh_k,\quad \mathcal{M}h_k=\mu_kh_k,\ \ k\ge1, \tag{3}$$

and also denote by $\alpha_k(\theta):=\rho_k+\theta\nu_k, k\ge1$, the eigenvalues of operator $\mathcal{A}_0+\theta\mathcal{A}_1$. Without loss of generality we assume that the operator Λ has the same eigenfunctions as operators $\mathcal{A}_0,\ \mathcal{A}_1,\ \mathcal{M}$: $\Lambda h_k=\lambda_kh_k,\ k\ge1$.

Definition 3. The equation (2) is called *parabolic* in the triple $(\mathbf{H}^{\gamma+m},\mathbf{H}^\gamma,\mathbf{H}^{\gamma-m})$, for some positive m and real γ, if there exists positive real numbers $\delta,\ C_1$ and a real number C_2 such that, for all $k\ge1$ and all $\theta\in\Theta$,

$$\lambda_k^{-2m}|\rho_k+\theta\nu_k|\le C_1; \tag{4}$$

$$2(\rho_k+\theta\nu_k)+\mu_k^2+\delta\lambda_k^{2m}\le C_2. \tag{5}$$

This definition is equivalent to the classical definition of parabolic equations, but written in terms of eigenvalues of corresponding operators.

Theorem 4. *Assume that equation* (2) *is diagonalizable and parabolic in the triple* $(\mathbf{H}^{\gamma+m},\mathbf{H}^\gamma,\mathbf{H}^{\gamma-m})$*, the initial conditions* u_0 *is deterministic and belongs to* $\mathbf{H}^\gamma$*, the process* $f=f(t)$ *is* $\mathcal{F}_t$*-adapted with* *values in* $\mathbf{H}^{\gamma-m}$*, and* $\mathbb{E}\int_0^T \|f(t)\|_{\gamma-m}^2 dt < \infty$*, the process* $g = g(t)$ *is* $\mathcal{F}_t$*-adapted with values in* $\mathbf{H}^\gamma$ *and* $\mathbb{E}\int_0^T \|g(t)\|_\gamma^2 \, dt < \infty$*. Then the process* u *defined by*

$$u(t) = \sum_{k\geq 1} u_k(t) h_k, \tag{6}$$

where

$$u_k(t) = u_k(0) \exp\left([\alpha_k(\theta) + f_k(t)]t - \frac{1}{2}[\mu_k + g_k(t)]^2 t^{2H} + [\mu_k + g_k(t)] W^H(t) \right), \tag{7}$$
$$f(t) = \sum_{k\geq 1} f_k(t) h_k, \;\; g(t) = \sum_{k\geq 1} g_k(t) h_k,$$

is an $\mathbf{H}^\gamma$*-valued stochastic process.*

Proof. Since $W^H(t)$ is a Gaussian random variable with zero mean and variance t^{2H}, we have

$$\mathbb{E}|u_k(t)|^2 = u_k^2(0) \exp\left(2(\alpha_k(\theta) + f_k(t))t + (\mu_k + g_k(t))^2 t^{2H}\right).$$

Hence,

$$\mathbb{E}\|u(t)\|_\gamma^2 = \sum_{k\geq 1} \lambda_k^{2\gamma} |u_k(t)|^2 \leq C \sum_{k\geq 1} \exp\left(2\alpha_k(\theta)t + \mu_k^2 t^{2H}\right).$$

By parabolicity condition (5), the last series converges uniformly in t, and the theorem follows. □

The functions u_k formally represent the Fourier coefficients of the solution of equation (2) with respect to the basis $\{h_k\}_{k\geq 1}$ and the uniqueness of u follows. Since the equation is diagonalizable, naturally we conclude that formally u_k has the following dynamics

$$du_k(t) = (\theta\nu_k + \rho_k) u_k(t) dt + \mu_k u_k(t) dW^H(t), \quad k \geq 1, t \geq 0. \tag{8}$$

Specifying the stochastic integration in (2) is equivalent to specifying in what sense we understand the integration with respect to fractional Brownian Motion for the Fourier coefficients (8). Consequently, since the equation has constant coefficients, specifying the solution of (8) is the same as to stipulate the sense of stochastic integration in (8). If the integration is understood in Wick sense then $u_k, \; k \geq 1$, defined in (7) is the unique solution of equation (8) for all $H \in (0,1)$ (see for instance [4], Theorem 6.3.1). All results stated here are easily transferable to any other form of integration, by caring out the relationship between different form of integration and consequently adjusting the form of the solution of equation (8) (for comparison of various form of integration with respect to fBM see [4], Chapter 6). Our choice was just to have a unified theory and same formulas for all $H \in (0,1)$.

Definition 5. The process u constructed in Theorem 4 is called the solution of equation (2).

It should be mentioned that the above result, with some obvious adjustments, also holds true for diagonalizable equations driven by several independent fractional Brownian Motions, even with different Hurst parameters.

2.2. **Parameter estimation for geometrical fractional Brownian motion.** In this section we will present some auxiliary results about parameter estimation for one dimensional diffusion processes driven by multiplicative fractional noise. For similar results for equations with additive noise see for instance Kleptsyna and Le Breton [19], Tudor and Viens [32], or Mishura [28], Chapter 6. The results essentially follow from Girsanov type theorem for diffusions driven by fractional Brownian motion.

Let Γ and B denote the Euler Gamma-functions. Following Mishura [28] we introduce the following notations

$$
\begin{aligned}
C_H &= \left(\frac{\Gamma(3-2H)}{2H\Gamma(\frac{3}{2}-H)^3\Gamma(\frac{1}{2}+H)}\right)^{\frac{1}{2}}, \\
l_H(t,s) &= C_H s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\mathbb{I}_{0<s<t}, \qquad (9)\\
M_t^H &:= \int_0^t l_H(t,s)dW_s^H, \, ,
\end{aligned}
$$

where $H\in(0,1)$, and the integration with respect to fractional Brownian Motion is understood in Wiener sense (for more details see [28], Chapter 1). The process M_t^H is a martingale, also called the fundamental martingale associated with fractional Brownian motion W_t^H (see for instance [29] or [28], Theorem 1.8.1). M_t^H has quadratic characteristic $\langle M^H\rangle_t = t^{2-2H}$, and by Lévy theorem, there exists a Wiener process $\{B_t, t\geq 0\}$ on the same probability space such that

$$
M_t^H = (2-2H)^{\frac{1}{2}}\int_0^t s^{\frac{1}{2}-H}dB_s.
$$

Moreover, $\sigma(W_s^H, 0\leq s\leq t) = \sigma(B_s, 0\leq s\leq t)$.

Let us consider the stochastic process of the form

$$
X_t = X_0\exp\left(\theta t - \frac{1}{2}\sigma^2 t^{2H} + \sigma W^H(t)\right), \quad t\geq 1,
$$

which can be called the Geometric Fractional Brownian Motion, and as mentioned in the previous subsection it is the unique solution of the stochastic equation

$$
dX_t = \theta X_t dt + \sigma X_t dW_t^H\,, \; X_0 = x_0, \; t\in[0,T].
$$

Let $Y_t := \ln X_t/X_0 = \theta t - \frac{\sigma^2 t^{2H}}{2} + \sigma W_t^H$, and consider the process $\widetilde{Y}_t := \int_0^t l_H(t,s)dY_s$. Note that observing one path of the process $\{Y_s, \; 0\leq s\leq t\}$ implies that the one path of process $\{\widetilde{Y}_s, \; 0\leq s\leq t\}$ is observable too. By (9) we have

$$
\widetilde{Y}_t = \sigma M_t^H + \theta b_1 t^{2-2H} - \sigma^2 H b_2 t, \quad t>0, \qquad (10)
$$

where $b_1 = C_H\mathrm{B}(3/2-H, 3/2-H)$, $b_2 = C_H\mathrm{B}(1/2+H, 3/2-H)$.

For a fixed parameter $\theta\in\Theta$, let us denote by $\mathbb{P}_\theta$ the distribution of the process $\widetilde{Y}_t$ and by $\mathbb{P}_0$ the distribution of the process $\widetilde{Y}_t^0 := \sigma M_t^H = \sigma b_0\int_0^t s^{1/2-2H}dB_s$. The measure $\mathbb{P}_\theta$ is absolutely continuous with respect to $\mathbb{P}_0$ and the Radon-Nikodym derivative, or the likelihood ratio, has the following form (see for instance [23], Theorem 7.19 or apply classical Girsanov Theorem for martingales)

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0}(\widetilde{Y}_t) = \exp\left(-\int_0^t \frac{\theta(2-2H)b_1 s^{1-2H} - \sigma^2 H b_2}{\sigma^2 b_0^2 s^{1-2H}}\, d\widetilde{Y}_s + \frac{1}{2}\int_0^t \frac{[\theta(2-2H)b_1 s^{1-2H} - \sigma^2 H b_2]^2}{\sigma^2 b_0^2 s^{1-2H}}\, ds \right).$$

The MLE is obtained by maximizing the log-likelihood ratio with respect to θ. Since

$$\frac{\partial}{\partial\theta} \ln \frac{d\mathbb{P}_\theta}{d\mathbb{P}_0}(\widetilde{Y}_t) = -\frac{(2-2H)b_1}{\sigma^2 b_0}\widetilde{Y}_t + \theta\frac{(2-2H)b_1^2 t^{2-2H}}{\sigma^2 b_0^2} - \frac{(2-2H)b_1 H b_2 t}{b_0^2},$$

the MLE for parameter θ has the form

$$\widehat{\theta}_t = \frac{\widetilde{Y}_t}{b_1 t^{2-2H}} + \frac{\sigma^2 H b_2}{b_1 t^{1-2H}}. \tag{11}$$

Proposition 1. *The estimate $\widehat{\theta}_t,\ t>0$, is an unbiased estimate for parameter θ_0; $\lim_{t\to\infty}\widehat{\theta}_t = \theta_0$ with probability one, i.e. $\widehat{\theta}_t$ is a strong consistent estimate of θ_0; $t^{1-H}(\widehat{\theta}_t - \theta_0)$ converges in distribution to a Gaussian random variable with zero mean and variance σ^2/b_1^2.*

Proof. Using the definition of the process $\widetilde{Y}_t$, we represent the estimate $\widehat{\theta}_t$ as follows

$$\widehat{\theta}_t = \theta_0 + \frac{\sigma M_t^H}{b_1 t^{2-2H}}, \tag{12}$$

where θ_0 is the true parameter.

The unbiasedness and asymptotic normality follows immediately from (12) and the fact that M_t^H is a Gaussian random variables with zero mean and variance t^{2-2H}. Since M_t^H is a square integrable martingale with unbounded quadratic characteristic $t^{2-2H}\to\infty$, as $t\to\infty$ a.s., by Law of Large Numbers for Martingales [22], Theorem 2.6.10, $M_t^H/\langle M^H\rangle_t \to 0$ a.s., and hence consistency follows. ∎

Note that, in particular, for $H=1/2$ we have $b_1=b_2=1$, and we recover the classical estimate for the drift coefficient of geometric Brownian Motion

$$\widehat{\theta}_t = \frac{Y_t}{t} + \frac{\sigma^2}{2} = \frac{\ln(X_t/X_0)}{t} + \frac{\sigma^2}{2} = \theta_0 + \frac{\sigma W_t}{t}, \quad t>0,$$

and its corresponding asymptotic behavior.

3. Maximum Likelihood Estimator for SPDEs

Consider the diagonalizable equation

$$du(t) = (\mathcal{A}_0 + \mathcal{A}_1)u(t)dt + \mathcal{M}u(t)dW^H(t), \tag{13}$$

with solution $u(t) = \sum_{k\ge1} u_k(t)h_k$ given by (7). As mentioned in Introduction, if u is observable, then all its Fourier coefficients u_k can be computed. Thus, we assume that the processes $u_1(t),\dots,u_N(t)$ can be observed for all $t\in[0,T]$ and the problem is to estimate the parameter θ based on this observations. Also, we assume that the Hurst parameter $H\in(0,1)$ is known for now.

By Definition 5 of the solution of equation (13) the Fourier coefficients $u_k,\ k\in\mathbb{N}$, have the following dynamics

$$du_k(t) = \alpha_k(\theta)u_k(t)dt + \mu_k u_k(t)dW^H(t), \quad t\in[0,T], \tag{14}$$

where $\alpha_k(\theta) = \rho_k + \theta\nu_k,\ k\in\mathbb{N}$.

For every non-zero $u_k(0),\ k \in \mathbb{N}$, denote by $v_k(t) = \ln(u_k(t)/u_k(0))$, and $\widetilde{v}_k(t) = \int_0^t l(t,s)dv_k(s)$, where $l(\cdot,\cdot)$ is defined in (9). By results of Section 2.2 it follows that there exists a Maximum Likelihood Estimate for $\alpha_k(\theta)$ and it has the form

$$\widehat{\alpha_k(\theta)} = \frac{\widetilde{v}_k(t)}{b_1 t^{2-2H}} + \frac{Hb_2\mu_k^2}{b_1 t^{1-2H}}, \quad k \geq 1. \tag{15}$$

Since $\alpha_k(\theta)$ is a strictly monotone function in θ, by invariant principle of MLE under invertible transformations, we can find an MLE for the parameter θ

$$\widehat{\theta}_{k,t} = \frac{\widetilde{v}_k(t)}{\nu_k b_1 t^{2-2H}} + \frac{Hb_2\mu_k^2}{\nu_k b_1 t^{1-2H}} - \frac{\rho_k}{\nu_k}, \quad k \geq 1,\ t \in [0,T]. \tag{16}$$

Using the definition of the process $\widetilde{v}_k$, the estimate $\widehat{\theta}_{k,T}$ can be represented as follows

$$\widehat{\theta}_{k,T} = \theta_0 + \frac{\mu_k M_T^H}{b_1\nu_k T^{2-2H}}, \tag{17}$$

and by similar arguments to the proof of Proposition 1, we have the following result.

Theorem 6. *Assume that equation* (13) *is diagonalizable and parabolic in the triple* $(\mathbf{H}^{\gamma+m}, \mathbf{H}^{\gamma}, \mathbf{H}^{\gamma-m})$ *for some* $\gamma \in \mathbb{R},\ m > 0$ *and* $u_0 \in \mathbf{H}^{\gamma}$. *Then,*

(1) *For every* $k \geq 1$ *and* $T > 0$, $\widehat{\theta}_{k,T}$ *is an unbiased estimator of* θ_0.
(2) *For every fixed* $k \geq 1$, *as* $T \to \infty$, *the estimator* $\widehat{\theta}_{k,T}$ *converges to* θ_0 *with probability one and* $T^{1-H}(\widehat{\theta}_{k,T} - \theta_0)$ *converges in distribution to a Gaussian random variable with zero mean and variance* $\mu_k^2/b_1^2\nu_k^2$.
(3) *If, in addition,*

$$\lim_{k\to\infty} \left|\frac{\mu_k}{\nu_k}\right| = 0, \tag{18}$$

then for every fixed $T > 0$, $\lim_{k\to\infty} \widehat{\theta}_{k,T} = \theta_0$ *with probability one, and* $|\nu_k/\mu_k|(\widehat{\theta}_{k,T} - \theta_0)$ *converges in distribution to a Gaussian random variable with zero mean and variance* T^{2H-2}/b_1^2.

Remark 7. The parabolicity conditions (4)-(5) and MLE consistency condition (18) in general are not connected. In terms of operator's order, parabolicity states that the order of operator $\mathcal{M}$ from the diffusion term is smaller than half of the order of the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ from deterministic part. Condition (18), that guarantees the consistency of MLE as number of Fourier coefficients increases, assumes that the order of operator $\mathcal{M}$ from the diffusion part does not exceed the order of the operator $\mathcal{A}_1$ from deterministic part that contains the parameter of interest θ.

By Theorem 6 it follows that the consistency and asymptotic normality of the estimates $\widehat{\theta}_{k,T}$ can be achieved in two ways: by increasing time T or by increasing the number of Fourier coefficients k. In both cases the quality of the estimate is improved by decreasing its variance.

It is interesting to note that $\mathrm{Var}\left(\widehat{\theta}_{k,T} - \theta_0\right) = \mu_k^2 T^{2H-2}/b_1^2\nu_k^2$ also depends on Hurst parameter H. For $H > 1/2$ the constant $1/b_1$ is close to one, and increases as function of H for $H \in (0, 1/2)$. The function t^{2H-2} increases in H for any $t > 1$. The constants μ_k and $\nu_k,\ k \geq 1$, do not depend on H. Overall, T^{2H-2}/b_1^2 increases in H for any $t > 1$ and thus quality of the estimates is higher for smaller H.

As mentioned before, due to the fact that the probability measures generated by the solution u of the original SPDE are singular, it is possible to estimate θ exactly on any finite interval of time $[0, T]$. A natural question is wether we can improve the quality of the estimates by considering several Fourier coefficients $u_k(t)$. The answer is that by statistical methods used above this is not possible. First, note that the measures associated to any two or more processes u_k are singular, and thus MLE does not exist for such vector-valued functions. In other words, by considering two or more Fourier coefficients u_k, we get a singular model, a fact that will be explored in the next section. Also, since each process u_k is driven by the same noise, each individual Fourier coefficient u_k contains the same amount of information: the sigma-algebra generated by $u_k(t),\ t \in [0, T]$ coincides with the sigma-algebra generated by $W^H(t),\ t \in [0, T]$. However, the speed of convergence of the sequence $\widehat{\theta}_{k,T}$ can be improved by using accelerating convergence technics from numerical analysis. Two methods have been discussed into details in [6]: the weighted average method and Aitken's Δ^2 method. For sake of completeness, we will state here the corresponding results applied to the sequence $\{\widehat{\theta}_{k,T}\}_{k\geq 1}$.

Weighted averaging. Suppose that $\beta_k,\ k \geq 1$, is a sequence of non-negative numbers such that $\sum_{k\geq 1} \beta_k = +\infty$, and consider the weighted averaging estimator

$$\widehat{\theta}_{(N,T)} = \sum_{k=1}^{N} \beta_k \widehat{\theta}_{k,T} \Big/ \sum_{k=1}^{N} \beta_k \quad N \geq 1,\ T > 0\,. \tag{19}$$

Then (a) $\widehat{\theta}_{(N,T)}$ is an unbiased estimator of θ_0 for every $N \geq 1$ and $T > 0$; (b) $\lim\limits_{T\to\infty} \widehat{\theta}_{(N,T)} = \theta_0$ a.e. for every $N \geq 1$ (consistency in T); (c) if in addition the consistency condition (18) is fulfilled, then $\lim\limits_{N\to\infty} \widehat{\theta}_{(N,T)} = \theta_0$ with probability one for every $T > 0$ (consistency in N).

Aitken's Δ^2 method. Define the following sequence of estimates

$$\widetilde{\theta}_k = \widehat{\theta}_{k,T} - \frac{(\widehat{\theta}_{k+1,T} - \widehat{\theta}_{k,T})^2}{\widehat{\theta}_{k+2,T} + 2\widehat{\theta}_{k+1,T} - \widehat{\theta}_{k,T}}. \tag{20}$$

One can show that the new sequence $\widetilde{\theta}_k$ converges to the true parameter θ_0 with probability one. Moreover, if $\mu_k/\nu_k \sim \alpha k^{-\delta}$ for some $\alpha, \delta > 0$, then

$$\frac{\mathbb{E}(\widetilde{\theta}_k - \theta_0)^2}{(\widehat{\theta}_{k,T} - \theta_0)^2} \sim \frac{1}{(1+\delta_1)^2},$$

and if $\mu_k/\nu_k = (-1)^k/k$, then

$$\frac{\mathbb{E}(\widetilde{\theta}_k - \theta_0)^2}{(\widehat{\theta}_{k,T} - \theta_0)^2} \sim \frac{c}{k^2}, \quad c > 0.$$

In both cases, the new sequence $\widetilde{\theta}_k$ converges faster than $\widehat{\theta}_{k,T}$ to θ_0.

The proofs of the above results follows from Theorem 6 and some direct computations, which will be omitted here.

4. Exact Estimates

In regular models the unknown parameter can be found only approximatively, and the consistency is gained either in large sample or small noise regime. For singular models the parameter can be found exactly. For example, if all Fourier coefficients of the solution u of equation (2) are known, according to the results from previous sections, one can find the value of θ_0 exactly, on any interval of time $[0,T]$. The possibility to evaluate θ_0 exactly is based on singularity of the measures generated by u^θ for different values of θ. However, while theoretically it is possible to estimate the true parameter exactly, in practice we (or computer) can perform only a finite number of operations. Recall that the measures associated to an individual Fourier coefficient u_k^θ are regular, while a vector consisting of any two or more Fourier coefficients will produce measures that are singular. In this section we will explore this singularity, and show that in fact the true parameter can be estimated exactly from a finite number of Fourier coefficients. Moreover, the described method allow to find both parameters θ and H, either individually or simultaneously.

Following [6] we say that an estimator is *closed-form exact* if it produces the exact value of the parameter of interest after finite number of additions, substraction, multiplications, and divisions on the elementary functions of the observations.

Closed-form exact estimates exists for the model (2) if we assume that observations are $u_k(t), k\geq 1,\ t\in[0,T]$. For every non-zero Fourier coefficient u_k of the form (14), set $v_k(t)=\ln u_k(t)/u_k(0),\ t\in[0,T]$. Then

$$v_k(t)=(\rho_k+\theta\nu_k)t-\frac{1}{2}\mu_k^2t^{2H}+\mu_kW^H(t)\ . \tag{21}$$

Case 1. θ unknown, H known. Assume that $\nu_k\mu_m\neq\nu_m\mu_k$ for some $k,m\in\mathbb{N}$. Then, taking (21) for these k and m, by direct arithmetic evaluations, one gets the exact estimate of the parameter θ

$$\theta=\frac{\mu_mv_k-\mu_kv_m+(\rho_m\mu_k-\rho_k\mu_m)t+\frac{1}{2}(\mu_k^2\mu_m-\mu_m^2\mu_k)t^{2H}}{t(\nu_k\mu_m-\nu_m\mu_k)}\ , \tag{22}$$

for any $t>0$ and $k,m\in\mathbb{N}$ for which $\nu_k\mu_m\neq\nu_m\mu_k$.

Note that if $\mu_k=\mu_m$ then the above exact estimate does not depend on H, and θ can be evaluated even if H is unknown. This is the case, for example, if $\mathcal{M}$ is the identity operator (see Example 1 below).

Case 2. H unknown, θ known. Assume now that the parameter of interest is the Hurst parameter H and assume that θ is known. By the same arguments as above, one can solve for H the system of two equations generated by (21) for some k and m, and get the following exact estimate for H

$$H=\frac{1}{2\ln t}\ln\left[\frac{(\rho_k+\theta\nu_k)t\mu_m-(\rho_m+\theta\nu_m)t\mu_k-v_k\mu_m+v_m\mu_k}{2\mu_k\mu_m(\mu_k-\mu_m)}\right]\ , \tag{23}$$

for any $t>0$, $k\neq m$, and under assumption that the expression under logarithm is positive and finite.

Case 3. Both θ and H unknown. Denote by $\alpha_{k,m}:=(\nu_k\mu_m-\nu_m\mu_k)t,\ \beta_{k,m}:=1/2(\mu_m^2\mu_k-\mu_k^2\mu_m)$ and $\delta_{k,m}:=v_k\mu_m-v_m\mu_k-\rho_k\mu_mt-\rho_m\mu_kt$. Assume that for some $m,k,i,j\in\mathbb{N}$, $\alpha_{k,m}\beta_{i,j}\neq\alpha_{k,m}\beta_{i,j}$.

Then the following exact estimate for θ holds true

$$\theta = \frac{\delta_{k,m}\beta_{i,j} - \delta_{i,j}\beta_{k,m}}{\alpha_{k,m}\beta_{i,j} - \alpha_{i,j}\beta_{k,m}}\,. \tag{24}$$

If in addition $\delta_{k,m}\alpha_{i,j} \neq \delta_{i,j}\alpha_{k,m}$, then there exists an exact estimate for Hurst parameter H given by

$$H = \frac{1}{2}\log_t \frac{\delta_{k,m}\alpha_{i,j} - \delta_{i,j}\alpha_{k,m}}{\beta_{k,m}\alpha_{i,j} - \beta_{i,j}\alpha_{k,m}}\,. \tag{25}$$

Note that for this case, generally speaking, it is sufficient to know only three Fourier coefficients, i.e. some of the indices k, m, i, j can coincide.

Remark 8.

(a) Applying the above idea, closed-form exact estimates can be obtained for equations driven by several fractional Brownian motions, even with different Hurst parameters. If we assume that the noise is driven by n fBMs, and that one of the parameters θ or H is known, then by considering $n+1$ Fourier coefficients we can eliminate all noises and get a closed-form estimate as a solution, under some non-degeneracy assumptions. Respectively, if both parameters are unknown, then one can estimate them by considering $n+2$ Fourier coefficients.

(b) Note that the construction of the exact estimates assumed only the existence of the solution and did not impose any additional assumptions on the order of the operators $\mathcal{A}_0,\ \mathcal{A}_1,\ \mathcal{M}$, in contrast to MLE estimates where the consistency holds only under additional assumptions on order of corresponding operators.

(c) The MLE $\widehat{\theta}_{k,T}$ depend on the whole trajectory of the Fourier coefficient $u_k(t),\ t \in [0,T]$. All exact estimates depend only on initial and terminal value of u_k's.

5. Examples

We conclude the paper with two practical examples where we explore some of the estimates proposed above.

Example 1. *Stochastic heat equation.* Let θ be a positive number, and consider the following equation

$$du(t,x) = \theta u_{xx}(t,x)dt + u(t,x)dW^H(t), \quad t > 0,\ x \in (0,1), \tag{26}$$

with zero boundary conditions and some nonzero initial value $u(0) \in L_2(0,1)$. In this case the operator $\mathcal{A}_1$ is the Laplace operator on $(0,1)$ with zero boundary conditions that has the eigenfunctions $h_k(x) = \sqrt{2/\pi}\sin(kx),\ k > 0$, and eigenvalues $\nu_k = -k^2,\ \rho_k = 0,\ \mu_k = 1,\ k > 0$. Assume that $u(t,x)$ is known for $x \in [0,1]$ and $t \in [0,T]$, hence $u_k(t) := \int_0^1 h_k(x)u(t,x)dx,\ k \in \mathbb{N}$, is observable. Denote by $v_k(t) := \log(u_k(t)/u_k(0))$ for every $k \in \mathbb{N}$, and $u_k(0) \neq 0$. By Theorem 6, the MLE for θ has the form

$$\widehat{\theta}_k = -\frac{\int_0^T l(T,s)dv_k(s)}{k^2 b_1 T^{2-2H}} - \frac{Hb_2}{k^2 b_1 T^{1-2H}},\ \ k \in \mathbb{N}.$$

The exact estimates (22) for θ are given by

$$\theta = \frac{1}{T(m^2 - k^2)} \ln \frac{u_k(T)u_m(0)}{u_m(T)u_k(0)} ,$$

for any $k \neq m$ and $T > 0$. Note that the exact estimates do not depend on H. However there are no exact-type estimates for H.

Example 2. Assume that G is a bounded domain in $\mathbb{R}^d$, and let Δ be the Laplace operator on G with zero boundary conditions. Then Δ has only point spectrum with countable many eigenvalues, call them $\sigma_k, k \in \mathbb{N}$. Moreover, the set of corresponding eigenvalues forms an orthonormal basis in $L_2(G)$; the eigenvalues can be arranged so that $0 < -\sigma_1 \leq -\sigma_2 \leq \ldots$; the eigenvalues have the asymptotic $\sigma_k \sim k^{2/d}$. In the space $\mathbf{H}^0(G)$ let us consider the following stochastic evolution equation

$$du(t) = [\Delta u(t) + \theta u(t)]dt + (1-\Delta)^r u(t) dW^H(t) \ ,$$

with some nonzero initial values in $\mathbf{H}^0(G)$, and some $r \in \mathbb{R}$. According to our notations we have the operators $\mathcal{A}_0 = \Delta,\ \mathcal{A}_1 = I,\ \mathcal{M} = (1-\Delta)^r$, with corresponding eigenvalues $\nu_k = 1,\ \rho_k = \sigma_k,\ \mu_k = (1+\sigma_k)^r$. The equation is diagonalizable, and by Theorem 4, it has a unique solution in the triple $(\mathbf{H}^1, \mathbf{H}^0, \mathbf{H}^{-1})$ for any $r \leq 1/2$.

The maximum likelihood estimate in this case has the form

$$\widehat{\theta}_{N,t} = \frac{\widetilde{v}_k(t)}{b_1 \sigma_k t^{2-2H}} + \frac{H b_2 (1-\sigma_k)^{2r}}{\sigma_k b_1 t^{1-2H}} - \frac{1}{\sigma_k},\ t > 0,\ k \in \mathbb{N},$$

which is an unbiased estimate of the parameter θ.

(2a) *Large time asymptotics.* $\lim\limits_{t\to\infty} \widehat{\theta}_{k,t} = \theta_0$ a.s. for all $k \geq 1$; $\lim\limits_{t\to\infty} t^{1-H}(\widehat{\theta}_{k,t} - \theta_0) \stackrel{d}{=} \xi$, where $\xi \sim \mathcal{N}(0, (1-\sigma_k)^2/b_1^2)$.

(2b) *Consistency in number of spatial Fourier coefficients.* Assume that $r < 0$. Then $\lim\limits_{k\to\infty} \widehat{\theta}_{k,t} = \theta_0$ a.s., for every $t > 0$, and the sequence $(1-\sigma_k)^{-1}(\widehat{\theta}_{k,t} - \theta_0)$ converges in distribution to a Gaussian random variable with mean zero and variance t^{2H-2}/b_1^2. If $r \in [0, 1/2]$ the solution still exists in the space $\mathbf{H}^0(G)$, while the estimate $\widehat{\theta}_{k,t}$ is not consistent in k.

(2b) *Exact estimates.* Let $v_k(t) = \ln(u_k(t)/u_k(0))$. Assume that Hurst parameter H is known. Then we have the following exact estimate for θ

$$\begin{aligned}\theta =& \frac{(1-\sigma_m)^r v_k - (1-\sigma_k)^r v_m}{t((1-\sigma_m)^r - (1-\sigma_k)^r)} + \frac{\sigma_m (1-\sigma_k)^r - \sigma_k (1-\sigma_m)^r}{(1-\sigma_m)^r - (1-\sigma_k)^r} \\ &+ \frac{t^{2H-1}}{2} \frac{(1-\sigma_k)^{2r}(1-\sigma_m)^r - (1-\sigma_m)^{2r}(1-\sigma_k)^r}{(1-\sigma_m)^r - (1-\sigma_k)^r},\end{aligned}$$

for any $k \neq m$ and $t > 0$.

If θ is known, then the Hurst parameter H can be found by

$$H = \frac{1}{2} \log_t \frac{[(\sigma_k + \theta)(1-\sigma_m)^r - (\sigma_m + \theta)(1-\sigma_k)^r]t - v_k (1-\sigma_k)^r + v_m (1-\sigma_m)^r}{2(1-\sigma_k)^{2r}(1-\sigma_m)^r - (1-\sigma_m)^{2r}(1-\sigma_k)^r},$$

for any $k \neq m,\ t > 0$.

Finally one can write the exact estimates (24) and (25) for the case when both parameters θ and H are unknown. Note that the exact estimates exists for all r as long as the solution exists (maybe in a larger space) and the Fourier coefficients $u_k(t)$ are computable.

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