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VARIATIONAL THEORY FOR 2-DIMENSIONAL FREE SURFACE FLOW: WHY ARE G.L. LIU'S VARIATIONAL PRINCIPLES INCORRECT?

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Abstract. Interests in variational theory of the problem discussed have grown rapidly in recent years, various variational formulae have appeared in literature. But some of the variational principles are wrong. The paper illustrates how to establish variational principles by the semi-inverse method step by step. Comparison with Liu's results reveals that the present technique is much more convenient and reliable. Liu's variational formulation is based on technical, theoretical and conceptual errors, including misrepresentations of the semi-inverse method.

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1. **Introduction**

The basic equations governing 2-D incompressible inviscid rotational flow under gravity can be written in the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{1.1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-g-\frac{1}{\rho}\frac{\partial P}{\partial y}\,, \tag{1.2}$$

$$\frac{1}{2}(u^2+v^2)+gy+\frac{P}{\rho}=B(\Psi)\,, \tag{1.3}$$

where B is the Bernoulli constant, which oes not change along the stream line, the stream function $\Psi =$ is also constant, u and v are velocity components in the x- and y-directions respectively, g is gravitational acceleration, P is pressure.

Difficulty arises when we apply the finite element method to free surface problems. In order to overcome the difficulty, an imaginary plane is introduced [1, Liu, 1995] [2, He, 1998] since the value of the stream function Ψ on the free surface should be given according to the inlet condition. Therefore it will be convenient for us to introduce an imaginary plane $\xi - \Psi$ defined as

$$\xi = x\,, \tag{1.4}$$

$$\psi = \psi(x, y)\,, \tag{1.5}$$

where the stream function ψ takes the form

$$\frac{\partial \psi}{\partial x} = -v, \qquad \frac{\partial \psi}{\partial y} = u\,. \tag{1.6}$$

It is easy to find that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \psi}\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial \xi} - v\frac{\partial}{\partial \psi}$$

and

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \psi}\frac{\partial \Psi}{\partial y} = u\frac{\partial}{\partial \psi}\,.$$

Consequently we have the following basic equations in the imaginary plane:

$$\frac{\partial}{\partial \xi}(\frac{1}{u}) - \frac{\partial}{\partial \psi}(\frac{v}{u}) = 0\,, \tag{1.7}$$

$$\frac{\partial v}{\partial \xi} + \frac{\partial}{\partial \psi}(\Pi) = 0\,, \tag{1.8}$$

$$\Pi + \frac{1}{2}(u^2 + v^2) = B\,, \tag{1.9}$$

where $\Pi = gy + P/\rho$.
Making use of equation (1.8) a general function Ω can be introduced [1, Liu, 1995]

$$\frac{\partial \Omega}{\partial \xi} = \Pi\,, \qquad \frac{\partial \Omega}{\partial \psi} = -v\,. \tag{1.10}$$

Luke [3, 1967] first studied the variational principle for fluids with free surface in a physical plane, and Liu [1, 1995] was the first to deduce variational principles in the imaginary plane. Recently Liu [5, 2001] re-studied the problem by Liu's systematic method [6, 2000], but, unfortunately, the variational principles obtained are proved to be wrong. We re-write two formulae for evaluation. Consider first the variational formulations obtained by Liu:

$$\begin{aligned} J_{Liu1}(\Omega, v, \Pi, u) = & \iint \frac{1}{\sqrt{2(B-\Pi) - v^2}}\left\{v\frac{\partial \Omega}{\partial \Psi} - \frac{\partial \Omega}{\partial \xi} + 2B - \Pi\right\}\mathrm{d}A \\ & - \iint \left\{au^{n-2}(u^2 + v^2 + 2\Pi - 2B) - \frac{2a}{n}u^n\right\}\mathrm{d}A\,, \end{aligned} \tag{1.11}$$

$$\begin{aligned} J_{Liu2}(\Omega, u, v, \Pi) = & \iint \left\{\frac{v}{u}\frac{\partial \Omega}{\partial \Psi} - \frac{1}{u}\frac{\partial \Omega}{\partial \xi} + \frac{u^2 + v^2}{u} + \frac{B}{u}\right\}\mathrm{d}\,A \\ & - \iint \left\{a\Pi^{n-1}(u^2 + v^2 + 2\Pi - 2B) - \frac{2a}{n}\Pi^n\right\}\mathrm{d}\,A\,. \end{aligned} \tag{1.12}$$

It is easy to prove that the above two functionals are wrong. As is pointed out by He [7, 2000], Liu's systematic method [6, 2000] contains a contradiction [8, 2000] leading to very limited validity of this approach [9, He, 2000]. Liu's method might result in incorrect functionals, for example, the variational functional obtained by Liu et al. in (Liu and Wang, [10, 1996]) is incorrect, which is corrected by He's semi-inverse method [11, 2000], [12, 1997].

2. Inverse problem of calculus of variations

In recent years the inverse problem of calculus of variations has brought about a renewed interest in continuum mechanics. It emanates from the powerful applications of the finite element methods (Zienkiewicz and Taylor [13], Liu [14]) and the meshfree particle methods (He [15, 1999]).

In 1997, the present author proposed a powerful tool called the semi-inverse method (He [12, 1997]) to search for various variational principles directly from the field equations and boundary conditions. Applications of the semi-inverse method can be found in the author's previous publications (He [16], [17], [18]).

In 2000, Liu [6] proposed a systematic approach to the derivation of variational principles from partial differential equations. Liu's method consists of two major lines. There are a number of books devoted to the issue of variational principles, e.g. the classical monographs by Chien [19, 1983] and Hildebrand [20, 1965]. The first line of Liu's approach is also discussed by Chien [19, 1983] and Hildebrand [20, 1965] in great detail. The question of determining whether a set of field equations can be derived from a functional may be systematically elucidated by recourse to Veinberg's theorem, which also provides a formula for the computation of the corresponding functional. Therefore the first line of Liu's approach offers nothing new. The application of this line can also be found in the literature (e.g. Meylan 2001). The key contribution of Liu's method lies in the second line, which provides a method for searching for a generalized variational principle directly from field equations. But the application of Liu's method might lead to incorrect results.

Consider the equation system

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{2.1}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0\,. \tag{2.2}$$

By Liu's approach one obtains the following functional [6, 2000a]

$$J(u,v) = \iint \left\{ v(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + u(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \right\} dxdy\,. \tag{2.3}$$

We cannot obtain any Euler equation from the above functional. Consequently, Liu's method has been proved to be incorrect for the above equations.

In view of the semi-inverse method [12, 1997], we can suppose that there exists an unknown functional [7, 2001]

$$J = \iint F dxdy \tag{2.4}$$

under the constraint (2.2). By the Lagrange multiplier method, we have

$$J(u, v, \Psi) = \iint \left\{ F + \Psi(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \right\} dxdy\,, \tag{2.5}$$

where Ψ is a Lagrange multiplier. The stationary conditions for the above functional are as follows

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0\,, \tag{2.6}$$

$$\frac{\delta F}{\delta u} + \frac{\partial \Psi}{\partial y} = 0\,, \tag{2.7}$$

$$\frac{\delta F}{\delta v} - \frac{\partial \Psi}{\partial x} = 0\,. \tag{2.8}$$

Here $\delta F/\delta u$ is called variational derivative of F with respect to u, and is defined as

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}) - \frac{\partial}{\partial y}(\frac{\partial F}{\partial u_y})\,.$$

From equations (2.7) and (2.8), we have

$$\frac{\partial}{\partial x}(\frac{\delta F}{\delta u}) + \frac{\partial}{\partial y}(\frac{\delta F}{\delta v}) = 0\,, \tag{2.9}$$

which should be the field equation $u_x + v_y = 0$. Hence we set

$$\frac{\delta F}{\delta u} = u\,,$$

and

$$\frac{\delta F}{\delta v} = v\,, \tag{2.10}$$

from which we identify the unknown F as follows

$$F = \frac{1}{2}(u^2 + v^2)\,. \tag{2.11}$$

Therefore we obtain the following variational principle

$$J = \iint \frac{1}{2}(u^2 + v^2) dxdy \tag{2.12}$$

and the following generalized variational principle

$$J(u, v, \Psi) = \iint \left\{ \frac{1}{2}(u^2 + v^2) + \Psi(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \right\} dxdy\,. \tag{2.13}$$

The Lagrange multiplier now has a physical meaning, i.e., the stream function. To search for a generalized variational principle, we always begin with an energy-like trial functional with an unknown function F. For example, we can construct a trial functional in the form

$$J(u,v,\Phi)=\iint\left\{u\frac{\partial\Phi}{\partial x}+v\frac{\partial\Phi}{\partial y}+F\right\}dxdy\,, \tag{2.14}$$

where Φ is the potential function for which

$$\partial\Phi/\partial x=u,\qquad \partial\Phi/\partial y=v$$

while F is an unknown function of u, v, and their derivatives.

It is obvious that the stationary condition of the functional (2.14) with respect to Φ results in (2.9). Calculating variation of functional (2.14) with respect to u and v, we have

$$\frac{\partial\Phi}{\partial x}+\frac{\delta F}{\delta u}=0\,, \tag{2.15}$$

$$\frac{\partial\Phi}{\partial y}+\frac{\delta F}{\delta u}=0\,. \tag{2.16}$$

We search for such an F that the above two equations should become

$$\partial\Phi/\partial x=u$$

and

$$\partial\Phi/\partial y=v\,,$$

respectively, so that we can immediately identify F as $F=-(u^2+v^2)/2$.

There exist many alternative approaches to the construction of the trial functionals. Illustrative examples can be found in the author's previous publications.

3. Semi-inverse method and variational principles

We will apply the semi-inverse method (He [12, 1997]) to search for a variational principle for the problem discussed above. The basic idea of the semi-inverse method is to construct a trial functional with an unknown function.

If we want to establish a generalized variational principle with 4 independent variables (u, v, Ω and Π), we can construct a trialfunctional in the form

$$J(u,v,\Omega,\Pi)=\iint\left\{\frac{1}{u}\frac{\partial\Omega}{\partial\xi}-\frac{v}{u}\frac{\partial\Omega}{\partial\psi}+F\right\}d\xi d\psi\,, \tag{3.1}$$

where F is the unknown function to be determined. We call the functional

$$L(u,v,\Omega,\Pi)=\frac{1}{u}\frac{\partial\Omega}{\partial\xi}-\frac{v}{u}\frac{\partial\Omega}{\partial\psi}+F \tag{3.2}$$

trial-Lagrangian.

The advantage of the above trial functional is that the Euler equation with respect to Ω is equation (1.7). Now calculating the variation of equation (3.1) with respect to u, we obtain the following trial-Euler equation

$$\delta u:\qquad -\frac{1}{u^2}\frac{\partial\Omega}{\partial\xi}+\frac{v}{u^2}\frac{\partial\Omega}{\partial\psi}+\frac{\partial F}{\partial u}=0\,. \tag{3.3}$$

In a view of equations (1.10), we have

$$\frac{\partial F}{\partial u} = \frac{1}{u^2}\frac{\partial \Omega}{\partial \xi} - \frac{v}{u^2}\frac{\partial \Omega}{\partial \psi} = \frac{1}{u^2}(\Pi + v^2)\,. \tag{3.4}$$

From equation (3.4), the unknown F can be identified as follows

$$F = -\frac{1}{u}(\Pi + v^2) + F_1\,, \tag{3.5}$$

where F_1 is a newly introduced unknown function, which should be free of the variables u and Ω. Inserting equation (3.5) into equation (3.2), we obtain a renewed trial-Lagrangian, which reads

$$L(u, v, \Omega, \Pi) = \frac{1}{u}\frac{\partial \Omega}{\partial \xi} - \frac{v}{u}\frac{\partial \Omega}{\partial \psi} - \frac{1}{u}(\Pi + v^2) + F_1\,. \tag{3.6}$$

Now the trial-Euler equations for δv and $\delta \Pi$ can be easily obtained

$$\delta v: \qquad -\frac{1}{u}\frac{\partial \Omega}{\partial \psi} - \frac{2v}{u} + \frac{\partial F_1}{\partial v} = 0\,, \tag{3.7}$$

$$\delta \Pi: \qquad -\frac{1}{u} + \frac{\partial F_1}{\partial \Pi} = 0\,. \tag{3.8}$$

By means of the field equations (1.10) and (1.9), we have

$$\frac{\partial F_1}{\partial v} = \frac{1}{u}\frac{\partial \Omega}{\partial \psi} + \frac{2v}{u} = \frac{v}{u} = \frac{v}{\sqrt{2B - v^2 - \Pi}}\,, \tag{3.9}$$

$$\frac{\partial F_1}{\partial \Pi} = \frac{1}{u} = \frac{1}{\sqrt{2B - v^2 - \Pi}}\,. \tag{3.10}$$

From the above relations (3.9) and (3.10), we can immediately identify the unknown F_1, which reads

$$F_1 = -\sqrt{2B - v^2 - \Pi}\,. \tag{3.11}$$

Finally we obtain the following Lagrangian

$$L(u, v, \Omega, \Pi) = \frac{1}{u}\frac{\partial \Omega}{\partial \xi} - \frac{v}{u}\frac{\partial \Omega}{\partial \psi} - \frac{1}{u}(\Pi + v^2) - \sqrt{2B - v^2 - \Pi}\,. \tag{3.12}$$

Liu (1995) obtained a similar Lagrangian, which reads

$$L_{Liu}(u, v, \Omega, \Pi) = \frac{1}{u}\frac{\partial \Omega}{\partial \xi} - \frac{v}{u}\frac{\partial \Omega}{\partial \psi} - \frac{u}{2} - \frac{v\left[2(B - \Pi) - u^2 + v^2\right]}{u\sqrt{2(B - \Pi) - u^2}}\,. \tag{3.13}$$

Supplementing the Lagrangian (3.12) or (3.13) by the field equation (1.9) as a side condition, we obtain a constrained functional

$$\tilde{J}_{Liu1}(\Omega, v, \Pi) = \iint \frac{1}{\sqrt{2(B - \Pi) - v^2}}\left\{v\frac{\partial \Omega}{\partial \Psi} - \frac{\partial \Omega}{\partial \xi} + 2B - \Pi\right\} \mathrm{d}\,A \tag{3.14}$$

and

$$\tilde{J}_{Liu2}(\Omega, u, v) = \iint \left\{\frac{v}{u}\frac{\partial \Omega}{\partial \Psi} - \frac{1}{u}\frac{\partial \Omega}{\partial \xi} + \frac{u^2 + v^2}{u} + \frac{B}{u}\right\} \mathrm{d}\,A\,. \tag{3.15}$$

The above two functionals are under the constrain of equation (1.9). Liu obtained functionals (1.11) and (1.12), respectively, from the above functional (3.14) and (3.15) by eliminating the constraint of equation (1.9) through the so-called Liu's systematic method, which leads to incorrect results hereby. Our approach seems to be much more straightforward and reliable. We can also readily obtain a variational principle with three independent variables. For example, if we want to establish a sub-generalized variational principle with 3 independent variables (u, v, and Ω), a trial-Lagrangian can be constructed as follows

$$L_1(u, v, \Omega) = \frac{1}{u}\frac{\partial \Omega}{\partial \xi} - \frac{v}{u}\frac{\partial \Omega}{\partial \psi} + F \,, \tag{3.16}$$

which is assumed to be under the constraint of equation (1.9).

The trial-Lagrangian (3.16) is similar to equation (3.2). The difference is that the variable Π in equation (3.2) is an independent variable, while it is not involved in equation (3.16). The variation of Π depends upon equation (1.9), i.e.,

$$\delta \Pi = -u\delta u - v\delta v \,.$$

The stationary conditions can be readily obtained:

$$\delta u: \qquad -\frac{1}{u^2}\frac{\partial \Omega}{\partial \xi} + \frac{v}{u^2}\frac{\partial \Omega}{\partial \psi} + \frac{\partial F}{\partial u} = 0 \,, \tag{3.17}$$

$$\delta v: \qquad \frac{1}{u}\frac{\partial \Omega}{\partial \psi} + \frac{\partial F}{\partial v} = 0 \,. \tag{3.18}$$

In view of the field equations, we have

$$\frac{\partial F}{\partial u} = \frac{1}{u^2}\frac{\partial \Omega}{\partial \xi} - \frac{v}{u^2}\frac{\partial \Omega}{\partial \psi} = \frac{1}{u^2}(\Pi + v^2) = \frac{1}{u^2}(B - \frac{1}{2}u^2 + \frac{1}{2}v^2) \,, \tag{3.19}$$

$$\frac{\partial F}{\partial v} = -\frac{v}{u} \,. \tag{3.20}$$

Hence the unknown function F can be identified as follows

$$F = -\frac{B}{u} - \frac{1}{2}u - \frac{v^2}{2u} \,. \tag{3.21}$$

Substituting equation (3.21) into equation (3.16), we obtain the following Lagrangian:

$$L_1(u, v, \Omega) = \frac{1}{u}\frac{\partial \Omega}{\partial \xi} - \frac{v}{u}\frac{\partial \Omega}{\partial \psi} - \frac{B}{u} - \frac{1}{2}u - \frac{v^2}{2u} = \frac{1}{u}\left[\frac{\partial \Omega}{\partial \xi} - v\frac{\partial \Omega}{\partial \psi} - B - \frac{1}{2}(u^2 + v^2)\right] \,. \tag{3.22}$$

Constraining the Lagrangian (3.22) by equations (1.10), we obtain

$$L_2(\Omega) = u = \sqrt{2(B - \frac{\partial \Omega}{\partial \xi}) - (\frac{\partial \Omega}{\partial \psi})^2} \,, \tag{3.23}$$

which is valid under the constraints formed by equations (1.10) and (1.9).

4. **Lagrange multiplier method and variational crises**

Liu tried his best to remove the constraint of the functionals (3.14) and (3.15) by Liu's systematic method, but in vain. In this section we discuss the Lagrange multiplier and its crises [19, 1, 21, 22, 23].

Now eliminating the constraints of equations (1.10) in equation (3.23), we obtain

$$\tilde{L}_2(\Omega, u, v, \lambda_1, \lambda_2) = u + \lambda_1(\frac{\partial\Omega}{\partial\xi} - \Pi) + \lambda_2(\frac{\partial\Omega}{\partial\psi} + v)\ , \quad (4.1)$$

where λ_1 and λ_2 are multipliers to be further determined, and the variation of Π depends upon equation (1.9), i.e., it follows that

$$\delta\Pi = -u\delta u - v\delta v\,.$$

According to the Lagrange multiplier method, the multipliers are considered as independent variables. Thus we obtain the following Euler equations:

$$\delta\lambda_1: \qquad \frac{\partial\Omega}{\partial\xi} = \Pi\,, \quad (4.2)$$

$$\delta\lambda_2: \qquad \frac{\partial\Omega}{\partial\psi} = -v \quad (4.3)$$

$$\delta\Omega: \qquad -\frac{\partial\lambda_1}{\partial\xi} - \frac{\partial\lambda_2}{\partial\psi} = 0\,, \quad (4.4)$$

$$\delta u: \qquad 1 + \lambda_1 u = 0\,, \quad (4.5)$$

$$\delta v: \qquad \lambda_1 v + \lambda_2 = 0\,. \quad (4.6)$$

Consequently, the multipliers can be determined as

$$\lambda_1 = -\frac{1}{u}, \qquad \lambda_2 = \frac{v}{u}\,. \quad (4.7)$$

Substituting the identified Lagrange multipliers into equation (4.1) results in

$$\tilde{L}_2(\Omega, u, v) = u - \frac{1}{u}(\frac{\partial\Omega}{\partial\xi} - \Pi) + \frac{v}{u}(\frac{\partial\Omega}{\partial\psi} + v)\,, \quad (4.8)$$

which is under the constraint of equation (1.9). Further eliminating the constraint (1.9), we obtain

$$\tilde{\tilde{L}}_2(\Omega, u, v, \Pi, \lambda_3) = u - \frac{1}{u}(\frac{\partial\Omega}{\partial\xi} - \Pi) + \frac{v}{u}(\frac{\partial\Omega}{\partial\psi} + v) + \lambda_3\left[\Pi + \frac{1}{2}(u^2 + v^2) - B\right]\,. \quad (4.9)$$

Calculating variation with respect to Π, we can easily identify the multiplier, which reads

$$\lambda_3 = -1/u\,. \quad (4.10)$$

Thus we have

$$\tilde{\tilde{L}}_2(\Omega, u, v, \Pi) = u - \frac{1}{u}\left[\frac{\partial\Omega}{\partial\xi} - B + \frac{1}{2}(u^2 + v^2)\right] + \frac{v}{u}(\frac{\partial\Omega}{\partial\psi} + v)\,. \quad (4.11)$$

According to the Lagrange multiplier method, the above Lagrangian contains four independent variables (Ω, u, v, Π). But by a careful inspection, we find the constraint, equation (1.9), is still kept as a non-variational constraint. So the Lagrange multiplier method is not valid in this case, and it is called by He the second variational crisis [21, 22, 23].

Now we apply the Lagrange multiplier method to eliminate the constraint (1.9) of equation (3.22):

$$\tilde{L}_1(u, v, \Omega, \Pi, \lambda_3) =$$
$$\frac{1}{u}\left[\frac{\partial \Omega}{\partial \xi} - v\frac{\partial \Omega}{\partial \psi} - B - \frac{1}{2}(u^2+v^2)\right] + \lambda_3\left[\Pi + \frac{1}{2}(u^2+v^2) - B\right] . \tag{4.12}$$

The stationary condition with respect to Π is

$$\lambda_3 = 0 . \tag{4.13}$$

Consequently the constraint cannot be eliminated by the multiplier either. This phenomenon is called the first variational crisis [19]. The same phenomenon will appear if we use a multiplier to eliminate the constraint of equation (1.9) or of the functionals (3.22) and (3.23).

As it was pointed out by He [12, 21, 16] the Lagrange multiplier can finally be expressed in the form

$$\lambda = \lambda(u, v, \Omega, \Phi) . \tag{4.14}$$

Thus we can introduce an unknown function F:

$$F = \lambda(u^2 + v^2 + 2\Pi - 2B) . \tag{4.15}$$

The augmented functional (4.12), therefore, can be rewritten in the form

$$J_{He1}(\Omega, v, \Pi, u) = \iint \frac{1}{\sqrt{2(B-\Pi) - v^2}}\left\{v\frac{\partial \Omega}{\partial \Psi} - \frac{\partial \Omega}{\partial \xi} + 2B - \Pi\right\} \mathrm{d}A + \iint F(u, v, \Phi, \Pi) dA , \tag{4.16}$$

where F is the function of the variables u, v, Φ, and Π.

To eliminate the constraint of the functional (3.22), a similar augmented functional can be constructed as follows

$$J_{He2}(\Omega, u, v, \Pi) = \iint \left\{\frac{v}{u}\frac{\partial \Omega}{\partial \Psi} - \frac{1}{u}\frac{\partial \Omega}{\partial \xi} + \frac{u^2+v^2}{u} + \frac{B}{u}\right\} \mathrm{d}\,A + \iint F(u, v, \Phi, \Pi) dA . \tag{4.17}$$

The unknown F can be identified by the same procedure as illustrated before. The Euler equations of the functional (4.17) are

$$-\frac{\partial}{\partial \Psi}(\frac{v}{u}) + \frac{\partial}{\partial \xi}(\frac{1}{u}) + \frac{\delta F}{\delta \Omega} = 0 , \tag{4.18}$$

$$-\frac{v}{u^2}\frac{\partial \Omega}{\partial \Psi} + \frac{1}{u^2}\frac{\partial \Omega}{\partial \xi} + \frac{u^2 - v^2 - 2B}{2u^2} + \frac{\delta F}{\delta u} = 0 , \tag{4.19}$$

$$\frac{1}{u}\frac{\partial \Omega}{\partial \Psi} + \frac{v}{u} + \frac{\delta F}{\delta v} = 0 , \tag{4.20}$$

$$\frac{\delta F}{\delta \Pi} = 0 . \tag{4.21}$$

We search an F such that the above 4 equations turn out to be the 4 field equations , i.e., equations (1.7), (1.9) and (1.10). To this end, we set

$$\frac{\delta F}{\delta \Pi} = aH^n = a\left[\frac{1}{2}(u^2+v^2)+\Pi - B(\Psi)\right]^n , \tag{4.22}$$

where a is a nonzero constant, and $n > 0$. So the unknown F can be identified as

$$F = \frac{a}{n+1}\left[\frac{1}{2}(u^2+v^2)+\Pi - B(\Psi)\right]^{n+1} + F_1(u, v, \Phi) , \tag{4.23}$$

where F_1 is an unknown function of u, v, and Φ. Substituting F into (4.18)−−(4.20) we search for an F_1 that the left equations (4.18)−−(4.20) satisfy the left field equations (1.7) and (1.10). It is clear that $F_1 = 0$. Therefore we obtain the following generalized variational principle:

$$J_{HE2}(\Omega, u, v, \Pi) = \iint \left\{ \frac{v}{u}\frac{\partial\Omega}{\partial\Psi} - \frac{1}{u}\frac{\partial\Omega}{\partial\xi} + \frac{u^2+v^2}{u} + \frac{B}{u} + \right. \\ \left. + \frac{a}{n+1}\left[\frac{1}{2}(u^2+v^2)+\Pi - B(\Psi)\right]^{n+1} \right\} \mathrm{d}\,A . \tag{4.24}$$

Similarly the unknown F in (4.16) can be easily determined, and the following functional is arrived at:

$$J_{HE1}(\Omega, v, \Pi, u) = \iint \left\{ \frac{1}{\sqrt{2(B-\Pi)-v^2}}\left[v\frac{\partial\Omega}{\partial\Psi} - \frac{\partial\Omega}{\partial\xi} + 2B - \Pi\right] + \right. \\ \left. + \frac{a}{n+1}\left[\frac{1}{2}(u^2+v^2)+\Pi - B(\Psi)\right]^{n+1} \right\} \mathrm{d}\,A . \qquad (a \neq 0, n > 1) \tag{4.25}$$

In view of equation (1.3), functional (4.17) can be re-written in the form

$$J_{HE3}(\Omega, v, \Pi, u) = \iint \left\{ \frac{1}{u}(v\frac{\partial\Omega}{\partial\Psi} - \frac{\partial\Omega}{\partial\xi} + 2B - \Pi) + F \right\} \mathrm{d}\,A . \tag{4.26}$$

The corresponding Euler equations are of the form

$$-\frac{\partial}{\partial\Psi}(\frac{v}{u}) + \frac{\partial}{\partial\xi}(\frac{1}{u}) + \frac{\delta F}{\delta\Omega} = 0 , \tag{4.27}$$

$$-\frac{1}{u^2}(v\frac{\partial\Omega}{\partial\Psi} - \frac{\partial\Omega}{\partial\xi} + 2B - \Pi) + \frac{\delta F}{\delta u} = 0 , \tag{4.28}$$

$$\frac{1}{u}\frac{\partial\Omega}{\partial\Psi} + \frac{\delta F}{\delta v} = 0 , \tag{4.29}$$

$$-\frac{1}{u} + \frac{\delta F}{\delta \Pi} = 0 . \tag{4.30}$$

Since the above equations should satisfy the field equations, we set

$$\frac{\delta F}{\delta\Omega} = \frac{\partial}{\partial\Psi}(\frac{v}{u}) - \frac{\partial}{\partial\xi}(\frac{1}{u}) = 0 , \tag{4.31}$$

$$\frac{\delta F}{\delta u} = \frac{1}{u^2}(v\frac{\partial \Omega}{\partial \Psi} - \frac{\partial \Omega}{\partial \xi} + 2B - \Pi) = \frac{1}{u^2}(-v^2 + 2B - 2\Pi) = 1\,, \tag{4.32}$$

$$\frac{\delta F}{\delta v} = -\frac{1}{u}\frac{\partial \Omega}{\partial \Psi} = \frac{v}{u} = \frac{v}{\sqrt{2(B-\Pi) - v^2}}\,, \tag{4.33}$$

$$\frac{\delta F}{\delta \Pi} = \frac{1}{u} = \frac{1}{\sqrt{2(B-\Pi) - v^2}}\,. \tag{4.34}$$

From the above relations, we have

$$F = u - \frac{1}{2}\sqrt{2(B-\Pi) - v^2}\,. \tag{4.35}$$

We obtain another variational principle in the form:

$$J_{HE3}(\Omega, v, \Pi, u) = \iint \left\{ \frac{1}{u}(v\frac{\partial \Omega}{\partial \Psi} - \frac{\partial \Omega}{\partial \xi} + 2B - \Pi) + u - \frac{1}{2}\sqrt{2(B-\Pi) - v^2} \right\} \mathrm{d}\,A\,. \tag{4.36}$$

5. **A modified Lagrange multiplier method**

In the procedure of variation, the multipliers are also considered to be independent variables. The present modification (He [21, 8]) considers the multipliers to be dependent functions. The problem of the independent Lagrange multipliers as well as the validity of the method are discussed in the paper [8] by He.

To overcome the problem the multipliers should be considered to be dependent functions during the identification of the multipliers.

Now re-consider equation (4.9), where λ_3 is not an independent variable. Thus the Euler equations can be expressed as follows

$$1 + \frac{1}{u^2}\left[\frac{\partial \Omega}{\partial \xi} - \Pi\right] - \frac{v}{u^2}(\frac{\partial \Omega}{\partial \psi} + v) + \frac{\partial \lambda_3}{\partial u}\left[\Pi + \frac{1}{2}(u^2+v^2) - B\right] + \lambda_3 u = 0\,, \tag{5.1}$$

$$\frac{1}{u}(\frac{\partial \Omega}{\partial \psi} + v) + \frac{v}{u} + \frac{\partial \lambda_3}{\partial v}\left[\Pi + \frac{1}{2}(u^2+v^2) - B\right] + \lambda_3 v = 0\,, \tag{5.2}$$

$$\frac{\partial}{\partial \xi}(\frac{1}{u}) - \frac{\partial \Omega}{\partial \psi}(\frac{v}{u}) + \frac{\partial \lambda_3}{\partial \Omega}\left[\Pi + \frac{1}{2}(u^2+v^2) - B\right] = 0\,, \tag{5.3}$$

$$\frac{1}{u} + \frac{\partial \lambda_3}{\partial \Pi}\left[\Pi + \frac{1}{2}(u^2+v^2) - B\right] + \lambda_3 = 0\,. \tag{5.4}$$

It is obvious that equation (5.4) vanishes completely if the multiplier is identified as $\lambda_3 = -1/u$ (See equation (4.10)). In order to recover equation (1.9) from equation (5.4), we can identify the multiplier in the following form

$$\lambda_3 = -\frac{1}{u} + C\left[\Pi + \frac{1}{2}(u^2+v^2) - B\right]\,. \tag{5.5}$$

where C is a nonzero constant. In this way we obtain the following modified Lagrangian

$$\tilde{\tilde{L}}_2(\Omega,u,v,\Pi)=u-\frac{1}{u}(\frac{\partial\Omega}{\partial\xi}-\Pi)+\frac{v}{u}(\frac{\partial\Omega}{\partial\psi}+v)-\frac{1}{u}\left[\Pi+\frac{1}{2}(u^2+v^2)-B\right]+ \\ +C\left[\Pi+\frac{1}{2}(u^2+v^2)-B\right]^2. \quad (5.6)$$

The multiplier in equations (4.12) and (4.13) can be identified in a similar way. The variational crisis can also be eliminated by the semi-inverse method, for example, we can re-write equation (4.9) in the form

$$\tilde{\tilde{L}}_2(\Omega,u,v,\Pi)=u-\frac{1}{u}(\frac{\partial\Omega}{\partial?\xi}-\Pi)+\frac{v}{u}(\frac{\partial\Omega}{\partial\psi}+v)+F\,. \quad (5.7)$$

where F is an unknown function to be determined.

6. Conclusion

We illustrate the effectiveness and convenience of the semi-inverse method in searching for variational principles for a physical problem, and also point out a difficulty in Liu's theory which leads to incorrect results. A modified Lagrange multiplier method is suggested, i.e., the multipliers cannot be considered to be independent variables during the procedure of their identification.

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