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FINITE ELEMENT ANALYSIS BASED ON HETEROGENEOUS MODELS

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Dedicated to Professor József FARKAS on the occasion of his seventyfifth birthday

Abstract. A heterogeneous mathematical model (elastic body - Timoshenko shell) has been applied for the analysis of some elastic structures. The boundary and variational formulation of the heterogeneous mathematical problem are presented. Bubble functions – Finite Element Method – have been applied for the numerical analysis.

Mathematical Subject Classification: 74K25, 74S05
Keywords: heterogeneous mathematical problem, finite element method, bubble functions

1. **Formulation of the problem**

1.1. **Introductory remarks.** Many structures encountered in engineering practice consist of shell parts linked to the solid continua. In the numerical analysis these structures cannot be approximated well with the lower dimensional theories of shells. If the reliability and accuracy of the computed data are to be ensured, one must use multifield modelling in numerical simulation of the structures [1]. An approach to the analysis of multistructures, based on the asymptotic theory of shells and the theory of elasticity was suggested in the work by P.Siarlet [2]. D - adaptive analysis of multistructures is considered in the work by E. Stein [3]. Here we suggest another approach to the analysis of multistructures, which is based on the Timoshenko shell theory and the theory of elasticity.

Let the elastic continuum occupy the bounded and connected domain $\Omega_1 \cap \Omega_2^*$, (Figure 1), where Ω_1, Ω_2^* are three-dimensional domains with the Lipshitz boundaries Γ_1, Γ_2^*. Let us suppose, that the three-dimensional domain Ω_1 is referred to a Cartesian coordinate system x_1, x_2, x_3. The three mutually orthogonal unit vectors on the

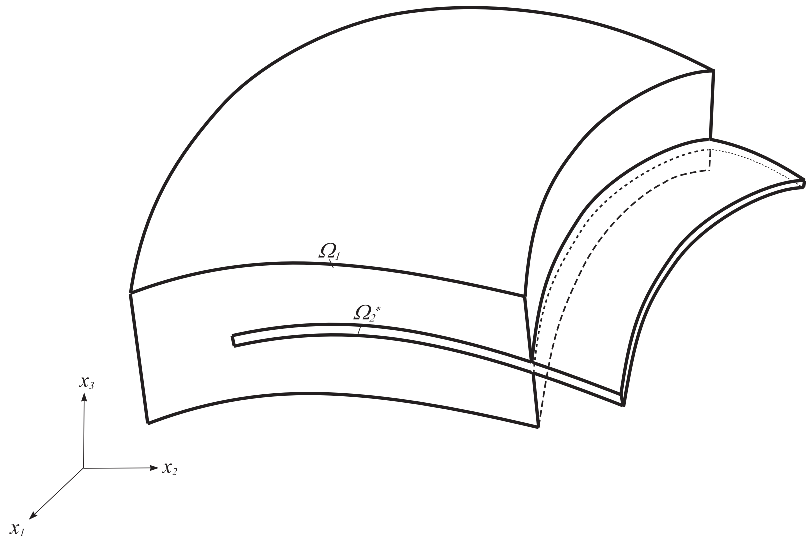

Figure 1. Thin shell embedded into a 3D body

boundary Γ_1 are denoted by $\overrightarrow{\boldsymbol{\nu}}_1$, $\overrightarrow{\boldsymbol{\nu}}_2$, $\overrightarrow{\boldsymbol{\nu}}_3$ ($\overrightarrow{\boldsymbol{\nu}}_1$ is the outer normal to Γ_1). We also suppose that the three-dimensional domain Ω_2^* is thin, i.e., one of its dimensions, the thickness h, is considerably smaller then the two others. We refer the domain Ω_2^* to the curvilinear coordinate system $\zeta_1, \zeta_2, \zeta_3$

$$\Omega_2^* = \left\{ \zeta_1, \zeta_2, \zeta_3 : \zeta_1, \zeta_2 \in \Omega_2, -\frac{h}{2} \leq \zeta_3 \leq \frac{h}{2} \right\},$$

defined on the middle surface $S \subset \mathbf{R}^3$, which is an image of the set $\Omega_2 \subset \mathbf{R}^2$ (with the boundary Γ_2) through a map

$$x_i = \varphi_i \left(\zeta_1, \zeta_2 \right), \ \zeta_1, \zeta_2 \in \Omega_2, \ i = 1, 2, 3. \tag{1}$$

Let us denote the three orthogonal right–handed unit vectors on the curve ∂S (∂S is the map of Γ_2 with respect to (1)) by $\overrightarrow{\mathbf{n}}_1$, $\overrightarrow{\mathbf{n}}_2$, $\overrightarrow{\mathbf{n}}_3$, where $\overrightarrow{\mathbf{n}}_1$ is the unit normal to $\partial\Gamma_2$ that lies in the tangent plane of the middle surface S; $\overrightarrow{\mathbf{n}}_2$ is the unit tangent to the curve Γ_2; and $\overrightarrow{\mathbf{n}}_3$ is a unit normal to the middle surface S.

1.2. **Equations of the theory of elasticity.** Let

$$\mathbf{u} = \left(u_1 \left(x \right), \ u_2 \left(x \right), \ u_3 \left(x \right) \right), \quad x = x_1, x_2, x_3, \tag{2}$$

be the displacement vector of the elastic continuum. The components $e_{ij}\left(u\right)$ of the deformation tensor are given by the relations

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \ i, j = 1, 2, 3. \tag{3}$$

The stress components are denoted by σ_{ij}. The stress strain relations are of the form

$$\sigma_{ij} = \sum_{k,l=1}^{3} c_{ijkl}\,(x)\,e_{kl}\,, \tag{4}$$

where c_{ijkl} stands for the elastic parameters. For a homogeneous and isotropic continuum equation (4) can be rewritten in the form

$$\begin{aligned} \sigma_{ii} &= \lambda\theta + 2\mu e_{ii}\,, \quad i = 1,2,3; \\ \sigma_{ij} &= 2\mu e_{ij}, \qquad i \neq j\,, \quad i,j = 1,2,3; \end{aligned} \tag{5}$$

where

$$\theta = e_{11} + e_{22} + e_{33}\,,$$

while

$$\lambda = \frac{E\nu}{(1+\nu)\,(1-2\nu)} \qquad \text{and} \qquad \mu = \frac{E}{2\,(1+\nu)}$$

are the Lame coefficients.

The components of the stress tensor satisfy the equilibrium equations

$$\sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} + f_i = 0,\; i = 1,2,3; \tag{6}$$

where f_i denote the components of the body forces applied to the elastic continuum in the domain Ω_1.

1.3. **Equations of the Timoshenko shell theory** [5]**.** The vector defined by the equation

$$\mathbf{v} = (v_1\,(\xi)\,, v_2\,(\xi)\,, w\,(\xi)\,, \gamma_1\,(\xi)\,, \gamma_2\,(\xi))\,,\; \xi = \xi_1, \xi_2, \tag{7}$$

involves the displacements $v_1\,(\xi)\,, v_2\,(\xi)\,, w\,(\xi)$ and the angles of rotations $\gamma_1\,(\xi)\,, \gamma_2\,(\xi)$ on the middle surface.

The deformation of a shell is described by the characteristics

$$\begin{aligned} \varepsilon_{\alpha\alpha} &= \frac{1}{A_\alpha}\partial_\alpha v_\alpha + \frac{1}{A_\alpha A_\beta} v_\beta \partial_\beta A_\alpha + k_\alpha w\,, \\ 2\varepsilon_{\alpha\beta} &= \frac{A_\alpha}{A_\beta}\partial_\beta \frac{v_\alpha}{A_\alpha} + \frac{A_\beta}{A_\alpha}\partial_\alpha \frac{v_\beta}{A_\beta}\,, \\ \varepsilon_{\alpha 3} &= -k_\alpha u_\alpha + \frac{1}{A_\alpha}\partial_\alpha w + \gamma_\alpha\,, \\ \chi_{\alpha\alpha} &= \frac{1}{A_\alpha}\partial_\alpha \gamma_\alpha + \frac{1}{A_\alpha A_\beta}\gamma_\beta \partial_\beta A_\alpha \\ 2\chi_{\alpha\beta} &= \frac{k_\alpha}{A_\beta}\partial_\beta \gamma_\alpha - \frac{k_\beta}{A_\alpha A_\beta} v_\alpha \partial_\beta A_\alpha + \frac{k_\beta}{A_\alpha}\partial_\alpha \gamma_\beta - \\ &\quad - \frac{k_\alpha}{A_\alpha A_\beta}\gamma_\beta \partial_\alpha A_\beta + \frac{A_\alpha}{A_\beta}\partial_\beta \frac{\gamma_\alpha}{A_\alpha} + \frac{A_\beta}{A_\alpha}\partial_\alpha \frac{\gamma_\beta}{A_\beta}\,, \end{aligned} \tag{8}$$

where $\alpha, \beta \in \{1,2\}\,; \alpha \neq \beta.$, $\partial_\alpha = \frac{\partial}{\partial \zeta_\alpha}$, A_α, k_α are Lame coefficients and main curvatures of the middle surface of the shell, respectively.

The force and moment characteristics $T_{\alpha\beta}, T_{\alpha 3}, M_{\alpha\beta}$ can be given in terms of the deformation characteristics of the shell $\varepsilon_{\alpha\beta},\ \varepsilon_{\alpha 3},\ \chi_{\alpha\beta},\ \alpha,\beta\in\{1,2\}$ via the material law:

$$\begin{aligned} T_{\alpha\alpha} &= \frac{Eh}{1-\nu^2}\left(\varepsilon_{\alpha\alpha}+\nu\varepsilon_{\beta\beta}\right), \\ T_{\alpha\beta} &= \frac{Eh}{2\left(1+\nu\right)}\varepsilon_{\alpha\beta}, \\ T_{\alpha 3} &= k'G'h\varepsilon_{\alpha 3}, \\ M_{\alpha\alpha} &= \frac{Eh^3}{12\left(1-\nu^2\right)}\left(\chi_{\alpha\alpha}+\nu\chi_{\beta\beta}\right), \\ M_{\alpha\beta} &= \frac{Eh^3}{12\left(1+\nu\right)}\chi_{\alpha\beta}, \end{aligned} \tag{9}$$

where k' is the shear coefficient, G' is the shear module. For isotropic materials

$$k'=\frac{5}{6}, \qquad G'=\frac{E}{2\left(1+\nu\right)}.$$

The force and moment characteristics introduced should satisfy the equilibrium equations

$$\begin{aligned} &\frac{1}{A_\alpha A_\beta}\partial_\alpha A_\beta T_{\alpha\alpha}-\frac{1}{A_\alpha A_\beta}\partial_\alpha(A_\beta)T_{\beta\beta}+\frac{1}{A_\alpha^2 A_\beta}\partial_\beta A_\alpha^2 T_{\alpha\beta}+k_\alpha T_{\alpha 3}+ \\ &\qquad +\frac{1}{A_\alpha A_\beta}\partial_\beta A_\alpha k_\alpha M_{\alpha\beta}+\frac{k_\beta}{A_\alpha A_\beta}\partial_\beta(A_\alpha)M_{\alpha\beta}+p_\alpha=0, \\ &-k_1T_1-k_2T_2+\frac{1}{A_1A_2}\partial_1 A_2 T_{13}+\frac{1}{A_1A_2}\partial_2 A_1 T_{23}+p_3=0, \\ &-T_{\alpha 3}+\frac{1}{A_\alpha A_\beta}\partial_\alpha A_\beta M_{\alpha\alpha}-\frac{1}{A_\alpha A_\beta}\partial_\alpha(A_\beta)M_{\beta\beta}+\frac{1}{A_\alpha^2 A_\beta}\partial_\beta A_\alpha^2 M_{\alpha\beta}+m_\alpha=0, \end{aligned} \tag{10}$$

where

$$\begin{aligned} p_i &= \left(1+k_1\frac{h}{2}\right)\left(1+k_2\frac{h}{2}\right)\sigma_{i3}^{+}+\left(1-k_1\frac{h}{2}\right)\left(1-k_2\frac{h}{2}\right)\sigma_{i3}^{-}+ \\ &\quad +\int_{-h/2}^{h/2}\left(1+k_1\zeta_3\right)\left(1+k_2\zeta_3\right)f_i d\zeta_3, \quad i=1,2,3; \\ m_i &= \left(1+k_1\frac{h}{2}\right)\left(1+k_2\frac{h}{2}\right)\frac{h}{2}\sigma_{j3}^{+}-\left(1-k_1\frac{h}{2}\right)\left(1-k_2\frac{h}{2}\right)\frac{h}{2}\sigma_{j3}^{-}+ \\ &\quad +\int_{-h/2}^{h/2}\left(1+k_1\zeta_3\right)\left(1+k_2\zeta_3\right)f_i\zeta_3 d\zeta_3, \quad j=1,2; \end{aligned} \tag{11}$$

in which f_i stands for the components of body forces in the domain Ω_2, σ_{i3}^{+}, σ_{i3}^{-} are the components of the surface forces on the shell surfaces $\zeta_3=+h/2,\ \zeta_3=-h/2$.

1.4. **Boundary and junction conditions.** We shall assume that boundary Γ_1 consists of parts $\Gamma_1^{(i)}$, that is

$$\Gamma_1 = \bigcup_{i=1}^{5} \Gamma_1^{(i)}, \ \Gamma_1^{(i)} \bigcap_{\substack{i,j=1,\\ i\neq j}}^{5} \Gamma_1^{(j)} = \emptyset\,. \tag{12}$$

Boundary Γ_2^* of the thin domain Ω_2^* consists of the side surface Γ_2^c and the two face surfaces Γ_2^+, Γ_2^-. The side surface Γ_2^c is a cylindrical one, which is generated by the motion of the normal to the middle surface S along the boundary ∂S of the middle surface. The boundary ∂S of the middle surface is the curve $\partial S \subset \mathbf{R}^3$, which is the map (1) $\Gamma_2 \subset \mathbf{R}^2$. Let us suppose that Γ_2 consists of parts $\Gamma_2^{(i)}$, which satisfy conditions

$$\Gamma_2 = \bigcup_{i=1}^{3} \Gamma_2^{(i)}, \ \Gamma_2^{(i)} \bigcap_{\substack{i,j=1,\\ i\neq j}}^{3} \Gamma_2^{(j)} = \emptyset. \tag{13}$$

The following boundary conditions are imposed on the parts of the boundary $\Gamma_1^{(1)}, \Gamma_1^{(2)}, \Gamma_2^{(1)}, \Gamma_2^{(2)}$:

$$u_1^\nu = 0\,,\ u_2^\nu = 0\,,\ u_3^\nu = 0\,, \quad x \in \Gamma_1^{(1)}; \tag{14}$$

$$\sigma_{11}^\nu = 0\,,\ \sigma_{12}^\nu = 0\,,\ \sigma_{13}^\nu = 0\,, \quad x \in \Gamma_1^{(2)}; \tag{15}$$

$$v_1^n = 0\,,\ v_2^n = 0\,,\ w = 0\,,\ \gamma_1^n = 0\,,\ \gamma_2^n = 0\,, \quad \zeta_1, \zeta_2 \in \Gamma_2^{(1)}; \tag{16}$$

$$T_{11}^n = 0\,,\ T_{12}^n = 0\,,\ T_{13}^n = 0\,,\ M_{11}^n = 0\,,\ M_{12}^n = 0\,, \quad \zeta_1, \zeta_2 \in \Gamma_2^{(2)}\,, \tag{17}$$

where u_1^ν, u_2^ν, u_3^ν, v_1^n, v_2^n, γ_1^n, γ_2^n are the normal deflections and rotation angles on the boundaries Γ_1 and Γ_2; σ_{ij}^ν, $T_{\alpha\beta}^n$, $T_{\alpha 3}^n$, $M_{\alpha\beta}^n$ are the normal stresses, forces and moments on the boundaries Γ_1 and Γ_2.

We shall also assume that $\Gamma_1^{(3)}$ and $\Gamma_2^{(3)}$ satisfy the relations

$$\Gamma_1^{(3)} = \left\{ \zeta_1, \zeta_2, \zeta_3 : \zeta_1, \zeta_2 \in \Gamma_2^{(3)}, -\frac{h}{2} \le \zeta_3 \le \frac{h}{2} \right\}.$$

On this part of the boundary perfect contact of two elastic continua, which occupy domains Ω_1, Ω_2^*, is carried out. On the part of the boundary the following relations exist (Figure 2)

$$\vec{\boldsymbol{\nu}}_1 = -\vec{\mathbf{n}}_1,\ \vec{\boldsymbol{\nu}}_2 = -\vec{\mathbf{n}}_2,\ \vec{\boldsymbol{\nu}}_3 = \vec{\mathbf{n}}_3\,.$$

On the boundary $\Gamma_1^{(3)}$ we specify the following junction conditions.

Geometrical conditions:

$$\begin{aligned} u_1^\nu &= -v_1^n - \zeta_3 \gamma_1^n\,, \\ u_2^\nu &= -v_2^n - \zeta_3 \gamma_2^n\,, \\ u_3^\nu &= w\,. \end{aligned} \tag{18}$$

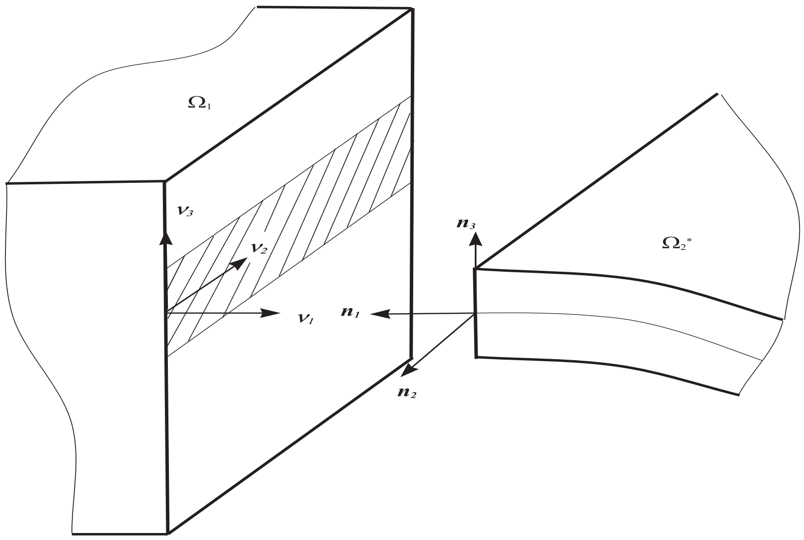

Figure 2. Junction of a thin shell to a 3D elastic body

Statical conditions:

$$-\int_{-h/2}^{h/2} \sigma_{11}^{\nu}\left(1+k_{\nu}\zeta_3\right)d\zeta_3 = T_{11}^{n}\,, \quad -\int_{-h/2}^{h/2} \sigma_{12}^{\nu}\left(1+k_{\nu}\zeta_3\right)d\zeta_3 = T_{12}^{n}\,,$$

$$\int_{-h/2}^{h/2} \sigma_{33}^{\nu}\left(1+k_{\nu}\zeta_3\right)d\zeta_3 = T_{13}^{n}\,, \tag{19}$$

$$-\int_{-h/2}^{h/2} \sigma_{11}^{\nu}\zeta_3\left(1+k_{\nu}\zeta_3\right)d\zeta_3 = M_{11}^{n}\,, \quad -\int_{-h/2}^{h/2} \sigma_{12}^{\nu}\zeta_3\left(1+k_{\nu}\zeta_3\right)d\zeta_3 = M_{12}^{n}\,,$$

where k_{ν} is the curvature of the normal section along the boundary curve of the shell.

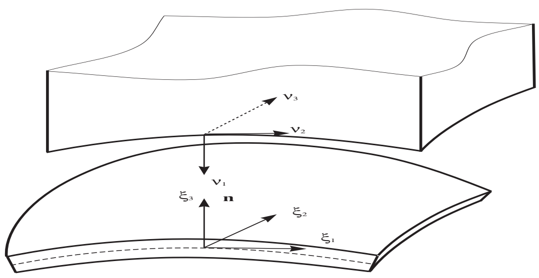

Figure 3. Embedding of a thin shell into a 3D elastic body on the upper face

Let us define junction conditions on the surfaces $\Gamma_1^{(4)} = \Gamma_2^+$ (see Figure 3). They have the following forms:

Geometrical conditions:

$$u_1^\nu = -w,\ u_2^\nu = u_1^n + \frac{h}{2}\gamma_1^n,\ u_3^\nu = u_2^n + \frac{h}{2}\gamma_2^n. \tag{20}$$

Statical conditions:

$$\sigma_{11}^\nu = -\sigma_{33}^+,\ \sigma_{12}^\nu = \sigma_{13}^+,\ \sigma_{13}^\nu = \sigma_{23}^+. \tag{21}$$

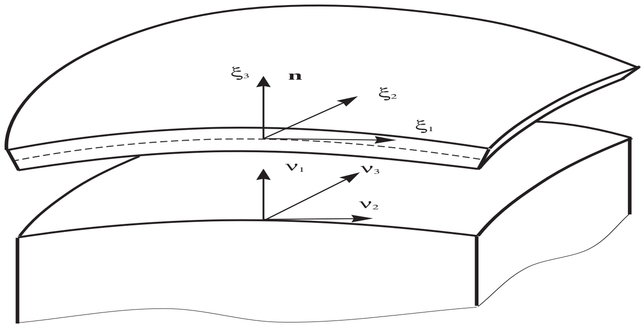

Figure 4. Embedding of a thin shell into a 3D elastic body on the lower face

We shall also define junction conditions on the surfaces $\Gamma_1^{(5)} = \Gamma_2^-$ (see Figure 4).

Geometrical conditions:

$$u_1^\nu = w,\ u_2^\nu = u_1^n - \frac{h}{2}\gamma_1^n,\ u_3^\nu = u_2^n - \frac{h}{2}\gamma_2^n. \tag{22}$$

Statical conditions:

$$\sigma_{11}^\nu = p_n^-,\ \sigma_{12}^\nu = p_1^-,\ \sigma_{13}^\nu = p_2^-. \tag{23}$$

Thus heterogeneous mathematical model [6] consists of the equations (6), (3) - (5), (8) - (10); boundary conditions (14) - (17); and junction conditions (18) - (23).

2. **Variational formulation**

Let us consider the function space

$$V = \Big\{ \mathbf{U} = (\mathbf{u}, \mathbf{v}), \mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, w, \gamma_1, \gamma_2), \mathbf{u} \in \mathbf{W}_2^{(1)}(\Omega_1), \mathbf{v} \in \mathbf{W}_2^{(1)}(\Omega_2), \text{conditions (14), (16), (18), (20), (22)} \Big\}.$$

We shall formulate two equivalent variational problems for the heterogeneous mathematical model: theory of elasticity and Timoshenko shell theory in displacements. Find a solution $\mathbf{U}$ which minimizes the functional (principle of the minimum of potential energy)

$$F(\mathbf{U}) \to \min,\ \mathbf{U} \in V \tag{24}$$

and find a $\mathbf{U}$, which satisfies the variational equation (weak formulation):

$$a_1(\mathbf{u},\widetilde{\mathbf{u}})+a_2(\mathbf{v},\widetilde{\mathbf{v}})=\left(\mathbf{P},\widetilde{\mathbf{U}}\right),\ \mathbf{U}=(\mathbf{u},\mathbf{v})\in\mathbf{V},\ \forall\widetilde{\mathbf{U}}=(\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})\in V\ . \tag{25}$$

Here

$$\begin{gathered}
F(\mathbf{U})=a_1(\mathbf{u},\mathbf{u})+a_2(\mathbf{v},\mathbf{v})-2(\mathbf{P},\mathbf{U})\ ,\\
a_1(\mathbf{u},\widetilde{\mathbf{u}})=2\int_{\Omega_1}W_1(\mathbf{u},\widetilde{\mathbf{u}})\,d\Omega_1\ ,\\
W_1(\mathbf{u},\widetilde{\mathbf{u}})=\frac{1}{2}\left[e_{11}(\mathbf{u})\,\sigma_{11}(\widetilde{\mathbf{u}})+...+e_{23}(\mathbf{u})\,\sigma_{23}(\widetilde{\mathbf{u}})\right]\ ,\\
a_2(\mathbf{v},\widetilde{\mathbf{v}})=2\int_{\Omega_2}W_2(\mathbf{v},\widetilde{\mathbf{v}})\,d\Omega_2\ ,\\
W_2(\mathbf{v},\widetilde{\mathbf{v}})=\frac{1}{2}\left[\varepsilon_{11}(\mathbf{v})\,T_{11}(\widetilde{\mathbf{v}})+\varepsilon_{22}(\mathbf{v})\,T_{22}(\widetilde{\mathbf{v}})+\varepsilon_{12}(\mathbf{v})\,T_{12}(\widetilde{\mathbf{v}})+\right.\\
\left.+\varepsilon_{13}(\mathbf{v})\,T_{13}(\widetilde{\mathbf{v}})+\varepsilon_{23}(\mathbf{v})\,T_{23}(\widetilde{\mathbf{v}})+\chi_{11}(\mathbf{v})\,M_{11}(\widetilde{\mathbf{v}})+2\chi_{12}(\mathbf{v})\,M_{12}(\widetilde{\mathbf{v}})\right]\ ,\\
\left(\mathbf{P},\widetilde{\mathbf{U}}\right)=\int_{\Omega_1}\sum_{i=1}^{3}u_if_id\Omega_1+\int_{\Omega_2}(v_1p_1+v_2p_2+wp_3+\gamma_1m_1+\gamma_2m_2)\,d\Omega_2\,.
\end{gathered} \tag{26}$$

3. **Penalty variational formulation**

Consider a penalty variational formulation of the heterogenous mathematical model in the following two forms:

$$F_\varepsilon(\mathbf{U}_\varepsilon)\to\min,\ \mathbf{U}_\varepsilon\in V_\varepsilon,\,\varepsilon\to 0 \tag{27}$$

and

$$a_1(\mathbf{u}_\varepsilon,\widetilde{\mathbf{u}})+a_2(\mathbf{v}_\varepsilon,\widetilde{\mathbf{v}})+\frac{1}{\varepsilon}a_3\left(\mathbf{U}_\varepsilon,\widetilde{\mathbf{U}}\right)=\left(\mathbf{P},\widetilde{\mathbf{U}}\right)\ , \tag{28}$$

$$\begin{gathered}
\mathbf{U}_\varepsilon=(\mathbf{u}_\varepsilon,\mathbf{v}_\varepsilon)\,,\ \mathbf{U}_\varepsilon\in V_\varepsilon,\,\varepsilon>0,\ \varepsilon\to 0\,,\\
\forall\widetilde{\mathbf{U}}\in V_\varepsilon\,,\\
V_\varepsilon=\Big\{\mathbf{U}=(\mathbf{u},\mathbf{v})\,,\mathbf{u}=(u_1,u_2,u_3)\,,\mathbf{v}=(v_1,v_2,w,\gamma_1,\gamma_2)\,,\mathbf{u}\in\mathbf{W}_2^{(1)}(\Omega_1)\,,\\
\mathbf{v}\in\mathbf{W}_2^{(1)}(\Omega_2)\ ,\text{conditions (14), (16), (20), (22)}\Big\}\,.
\end{gathered}$$

Here

$$\begin{gathered}
F_\varepsilon(\mathbf{U}_\varepsilon)=a_1(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon)+a_2(\mathbf{v}_\varepsilon,\mathbf{v}_\varepsilon)+\frac{1}{\varepsilon}a_3(\mathbf{U}_\varepsilon,\mathbf{U}_\varepsilon)-2(\mathbf{P},\mathbf{U}_\varepsilon)\,,\\
a_3\left(\mathbf{U}_\varepsilon,\widetilde{\mathbf{U}_\varepsilon}\right)=\int_{\Gamma_1^{(3)}}\{(u_1^\nu+v_1^n+\zeta_3\gamma_1^n)(\widetilde{u}_1^\nu+\widetilde{v}_1^\nu+\zeta_3\widetilde{\gamma}_1^\nu)+\\
+(u_2^\nu+v_2^n+\zeta_3\gamma_2^n)(\widetilde{u}_2^\nu+\widetilde{v}_2^\nu+\zeta_3\widetilde{\gamma}_2^\nu)+(u_3^\nu-w)(\widetilde{u}_3^\nu-\widetilde{v}_3^\nu)\}\,d\Gamma.
\end{gathered}$$

The penalty item $\frac{1}{\varepsilon}a_3(\mathbf{U}_\varepsilon,\mathbf{U}_\varepsilon)$ is introduced to avoid satisfying the geometrical conditions on the boundary $\Gamma_1^{(3)}$

$$u_1^\nu=-v_1^n-\zeta_3\gamma_1^n,$$

$$u_2^\nu = -v_2^n - \zeta_3 \gamma_2^n,$$
$$u_3^\nu = w,$$

which are difficult to satisfy in a finite element algorithm.

4. **Numerical examples**

4.1. **Example 1.** As a test problem we shall consider a one-dimensional problem, i.r. a plate subjected to uniform pressure $p_0 = const$. The plate is simply supported.

It‘s well known [5], [7] that there appears the locking effect if we use the FEM to the analysis of shells on the base of Timoshenko‘s shell theory, To remove the inaccuracy in results generated by the locking effect, we suggest to use bubble-approximation for the unknown displacements.

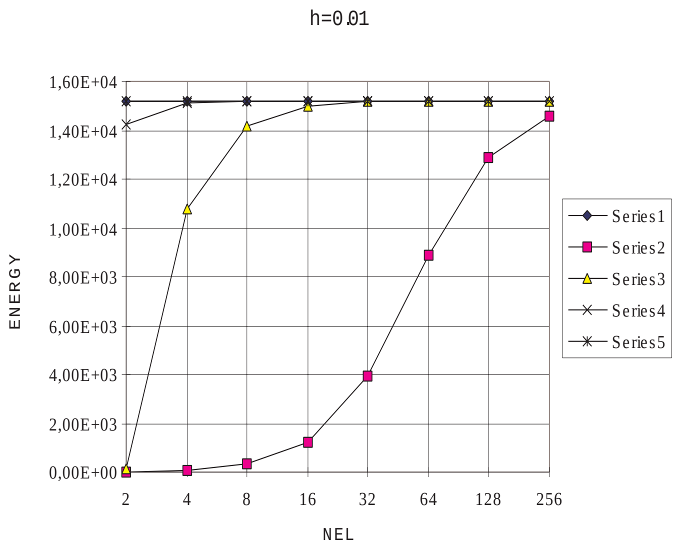

Figure 5. Convergence of the FEM solution

We map the finite element

$$\Omega_k = \left\{\zeta_1 : \zeta_1^{k-1} \leq \zeta_1 \leq \zeta_1^k\right\}$$

onto the standard element

$$\Omega_{st} = \{\xi : -1 \leq \xi \leq 1\}$$

with the mapping

$$\zeta_1 = \frac{1-\xi}{2}\zeta_1^{k-1} + \frac{1+\xi}{2}\zeta_1^k \ ,$$

and select the following shape functions [4]

$$\varphi_1 = \frac{1-\xi}{2}\ , \quad \varphi_2 = \frac{1+\xi}{2}\ , \quad \varphi_i = \Phi_{i-1}(\xi) \quad i = 3, 4, ...m,$$

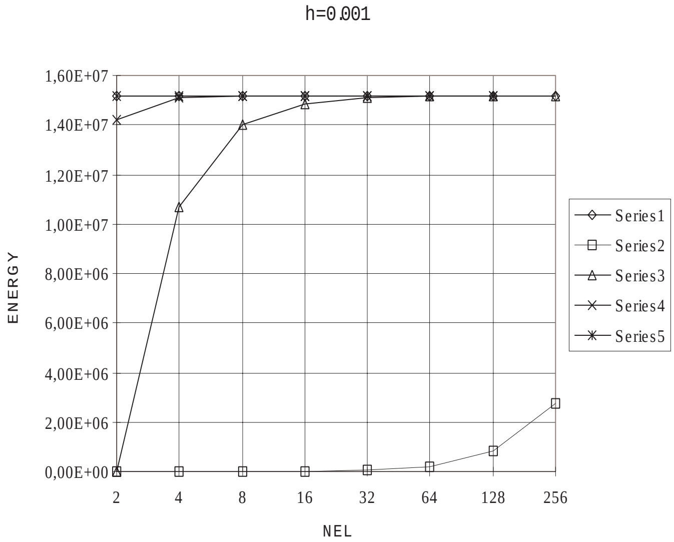

Figure 6. Convergence of the FEM solution

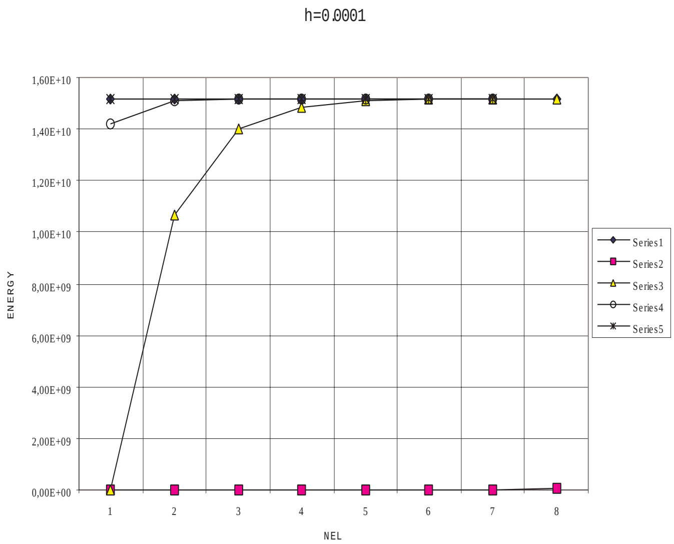

Figure 7. Convergence of the FEM solution

where

$$\Phi_j = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(t)dt \,, \quad j = 2, 3, \ldots$$

in which P_j is the j-th Legendre polynomial.

Figures 5-7 illustrate the convergence of the FEM solution in energy norm for different thicknesses (h/l) depending on the number of elements (Series 1: exact value; Series 2: $m = 2$; Series 3: $m = 3$; Series 4: $m = 4$; Series 5: $m = 5$).

4.2. **Example 2.** Let us apply the heterogeneous mathematical model for the numerical analysis of a tube junction (Figure 8) which is subjected to inner pressure[8].

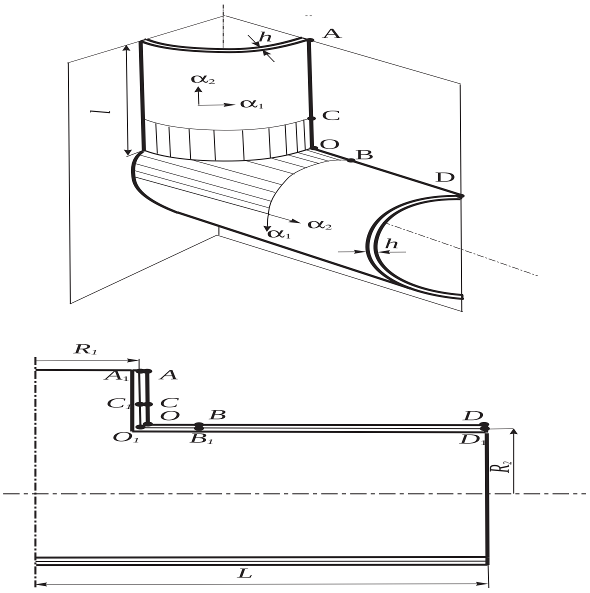

Figure 8. Tube junction

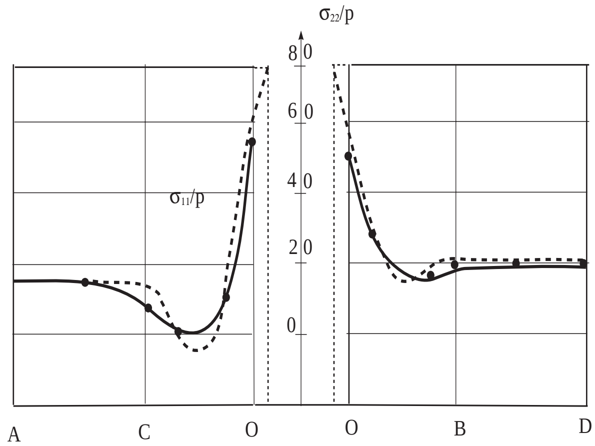

Figure 9. Graphs of stresses in the tube junction

The data are as follows:

$$R_1 = 0.5R_2\,,\quad h = 0.05R_2\,,\quad l = 1.7R_2\,,\quad L = 2.2R_2\,,$$
$$E = 2.1\times 10^5 p\,,\quad \nu = 0.3\,,\quad \varepsilon = 10^{-4}$$

For the analysis we used FEM with quadratic approximation on two-dimensional (theory of shell) and three-dimensional (theory of elasticity in shaded region).

In Figure 9 the graphs of stresses σ_{22}/p along the line ACOBD are shown. The solid line corresponds to the analysis by the FEM based on the heterogeneous model. The dotted line corresponds to the analysis by the FEM based on the theory of coupled shells. The point O_1 is the point where the middle surfaces meet. The value of stresses in this point obtained using the theory of coupled shells is not adequate. Bold dots represent the results of experimental data.

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