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MOBILITY AND STRESS ANALYSIS OF HIGHLY SYMMETRIC GENERALIZED BAR-AND-JOINT STRUCTURES

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Abstract. This paper discusses the possibility of detecting mechanisms with second-order stiffness (resistance to the excitation of an infinitesimal mechanism) imposed by self-stresses in highly symmetric structures. Coupled application of symmetry adapted first-order matrix analysis and a second-order stiffness analysis is performed, then the symmetry adapted form of that second-order analysis is presented, specifying conditions under which the stiffening effect of multiple states of self-stress can be analyzed. Finally, a generalized bar-and-joint model containing new kinematic scalar constraints and variables is proposed, with respect to their applicability in symmetry adapted and second-order analyses. The results are illustrated on structural models of viruses in biology with icosahedral symmetry.

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1. **Introduction**

Living systems in nature and engineering structures - consequently, their mechanical models as well - often show certain symmetry: it is enough to mention flowers, leaves, micro-organisms and architectonical solutions. These models, for having some degree of kinematical and statical indeterminacy, are usually highly indeterminate due to a high order of symmetry. Inclusion of symmetry properties in the computation can therefore be useful for two reasons: on the one hand, a given problem can often be reduced to a simpler one with less computational work by symmetry considerations, and on the other hand, clear description and physical interpretation of a multi-parameter system of self-stresses and displacements can hardly be made without using symmetry.

Since the first aim of this paper is to present a tool only for the detection and categorization of these mechanisms and states of self-stress, all further arguments and examples are based on the assumption of a perfectly rigid material behavior.

In the starting sections, a short review of existing analytical methods and a theoretical introduction of new ones are presented for classical bar-and-joint structures, even if this simple model is not always applicable (or practical) for many structures. Our second aim is therefore to extend these analytical methods to generalized models containing various kinematic constraints instead of the only classical constant bar length, as well as to set up basic conditions for the type of extension that can match the original symmetry adapted techniques.

2. First-order calculations in symmetry adapted coordinate systems

If a compatibility or equilibrium matrix of a structure is given in an arbitrary coordinate system, it is a very simple task to determine the number of independent infinitesimal mechanisms and states of self-stress, since it depends only on the rank and dimensions of the matrix in case. Difficulties arise, however, when an attempt is made to characterize these mechanisms given on a general basis, knowing that all their linear combinations constitute another infinitesimal mechanism. A possible solution to this problem can be the choice of a special, symmetry adapted basis in which the compatibility matrix **C** (consequently, equilibrium and stiffness matrices **G** and **R** as well) appears in a block-diagonal form according to symmetry properties.

The method of obtaining these bases was developed by Kangwai and Guest [1,2] and it is built upon the foundations of group representation theory. In order to justify some of the later arguments, it is necessary to reassume the essential definitions, theorems and notations in this field.

Connection to group theory comes from the fact that symmetry can be interpreted as a set of symmetry operations like rotation or reflection, etc, applied to a geometrical object. It is a trivial statement that there always exists the identity operation and if two of all existing operations are done successively, the resultant operation is always found to be equivalent to one operation of the original set. For any operation there must also exist an inverse operation such that its application to the original one results in the identity operation. Fulfilling these conditions, symmetry operations of an object constitutes a group, and it is possible to assemble the full multiplication table of all operations of the group [3].

Operations in general have different representations among which the most common one is the so-called matrix representation: an operation is represented by a matrix multiplication that can express in the most natural way, for example, a coordinate-transformation in 3D space. It is very important, however, that any set of square matrices that obeys the multiplication table forms a matrix representation of the group. Among the infinite number of representations, there is, for example, a natural representation of the geometrical object that expresses transformations of all specific coordinates (in bar-and-joint structures, these are nodal coordinates). This representation is called external, in contrast to a similar possible representation that concerns transformations among internal forces and internal deformations belonging to the respective constraints. For illustration, let us consider a simple planar structure in Figure 1 with a classical - and instructive - C_{3v} symmetry. In this symmetry group, there are six operations: two rotations (by 120^o and 240^o, denoted as C_3 and C_3^2), three reflections (σ_v) and the identity (E). Concentrating on a simple operation

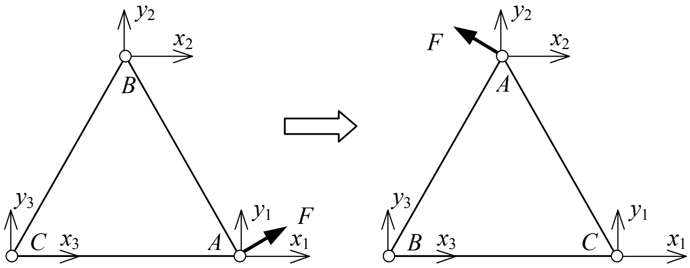

Figure 1. A structure with C_{3v} symmetry – rotation by 120^o counter-clockwise

(counter-clockwise rotation by 120 degrees), one can observe what happens, for instance, to force F acting at node A. Looking only at the direction, new coordinates can be obtained from a multiplication by the matrix of rotation by 120^o as follows:

$$\left[\begin{array}{c} x_{new} \\ y_{new} \end{array} \right] = \left[\begin{array}{cc} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{array} \right] \left[\begin{array}{c} x_{old} \\ y_{old} \end{array} \right] = \mathbf{M}_{C3} \left[\begin{array}{c} x_{old} \\ y_{old} \end{array} \right] . \tag{2.1}$$

Transformation matrices like in (2.1) can also be generated for all other operations in C_{3v} - these form a 2-by-2 representation of the group C_{3v} in general. Taking into account that a rotation of the object also shifts the nodes, equation (2.1) needs correction:

$$\left[\begin{array}{c} x_2^{new} \\ y_2^{new} \end{array} \right] = \mathbf{M}_{C3} \left[\begin{array}{c} x_1^{old} \\ y_1^{old} \end{array} \right] , \tag{2.2}$$

and since all the three nodes move, rotation of the whole object is described by

$$\left[\begin{array}{c} x_1^{new} \\ y_1^{new} \\ \vdots \\ y_3^{new} \end{array} \right] = \left[\begin{array}{ccc} 0 & 0 & \mathbf{M}_{C3} \\ \mathbf{M}_{C3} & 0 & 0 \\ 0 & \mathbf{M}_{C3} & 0 \end{array} \right] \left[\begin{array}{c} x_1^{old} \\ y_1^{old} \\ \vdots \\ y_3^{old} \end{array} \right] , \tag{2.3}$$

where the 6-by-6 matrix is an external representation of the object under C_3 operation.

If internal forces and deformations are considered, the same rotation moves bar b_1 into bar b_2 etc., hence

$$\left[\begin{array}{c} b_1^{new} \\ b_2^{new} \\ b_3^{new} \end{array} \right] = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] \left[\begin{array}{c} b_1^{old} \\ b_2^{old} \\ b_3^{old} \end{array} \right] \tag{2.4}$$

and the 3-by-3 matrix is an internal representation of the object under the same operation. Note that general symbol b can refer to bar forces or elongations as well, while x_i, y_i may equally denote nodal force or displacement coordinates.

Matrix representations - as it happens to quadratic matrices - can undergo unitary transformations that generate another representation of the group. Some of them have all matrices in block-diagonal form but usually it is impossible to diagonalise all matrices of a representation with the same transformation. A set of blocks that cannot be split into smaller blocks is called 'irreducible representation' but since a set of n-by-n matrices can be operated on by further unitary transformations, the number of irreducible representations is still infinite. Within this infinite set, there can be chosen only a few representations that cannot be transformed into each other by a unitary transformation. Their name is 'non-equivalent irreducible representation' [3] (note that the matrix forms of these few representations still depend on the vector basis, therefore it is not uniquely defined unless the dimension of matrices is 1 by 1).

From the character tables for group theory [4], the number and matrix dimensions of non-equivalent irreducible representations can be read, but beyond that, the table itself gives the characters (traces of matrices) of each representation row by row: in spite of the form of matrices, these are uniquely defined since left unchanged by unitary transformations. For example, in group C_{3v} there are three non-equivalent irreducible representations, two of them are 1-dimensional and denoted by A_1 and A_2, the third one (E) is 2-dimensional (dimension numbers of a representation are always equal to the trace under identity in that representation). Since in C_{3v} there are 6 symmetry operations, it means 6 matrices and therefore 6 character values, but some of the operations (those belonging to the same *class*) have regularly the same character that is given in a single column. In our example there are three classes: identity belongs to the first one, while two rotations and three reflections compose the second and third ones, respectively. Thus, the character table for C_{3v} is as shown in Table 1:

	E	$2C_3$	$3\sigma_v$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

Table 1. Character table of group C_{3v}

There exists a method for generating also matrix forms of multi-dimensional non-equivalent irreducible representations that are needed for further symmetry-analysis. It is shown in [1] that using the Great Orthogonality Theorem of group theory, it is possible to construct quadratic matrices that transform compatibility and rigidity matrices into a block-diagonal form, once a full set of non-equivalent irreducible representations of the group and an arbitrary internal and external representation of the object are given.

For instance, block-diagonalisation of a compatibility matrix can be written in the form

$$\mathbf{C}^S = \mathbf{V}_f^T \cdot \mathbf{C} \cdot \mathbf{V}_p, \tag{2.5}$$

where superscript S means symmetry adapted form, whilst $\mathbf{V}_f$ and $\mathbf{V}_p$ are orthogonal transformation matrices of internal and external quantities with subscripts f and p referring to internal bar force and external nodal load. The block structure of $\mathbf{C}^S$ is determined by the number and dimension of non-equivalent irreducible representations: each representation means as many blocks as the dimension of its matrices. For illustration, $\mathbf{C}^S$ of a structure with $\mathbf{C}_{3v}$ symmetry will assume the form shown in Figure 2:

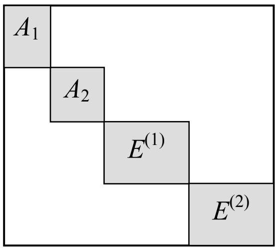

Figure 2. Block structure of compatibility matrix in symmetry adapted form with C_{3v} symmetry

In this symmetry adapted form it is possible to perform an independent analysis of each block that means practically a singular value decomposition: this is because any state of self-stress given by the left nullspace or mechanisms coming from the right nullspace of the matrix is within an invariant subspace associated with one of the blocks.

In addition to the reduced matrix calculations, mechanisms and self-stresses belonging to a given block display well-defined symmetry properties: for example, a mechanism or state of self-stress found in a block A must be left unchanged by any of the symmetry operations, which is why they are said to have full symmetry. Mechanisms and states of self-stress categorized by symmetry provide then a system where physical interpretation of mobility or possibilities of pre-stressing turns out to be more straightforward.

3. **Problems of higher-order rigidity**

Singular value decomposition accounts only for the existence of states of self-stress and infinitesimal mechanisms. This latter category, however, covers now three different cases. An infinitesimal mechanism can be a [5]

a) first-order infinitesimal mechanism with additional stiffness provided by pre-stressing,
b) first-order infinitesimal mechanism without additional stiffness or higher-order infinitesimal mechanism that can never be stiffened by pre-stressing,
c) finite mechanism, always without additional stiffness.

We notice that an infinitesimal mechanism is of n-th order if there is at least one bar for which in the expansion of the Taylor-series of its elongation, the order of the first non-vanishing term is $n+1$: a typical example for (a) is a linkage supported at two endpoints, with all nodes lying along a straight line; a first-order infinitesimal mechanism pertaining to case (b) is presented in [6]. In accordance with mechanisms, we define second-order stiffness: when there is a state of self-stress, and an infinitesimal mechanism is activated, unbalanced nodal forces appear. If the virtual work done by this force system on the mechanism is positive, then the self-stress is said to be able to impart second-order stiffness to the mechanism; if it holds for all possible mechanisms, then the whole structure has second-order stiffness.

Interestingly enough, some of the finite mechanisms can be detected even with first-order symmetry-analysis: if a block with full symmetry contains one or more mechanisms but no self-stress, then all displacements in the subspace spanned by vectors of these mechanisms must be finite [7]. Even if this procedure works only for full symmetry, it is possible to find an appropriate group for any symmetric mechanism where the respective mechanism is fully symmetric, and symmetry adapted diagonalization based on this group can also be performed. It is impossible to make a decision in this way about finiteness, however, when at least one fully symmetric state of self-stress appears.

Another approach to the question of rigidity leads to analyses of existence of additional stiffness: once it is proven, finiteness of the motion is ruled out. A method developed for this purpose by Pellegrino and Calladine [8] uses the concept of 'product force', defined as a nodal resultant of forces in adjacent bars when a single-parameter state of self-stress and a mechanism is activated to the structure. For example, a collinear linkage supported at two endpoints has a uniform tensional self-stress; moving internal nodes infinitesimally off the axis will induce also product force $\mathbf{F}_1$ shown in Figure 3.

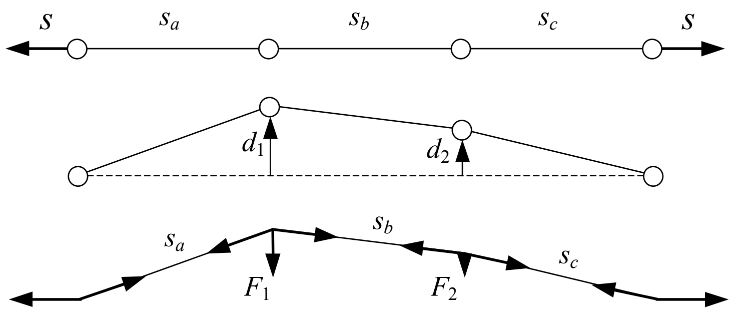

Figure 3. State of self-stress, mechanism and product forces

Magnitude of F_1 can be computed, based on the assumption of small displacements, as

$$F_1 = -d_1 \frac{s_a}{l_a} - d_1 \frac{s_b}{l_b} + d_2 \frac{s_b}{l_b}. \tag{3.1}$$

It can be proved in the same way that a general formula for product force at point P_i is

$$F_{ir} = -d_{ir} \sum_j \frac{s_{ij}}{l_{ij}} + \sum_j d_{jr} \frac{s_{ij}}{l_{ij}}, \tag{3.2}$$

where $r^o = {}^o x$, ${}^o y$, ${}^o z$, d_i denotes displacement components at point P_i, while j runs over all adjacent nodes; s_{ij} and l_{ij} are bar forces and lengths, respectively.

Consider a structure with n nodes and b bars. Let $\mathbf{S}$ denote a diagonal matrix of self-stresses containing b diagonal blocks of dimension dim equal to that of the Euclidean space that the structure is defined in: i-th block $\mathbf{S}^{(i)}$ can be obtained as

$$\mathbf{S}^{(i)}_{(dim \times dim)} = \frac{s_i}{l_i} \mathbf{E}_{(dim \times dim)}. \tag{3.3}$$

Now a $(dim \times b)$-by-$(dim \times n)$ matrix $\mathbf{T}$ can be constructed: let a block t_{ij} be -$\mathbf{E}_{(dim \times dim)}$ or $\mathbf{E}_{(dim \times dim)}$ if P_j is starting or endpoint of the i-th bar, respectively. With the help of $\mathbf{T}$ and $\mathbf{S}$, the complementary stiffness matrix ($\mathbf{Q}$) of the given structure can be defined:

$$\mathbf{Q} = \mathbf{T}^T \cdot \mathbf{S} \cdot \mathbf{T} \ , \tag{3.4}$$

and one can verify that if $\mathbf{d}$ is the vector of a mechanism, the product $\mathbf{d}^T\mathbf{Q}$ gives exactly the coordinates of all product forces.

Existence of stiffening by pre-stressing is tested by the following criterion: if the external work done by a displacement on the set of product forces generated by the same displacement vector is zero, no stiffening effect exists [9]. In terms of linear algebra, it means a quadratic form of zero value:

$$w = \mathbf{d}^T\mathbf{Q}\mathbf{d} = 0 \ . \tag{3.5}$$

The method is applicable also with k mechanisms: if these n column vectors are collected in a matrix $\mathbf{D}$, then the product

$$\mathbf{d} = \mathbf{D}\mathbf{a}, \tag{3.6}$$

gives their general linear combination ($\mathbf{a}^T = [\alpha_1^0 \dots {}^o\alpha_k]$). For this particular mechanism, the condition of zero external work (regarding that $(\mathbf{D}\mathbf{a})^T = \mathbf{a}^T\mathbf{D}^T$) yields

$$\mathbf{a}^T\mathbf{D}^T\mathbf{Q}\mathbf{D}\mathbf{a} = 0 \ . \tag{3.7}$$

Term $\mathbf{D}^T\mathbf{Q}\mathbf{D}$ is a symmetric n-by-n matrix called reduced complementary stiffness matrix [10] and denoted by $\mathbf{W}$, thus the left-hand side of (3.7) can again be written as a quadratic form $\mathbf{a}^T\mathbf{W}\mathbf{a}$. If coefficients in $\mathbf{a}$ are considered to be variables, the analysis can be extended to all possible mechanisms. In this case, there is additional stiffness for all mechanisms if and only if matrix $\mathbf{W}$ is definite.

Nevertheless, there is still an open question: what happens when self-stresses are multiple? For special two-parameter states of self-stress there can be found particular solutions in [9] but the problem is more complex when a structure has several states of self-stress with different symmetry properties.

4. Symmetry adapted second-order rigidity analysis

As shown, product force analysis does not require necessarily a symmetry adapted treatment but in some cases one can make use of it. In this section, a new symmetry adapted higher-order analysis will be described, pointing out some advantages and restrictions of its application.

To avoid confusion, matrices $\mathbf{S}$, $\mathbf{Q}$ and $\mathbf{W}$ will be indexed by serial numbers of independent states of self-stress found in the first-order analysis. Suppose that $\mathbf{S}_1$ is coming from the fully symmetric block: it is easy to see now that $\mathbf{Q}_1$ expresses full symmetry as well, consequently it can be block-diagonalised with the formula

$$\mathbf{Q}_1^S = \mathbf{V}_p^T \cdot \mathbf{Q}_1 \cdot \mathbf{V}_p \ , \tag{4.1}$$

while the transformation formula of displacement vectors $\mathbf{d}_i$ into symmetry adapted system is

$$\mathbf{d}_i^S = \mathbf{V}_p^T \cdot \mathbf{d}_i \ . \tag{4.2}$$

It is possible then to perform the described matrix analysis in symmetry adapted system where displacement vectors have all zero values except for those being in the block of the representation where the displacement was found: in a fully symmetric displacement vector, for instance, only the first few entries are nonzero.

Consider now a matrix $\mathbf{D}^S$ containing vectors that belong to different blocks. The structure of reduced complementary stiffness matrix can be illustrated by the scheme in Figure 4:

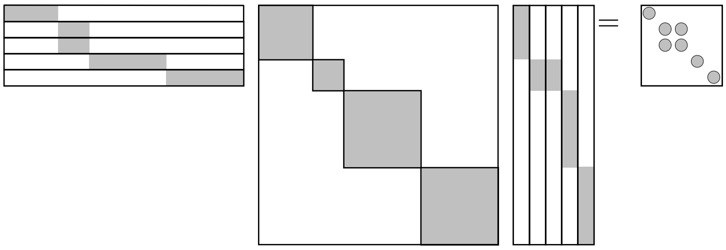

Figure 4. Block structure of product $(\mathbf{D}^S)^T \cdot \mathbf{Q}^S \cdot \mathbf{D}^S = \mathbf{W}^S$

An important conclusion may be drawn at this stage: having a fully symmetric state of self-stress, a reduced complementary stiffness matrix is always obtained in block-diagonal form (note that $\mathbf{W}^{S\,o} = {}^o\mathbf{W}$, irrespective of intermediate steps made in the symmetry adapted system), hence a necessary condition for the existence of a global stiffening effect can be formulated as each set of displacements belonging to a given block must be stiffened by self-stress $\mathbf{S}_1$, or in other words, blocks in $\mathbf{W}$ must be definite in themselves.

Altogether, it is a true objection that a simple check for definiteness does not justify such amount of symmetry calculations: real applicability of symmetry adapted analysis is experienced when there are multiple states of self-stress.

Imagine a state of self-stress $\mathbf{S}_k$ with lower symmetry than that of $\mathbf{S}_1$: formula (4.1) now will not diagonalise the original matrix $\mathbf{Q}$ but $\mathbf{Q}^S$ will show considerable regularity. If there is a mechanism $\mathbf{d}_l$ that belongs to a different representation

	A_1	A_2	E
A_1	A_1	A_2	E
A_2		A_1	E
E			$A_1^o + {}^o[A_2]^o + E$

Table 2. Direct product table of group C_{3v}

(therefore, certainly to a different block), the product force should 'mix' properties of the two symmetries. This effect can be read from direct product tables of groups [4] and it says what kind of representations can ever appear in a direct product of two objects belonging to two representations. For illustration, Table 2 gives the direct product table of group C_{3v}.

If, for example, $\mathbf{S}_k$ belongs to $\mathbf{A}_2$ and $\mathbf{d}_l$ to E, representation of the resultant product force system may contain E-blocks only. This property is reflected in the structure of $\mathbf{Q}_k$ in a way that in two block columns pertaining to E there are nonzero blocks only in block rows that belong to E. Matrix $\mathbf{Q}_k$ is therefore still sparse, and it can happen that blocks in the main block diagonal are all empty. This is not simple coincidence, since direct product tables often rule out nonzero blocks in rows and columns pertaining to the same representation (e.g. from Table 2, it follows that a block pertaining to block row and column of A_2 must be empty, since a mechanism that belongs to A_2 cannot generate a product force system belonging to A_2 but A_1). The second reason for frequent appearance of zero diagonal blocks is that in case of representations with multiplicity μ there is a hyperblock of $\mu \times \mu$ blocks pertaining all to the same representation, hence non-empty block(s) can be located off the diagonal.

Finding a matrix $\mathbf{Q}_k$ with empty diagonal blocks, it is easy to see from arguments like in Figure 4 that the main diagonal of $\mathbf{W}$ is empty as well. Since unitary transformations used for the digitalization cannot modify the trace of matrix $\mathbf{W}$, among the eigenvalues of $\mathbf{W}$ there must appear both positive and negative numbers, which is a proof of indefiniteness.

Consider now a set of states of self-stress $(\mathbf{S}_2, {}^o, \ldots, {}^o\mathbf{S}_\nu)$ for which all complementary stiffness matrices in symmetry adapted form $\mathbf{Q}^S{}_2, {}^o \ldots, {}^o\mathbf{Q}^S{}_\nu$ have empty diagonal blocks. In this case any linear combination of matrices $\mathbf{Q}_i^S$ gives a resultant matrix with empty block-diagonal, therefore the respective matrix $\mathbf{W}$ must also be indefinite. In mechanical aspect it means that *any* linear combination of these self-stresses is insufficient to provide additional stiffness to *any* linear combination of independent displacement vectors included in $\mathbf{D}^S$.

5. **Generalized bar-and-joint structures: extension of results**

The higher-order symmetry analysis presented in the previous section uses the supposition of kinematic constraints being constant bar lengths. This section deals with possible extensions of constraint types that fit both symmetry adapted and product force analyses in order that the analysis under Section 4 can also be performed.

Theoretically, symmetry adapted first-order computations are applicable to an arbitrary type of constraints, provided it does not break the symmetry of the whole object, or in other words, if there is an internal matrix representation for the object that gives full account of the topology. Since bar lengths are given by scalars, this representation contained only ones and zeros, but if a definite direction had been associated with the constraints, internal representations should contain minus ones as well (for example, rotation of a straight line segment about a perpendicular bisector seems to do nothing, while the same operation applied to an arrow reverses its direction). From the aspect of nodal coordinates, a generalization is possible by introduction of single vectors. It will be useful when a folded structure is modelled: folding lines are inclined to each other with a given angle that can be prescribed by constant difference (or constant scalar product) of two vectors of fixed length, lying in the direction of folds. Similarly, an angle between a vector and a bar can also be prescribed by a scalar product or vector difference. There is, however, a necessary additional condition that vector lengths must be kept fixed. Numerically it means a constraint of constant vector norm.

An application of generalized constraints and nodes in first-order analysis is presented in [11, 12], here we restrict ourselves only to presenting a single example.

Consider two nodes P_i and P_j connected by a bar. Let this bar be an edge of a rectangular plate that is determined by a vector $\mathbf{v}_k$ in the model: length and direction of $\mathbf{v}_k$ is equal and parallel to the other edge of the modelled plate (Figure 5).

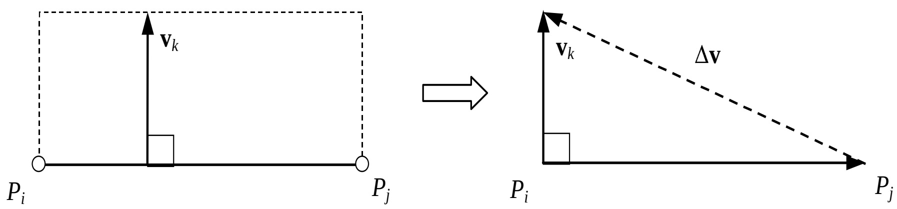

Figure 5. Bar and vector modelling a rectangular plate: difference of the vectors $\mathbf{v}_k$ and P_iP_j

It can be guaranteed by three types of constraint functions:

$$f_b = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2} - l_{ij} \tag{5.1}$$

is for constant bar length P_iP_j,

$$f_v = \sqrt{x_k^2 + y_k^2 + z_k^2} - l_k \tag{5.2}$$

fixes Euclidean norm of $\mathbf{v}_k$ (here and in the sequel, x_k, y_k and z_k are relative vector coordinates, in contrast to absolute coordinates indexed by i and j), while

$$f_d = \sqrt{(x_k - (x_j - x_i))^2 + (y_k - (y_j - y_i))^2 + (z_k - (z_j - z_i))^2} - l_{ij,k} \tag{5.3}$$

expresses the constant difference of vectors P_iP_j and $\mathbf{v}_k$, denoted as $\Delta\mathbf{v}$ in Figure 5. Note that if there is a C_2 axis within the plane of the rectangle that shifts P_i and P_j, representation of f_b and f_v is +1 under this C_2 operation but it is -1 for f_d, otherwise the direction of the difference vector would break the symmetry.

Constraint functions in (5.1-5.2) were all generated in a form of vector difference. This is useful when a product force test is intended to be done. Method of constructing matrix $\mathbf{Q}$ is based now on the same principles as in Section 3: vector norms and differences in constraints of type f_v and f_d generate force-like quantities along the respective directions. Their effect can now be taken into account by assembling matrices $\mathbf{T}$ and $\mathbf{S}$ in the same way as in the case of constraints f_b: a 3-by-3 diagonal block in $\mathbf{T}$, associated with a function f_v and f_d are filled with +1 if columns refer to vector coordinates, while constraint of type f_d will mean an additional diagonal block of plus or minus ones in columns of edge starting or endpoints, respectively.

Note that entries of $\mathbf{T}$ divided by vector norms or difference vector lengths can also be obtained as first-order approximation of second derivatives of the respective constraint function according to the variable in question. This approach to the problem of second-order stiffness is based on a regular second-order analysis of the original compatibility matrix of bar-and-joint structures [13]. For example, if f_d and y_k belong to m-th row and n-th column, respectively, of the compatibility matrix of a structure, entry t_{mn} can be obtained from

$$t_{mn} = l_{ij,k} \frac{\partial^2 f_d}{\partial y_k^2} \approx l_{ij,k} \frac{\partial \frac{y_k - (y_j - y_i)}{l_{ij,k}}}{\partial y_k} = +1. \tag{5.4}$$

An important remark: a similar extension of higher-order symmetry analysis is possible by using other constraint functions instead of vector differences (e.g. a scalar product), but the applicability of a product force test requires an exact statical interpretation of self-stress induced by the respective constraint, which is not always an easy problem to solve.

6. Expandohedra: a numerical example

For better understanding, in this chapter two sample problems will be presented to illustrate practical applications for the theory above. The object of the analysis will be in both cases an assembly with icosahedral symmetry, called expandohedron [14]. Expandohedra are constructed to model the swelling of some viruses, and the denomination refers to a fully symmetric finite expansion.

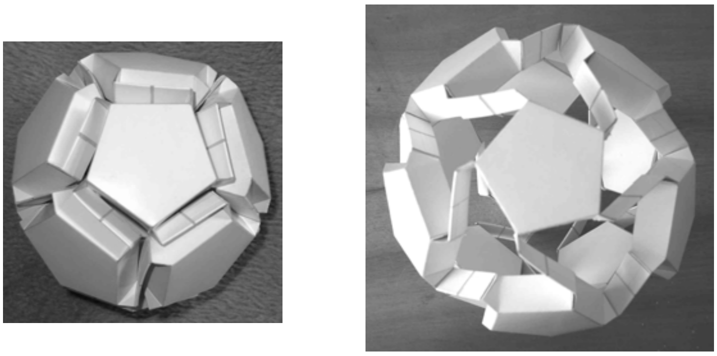

Figure 6. Cardboard model of an icosahedral expandohedron

6.1. **Single-link icosahedral expandohedron.** The assembly consists of rigid pentagonal prisms connected by a triangle-rectangle-triangle folded linkage of C_2 symmetry (connections between rigid elements are all revolute hinges). In the mechanical model, prisms were substituted by determinate bipyramidal bar-and-joint networks built upon the inner pentagonal faces, and new constraints shown in Section 5 were used to reduce matrix dimensions. The physical and mechanical models are sketched in Figure 7, B and K are vertices of a bypiramid lying on its C_5-axis, the constant length of dotted difference vector $\Delta\mathbf{v}$ fixes the constant angle of revolute hinges along the two edges of the triangular plates.

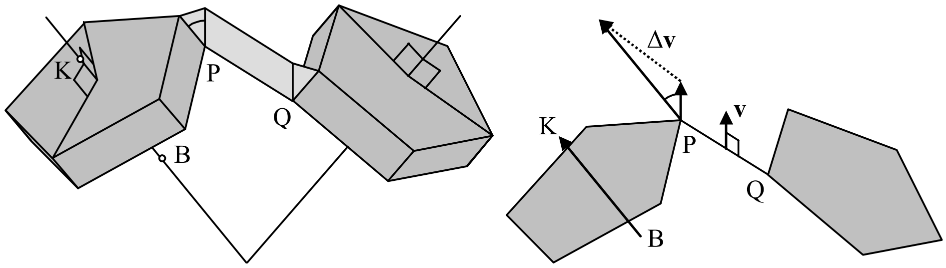

Figure 7. Physical (folded) and mechanical model for a single-link icosahedral expandohedron

Irrespective of the applied numerical model, this expandohedron must have a compatibility matrix with 12 rows less than the number of its columns, which means at least 12 independent displacement systems (six of them are due to rigid body motions). Assuming a general - not fully open - configuration, first-order symmetry adapted analysis showed that there are 9 extra mechanisms with 9 states of self-stress, in the following distribution:

Representation	Number of Blocks	General Configuration: Number of Mechanisms	General Configuration: Number of States of Self-stress
A	1	1	1
T_1	3	3x2 = 6	0
T_2	3	3x1 = 3	3x1 = 3
G	4	0	0
H	5	5x1 = 5	5x1 = 5

Table 3. Mechanisms and states of self-stress of a single-link icosahedral expandohedron

Since there are both a fully symmetric mechanism and a state of self-stress, the finite character of swelling motion cannot be proved by symmetry arguments (nevertheless, there exists a geometrical proof). A simple product force test based on the fully symmetric state of self-stress, however, accounts for the existence of additional stiffness pertaining to all linear combinations of mechanisms except for that containing only the fully symmetric one. In other words: without using serious higher-order symmetry considerations we have proved the existence of exactly one finite (swelling) mechanism, all others can be blocked by self-stresses.

6.2. **Double-link icosahedral expandohedron.** In this model adjacent pentagonal bypiramids are connected by pairs of ball-jointed bars with C_2 symmetry.

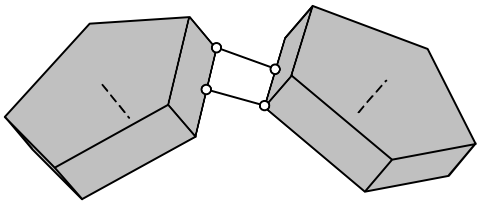

Figure 8. Ball-jointed connection of a double-link icosahedral expandohedron

Apart from rigid body motions, here are obtained six mechanisms again only from counting rows and columns of matrix **C**. The result of first-order analysis is as follows:

Represen-tation	Number of Blocks	General Configuration	
		Number of Mechanisms	Number of States of Self-stress
A	1	1	0
T_1	3	3x1 = 3	0
T_2	3	0	3x1 = 3
G	4	0	0
H	5	5x1 = 5	0

Table 4. Mechanisms and states of self-stress of a double-link icosahedral expandohedron

Lack of fully symmetric self-stress indicates now finite expansion directly. The symmetry adapted form of matrices **Q** based on each state of self-stress has all the empty diagonal blocks, therefore *neither* of the linear combinations of self-stresses can stiffen *any* linear combinations of mechanisms.

7. Conclusions

Symmetry-adapted higher-order mobility and stiffness analysis covers several submethods that were partially developed earlier: the first-order matrix analysis in symmetry adapted coordinate system and the product force test also for asymmetric structures are robust tools for investigation of bar-and-joint structures. It was shown, however, that efficiency can be increased by coupling the two methods: if there exists any, a fully symmetric state of self-stress - which is the most likely to impart second-order stiffness to a structure - can be identified by symmetry analysis. Existence or lack of second-order stiffness, however, can only be decided in a general case for a given state of self-stress by second-order analysis. The symmetry adapted version of this latter method simplifies calculations with fully symmetric states of self-stress and in some cases it accounts for the non-existence of stiffening effect for arbitrary linear combinations of self-stresses with lower symmetry.

A coupled symmetry and second-order stiffness analysis for bar-and-joint structures could be generalized to more complex mechanical models containing free vectors and kinematic constraints formulated by a vector difference norm but it is possible to use another type of scalar constraints once a product force can be defined and symmetry group representations can be found.

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